SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2010

Year 11 Half Yearly

Mathematics Accelerated

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Start each **NEW** question in a separate answer booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All answers must be given in exact simplified form unless otherwise stated.
- All necessary working should be shown in every question.

Total Marks – 83

- Attempt questions 1-7.

Examiner: *D.McQuillan*

Start a new answer booklet

Question One (12 marks)

(a) Factorise $64a^6 - b^3$. **1**

(b) Write $8x^2 - 11x + 2$ in the form $Ax(x-1) + Bx + C$. **2**

(c) Solve and graph $|x-5| \geq 3$. **2**

(d) Solve the equation $x^6 - 3x^3 - 40 = 0$. **3**

(e) Find the equation of the line that passes through the point (–1, 1) and the intersection of $x + 2y = 8$ and $3x - 4y = 4$. **2**

(f) Is the function $f(x) = |x-2| - |x+2|$ ODD, EVEN or NEITHER. Justify your answer. **2**

End of Question One

Start a new answer booklet

Question Two (11 marks)

(a) Sketch $y = \tan x$ for $-180° \le x \le 180°$. **2**

(b) Solve **3**

$$a + b + c = 6$$
$$5a - b + 4c = 32$$
$$a - 3b - 2c = -2$$

(c) Find the exact value of **3**

(i) $\sin\left(\dfrac{\pi}{4}\right)$

(ii) $\sec\left(-\dfrac{4\pi}{3}\right)$

(d) Show that $\dfrac{1-\cos^4 A}{\sin^4 A} = 2\cot^2 A + 1$ **3**

End of Question Two

Start a new answer booklet

Question Three (12 marks)

(a) Graph the following functions indicating domain, range and other main features. **12**

(i) $f(x) = x^2 - 3x - 10$

(ii) $h(x) = \sqrt{5 - x}$

(iii) $g(x) = \dfrac{x+1}{x-1}$

End of Question Three

Start a new answer booklet

Question Four (12 marks)

(a) Given the parabola $2y = 3 + 2x - x^2$. **5**

(i) Rewrite in the form $(x-h)^2 = 4a(y-k)$.

(ii) Write down the

(α) Axis of symmetry

(β) Focus

(γ) Directrix

(b) Evaluate $\sum_{r=2}^{\infty} 3 \times \left(\frac{1}{2}\right)^r$. **2**

(c) Given that $\cot\alpha = 3$ for $180° \le \alpha \le 360°$ find the value of **3**

(i) $\tan\alpha$

(ii) $\sin\alpha$

(d) Find all possible values of a, b and c if $12 + a + b + c + 3\frac{51}{64}$ is a geometric series. **2**

End of Question Four

Start a new answer booklet

Question Five (13 marks)

(a) Find the sum of n terms of the series $\frac{n+1}{n}+\frac{n+2}{n}+\frac{n+3}{n}+\ldots$ **3**

(b) Use the triangle inequality to prove that $|x-y| \geq |x|-|y|$. **2**

(c) Express the hypotenuse h of a right triangle in terms if its area A and its perimeter P. **3**

(d) **5**

(i) Find the intersection of $xy=4$ and $y=x+3$.

(ii) Graph the region defined by the intersection of $xy<4$ and $y \geq x+3$.

End of Question Five

Start a new answer booklet

Question Six (13 marks)

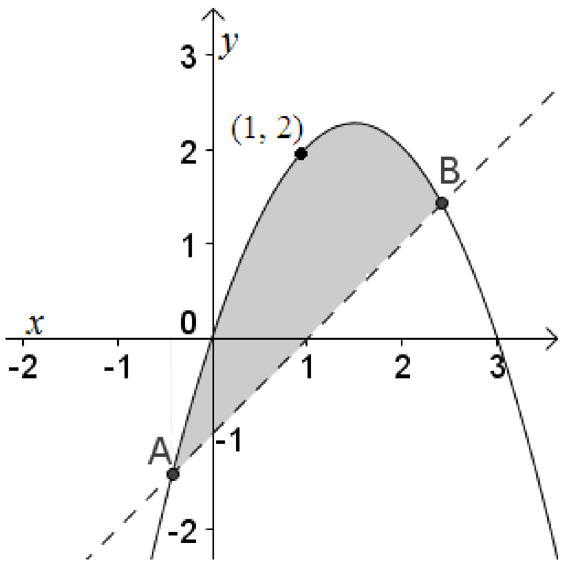
(a) Given the quadratic $2x^2 + kx + 5$, **6**

(i) For what range of values of k is the quadratic positive definite?

(ii) For what values of k does the quadratic have equal roots?

(iii) Find two integral values of k such that the quadratic has rational roots?

(b) Describe the region in the graph. **4**



(c) Write down the next two terms of the series **3**

$$\frac{1}{\sqrt{2}-1} + \frac{1}{3-2\sqrt{2}} + \frac{1}{5\sqrt{2}-7} + \ldots$$

End of Question Six

Start a new answer booklet

Question Seven (10 marks)

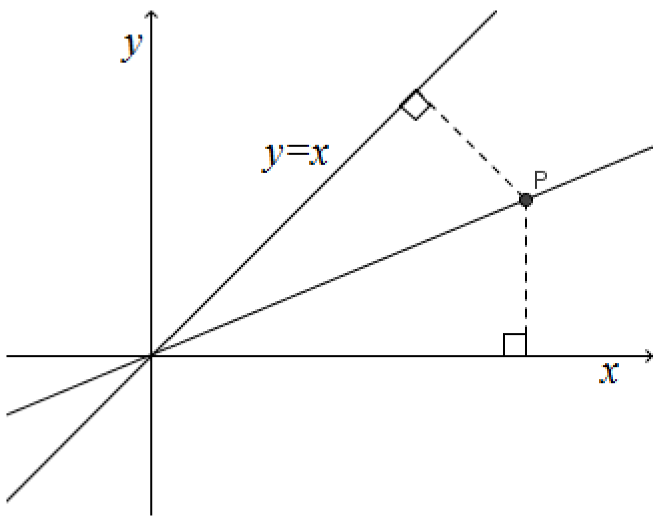
(a) Anna borrows \$3000 from a bank at 1.4% per month reducible interest. **6**

(i) She repays the loan in 36 equal monthly instalments. What are her repayments each month?

(ii) If she increases her repayments to \$150 per month how many instalments will it take to repay the loan.

(b) **4**

(i) What is the expression for the perpendicular distance of the point $P(x, y)$ from the line $y = x$?



(ii) What is the equation of the locus of a point which moves so that its perpendicular distance from the line $y = x$ is equal to its perpendicular distance from the x-axis?

End of Exam

(a) $64a^6 - b^3$ 1

$= (4a^2)^3 - b^3$

$= (4a^2 - b)(16a^4 + 4a^2b + b^2)$

(b) $8x^2 - 11x + 2$ 2

$= \underset{A}{8}x(x-1)\underset{B}{(-3)}x + \underset{C}{2}$

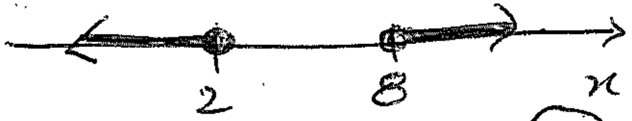
(c) $|x-5| \geq 3$ 2

$x - 5 \geq 3$

$x \geq 8$

or $x - 5 \leq -3$

$x \leq 2$



(d) $x^6 - 3x^3 - 40 = 0$ 3

$m = x^3$

$m^2 - 3m - 40 = 0$

$(m-8)(m+5) = 0$

$\therefore x^3 = 8 \Rightarrow x = 2$

$x^3 = -5 \Rightarrow x = \sqrt[3]{-5}$

(e) 2

$x + 2y - 8 + k(3x - 4y - 4) = 0$

$(-1, 1)$

$-1 + 2 - 8 + k(-3 - 8) = 0$

$-7 - 11k = 0$

$\therefore -11k = 7$

$\Rightarrow k = -\frac{7}{11}, \frac{-11}{7}$

$\therefore 11x + 22y - 88$

$-21x + 28y + 28 = 0$

$-10x + 50y - 60 = 0$

$x - 5y + 6 = 0$

(f) 2

$f(x) = |x-2| - |x+2|$

$f(-x) = |-(x+2)| - |2-x|$

$= |x+2| - |-(x-2)|$

$= |x+2| - |x-2|$

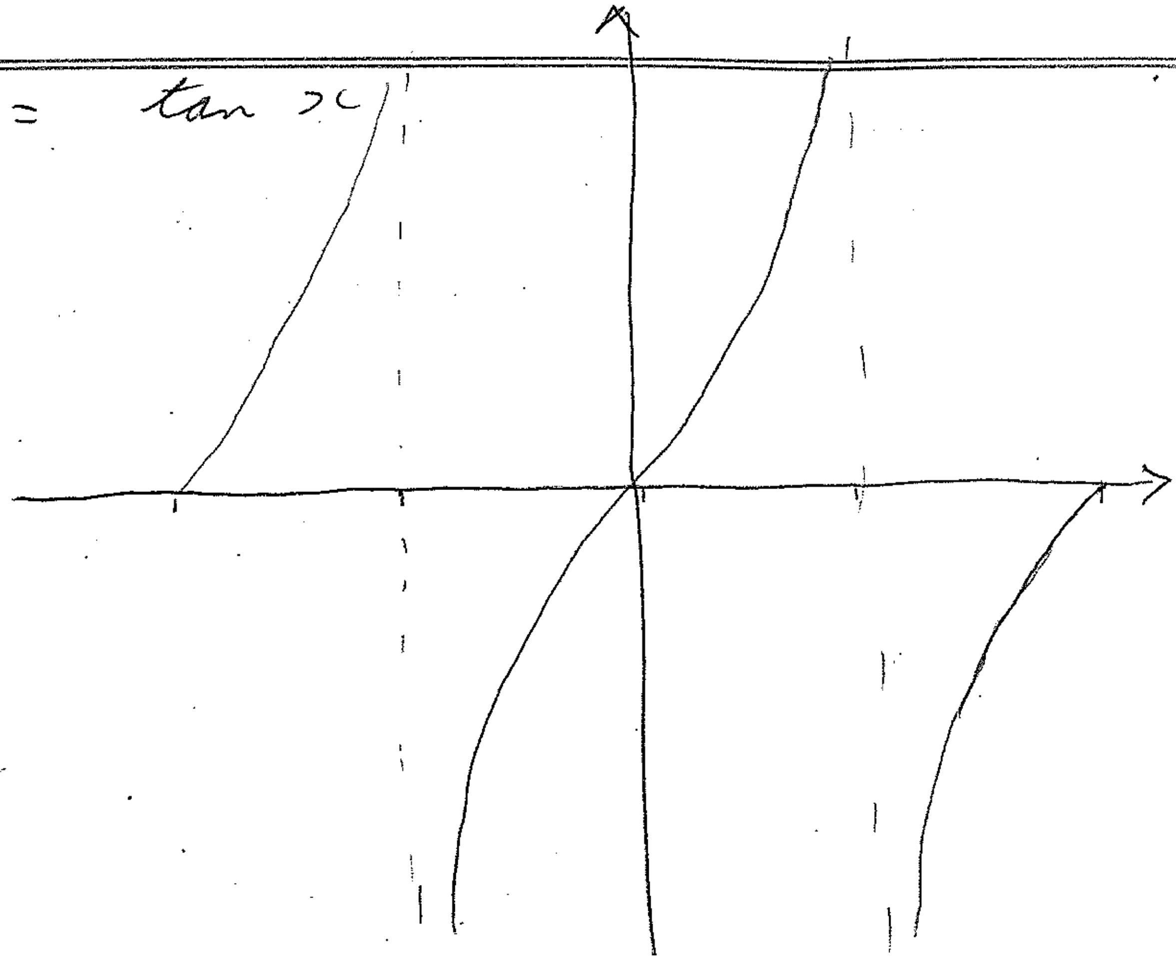
$\therefore -f(-x)$

$= |x-2| - |x+2|$

$\therefore f(x) = -f(-x)$

i.e. f is odd.

(a) $y = \tan x$



(b)

$$a + b + c = 6 \quad -(i)$$
$$5a - b + 4c = 32 \quad -(ii)$$
$$a - 3b - 2c = -2 \quad -(iii)$$

$(i)+(ii)$ $\quad 6a + 5c = 38 \quad (iv)$

$3\times(ii) - (iii)$ $\quad 14a + 14c = 98$

$$a + c = 7 \quad (v)$$

sub (v) in (iv) $\quad 6(7-c) + 5c = 38$

$$42 - 6c + 5c = 38$$
$$42 - c = 38$$
$$c = 4$$
$$\therefore a = 3$$
$$b = -1$$

(c) (i) $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

(ii) $\sec\left(\frac{-4\pi}{3}\right) = \sec\left(\frac{4\pi}{3}\right) = \frac{1}{\cos\left(\frac{4\pi}{3}\right)} = \frac{1}{-\left(\frac{1}{2}\right)} = -$

(ii) Show $\dfrac{1-\cos^4 A}{\sin^4 A} = 2\cot^2 A + 1$

$$LHS = \frac{(1-\cos^2 A)(1+\cos^2 A)}{\sin^4 A}$$

$$= \frac{\sin^2 A\,(1+\cos^2 A)}{\sin^4 A}$$

$$= \frac{1+\cos^2 A}{\sin^2 A}$$

$$= \operatorname{cosec}^2 A + \cot^2 A$$

$$= (1+\cot^2 A) + \cot^2 A$$

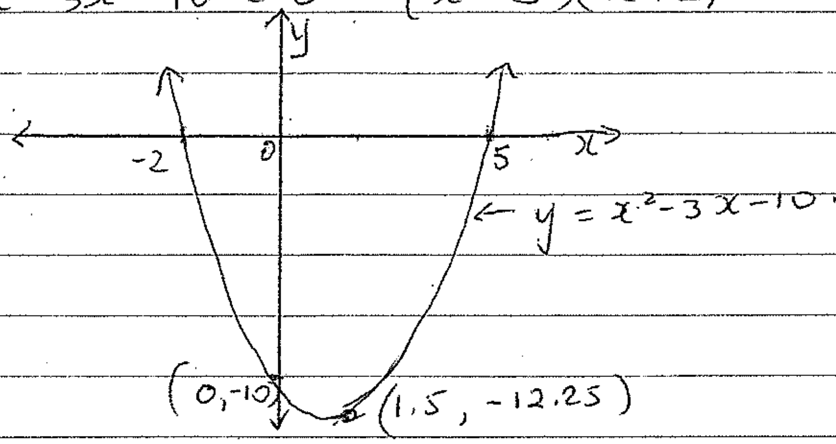
$$= RHS$$

QUESTION 3

(i)

$f(x) = x^2 - 3x - 10$

$x^2 - 3x - 10 = 0 \qquad (x-5)(x+2) = 0$

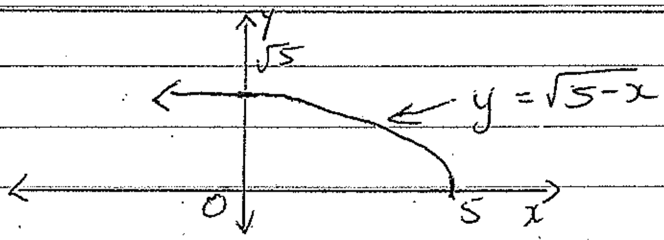


Vertex $\left(\frac{3}{2}, -12\frac{1}{4}\right)$

Domain: All real x

Range $y \geq -12\frac{1}{4}$

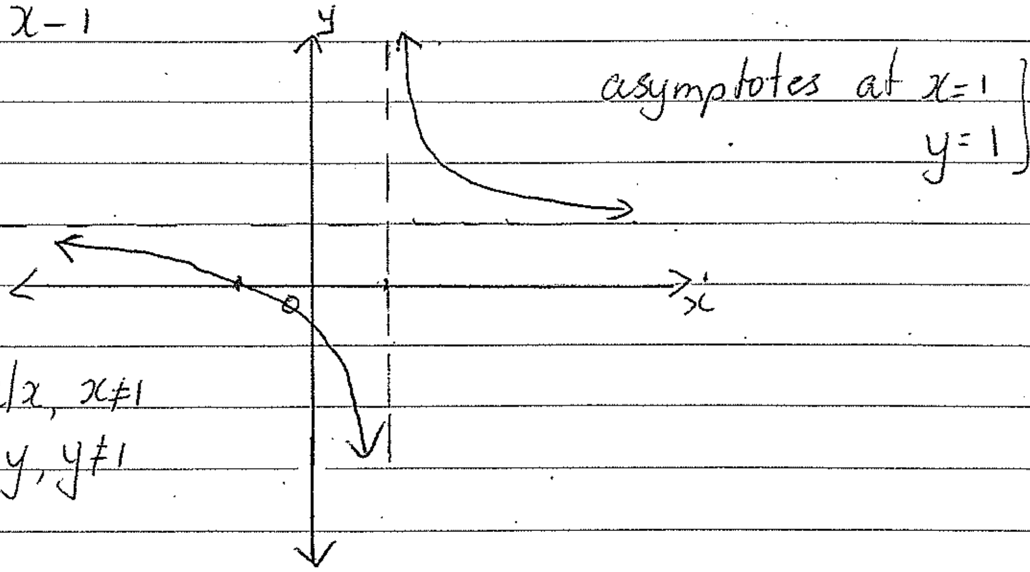
(ii) $h(x) = \sqrt{5-x}$



Domain: $x \leq 5$

Range: $y \geq 0$

(iii) $g(x) = \dfrac{x+1}{x-1}$



Domain: All real x, $x \neq 1$

Range: All real y, $y \neq 1$

Question 4

a) i) $2y = 3 + 2x - x^2$

$x^2 - 2x = -2y + 3$

$x^2 - 2x + 1 = -2y + 4$

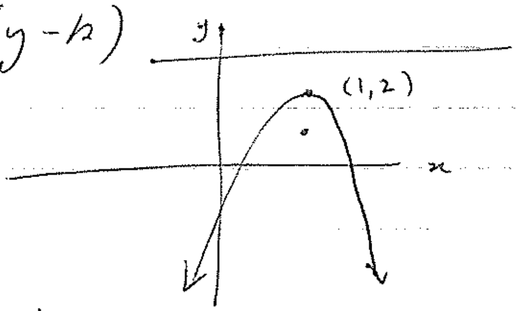
$(x-1)^2 = -2(y-2)$ $\qquad\qquad (x-1)^2 = 4\left(-\frac{1}{2}\right)(y-2)$

(ii) In the form $(x-h)^2 = -4a(y-k)$

vertex $(1, 2)$

$-4a = -2$

$a = \frac{1}{2}$



(α) axis of symmetry: $x = 1$

(β) Focus: $\left(1, \frac{3}{2}\right)$

(γ) Directrix: $y = \frac{5}{2}$

b) $\sum_{r=2}^{\infty} 3 \times \left(\frac{1}{2}\right)^r = 3 \times \left(\frac{1}{2}\right)^2 + 3 \times \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^4 + \ldots\ldots$

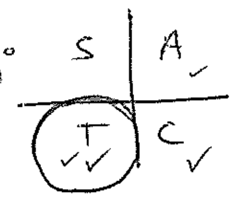
$= \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \ldots\ldots$

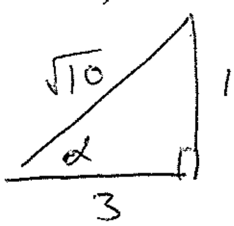
$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{\frac{3}{4}}{1 - \frac{1}{2}}$

$= \frac{3}{2}$

c) $\cot\alpha = 3$, $180^\circ \le \alpha \le 360^\circ$





(i) $\tan\alpha = \frac{1}{3}$

(ii) $\sin\alpha = -\frac{1}{\sqrt{10}}$ or $-\frac{\sqrt{10}}{10}$

d) $12 + a + b + c + 3\frac{51}{64}$

$T_1 = 12 (= A)$

$T_5 = 3\frac{51}{64}$

$T_N = AR^{N-1}$

$T_5 = 12R^4 = 3\frac{51}{64}$

$R^4 = \frac{81}{256}$

$R = \pm\frac{3}{4}$

$\therefore \left.\begin{aligned} a &= 9 \\ b &= 6\tfrac{3}{4} \\ c &= 5\tfrac{1}{16} \end{aligned}\right\} \qquad \left.\begin{aligned} a &= -9 \\ b &= 6\tfrac{3}{4} \\ c &= -5\tfrac{1}{16} \end{aligned}\right\}$

Q5 (a) $T_2 - T_1 = \frac{n+2}{n} - \frac{n+1}{n} = \frac{1}{n} = T_3 - T_2 =$

ie the series is an AP with $a = \frac{n+1}{n}$ and $d = \frac{1}{n}$

Let the number of terms be N.

Then $S_N = \frac{N}{2}\left[2a + (N-1)d\right]$

$$= \frac{N}{2}\left[2 \times \frac{n+1}{n} + (N-1) \times \frac{1}{n}\right]$$

$$= \frac{N}{2n}\left[2n + 2 + N - 1\right]$$

$$= \frac{N}{2n}\left[N + 2n + 1\right]$$

(b) The triangle inequality is $|a+b| \leq |a| + |b|$

To establish the required inequality, let $a+b = x$ and $b = y$. We use these to rewrite the inequality in x & y.

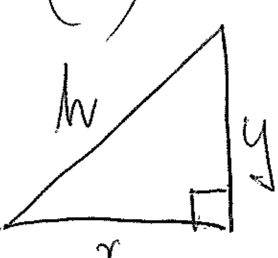
Then the triangle inequality becomes:-

$$|x| \leq |x-y| + |y| \qquad \left[x - y = (a+b) - b = a\right]$$

ie $|x-y| + |y| \geq |x|$ [reversing inequality.

$\therefore \quad |x-y| \geq |x| - |y|$.

(c)



$A = \frac{1}{2}xy \longrightarrow 2A = xy$ — (A)

$P = x + y + h \longrightarrow (x+y) = P - h$ — (B)

Now by Pythagoras' Theorem

$$h^2 = x^2 + y^2$$

$\therefore h^2 = (x+y)^2 - 2xy$

Substituting from (A) & (B) for xy and $(x+y)$ we obtain:-

$h^2 = (P-h)^2 - 2\times 2A$

ie $h^2 = P^2 - 2Ph + h^2 - 4A$.

$\therefore 2Ph = P^2 + h^2 - 4A - h^2$

$= P^2 - 4A$

ie $h = \dfrac{P^2 - 4A}{2P}$

(d) $xy < 4$ and $y \geq x+3$

(i) If $xy = 4$ and we substitute $y = x+3$ we get:-

$x(x+3) = 4 \rightarrow x^2 + 3x - 4 = 0$

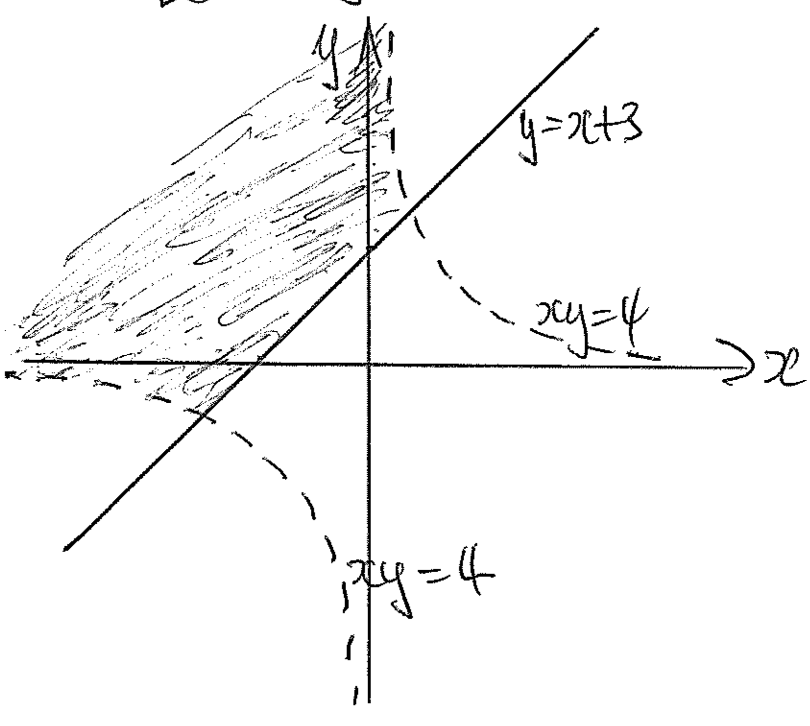
$(x-1)(x+4) = 0$

$\therefore x = 1, -4$

$x = 1 \rightarrow y = 1+3 = 4$ and $x = -4 \rightarrow y = -4+3 = -1$

ie. Intersections are at $(1,4)$ and $(-4,-1)$

(ii) NB In graphing we must use dotted lines for the hyperbolas for $xy=4$ since the inequality is < only



By substituting $(0,0)$ into each inequality we find that:-

1) $0.0 < 4$ is true so the region of the plane where $xy < 4$ is between the 2 arms of the hyperbola $x.y \leq 4$

2) $0 \geq 0+3$ is not true so the region for $y \geq x+3$ is to the left of the line $y = x+3$

Question 6 (13 Marks)

(a) Given the quadratic $2x^2 + kx + 5$, $\boxed{6}$

(i) For what range of values of k is the quadratic positive definite?

> **Solution:** $\triangle = k^2 - 40,$
> $\quad < 0$ when $k^2 < 40$,
> *i.e.* $|k| < 2\sqrt{10}$ or $-2\sqrt{10} < k < 2\sqrt{10}$.

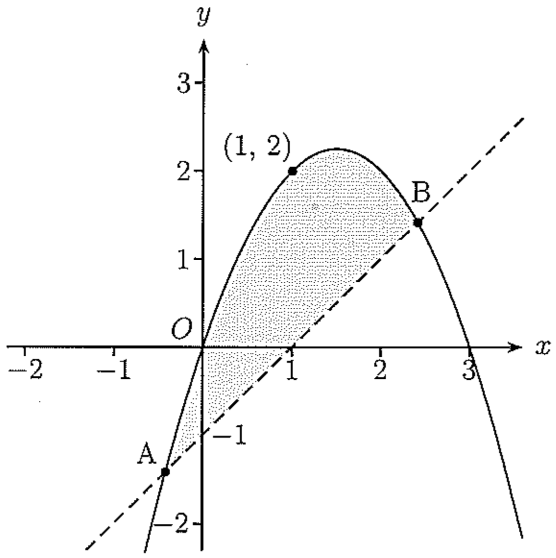
(ii) For what values of k does the quadratic have equal roots?

> **Solution:** $k = \pm 2\sqrt{10}$.

(iii) Find two integral values of k such that the quadratic has rational roots.

> **Solution:** We need $\triangle$ to be a perfect square so,
> after some trial and error: $49 - 40 = 9$,
> $121 - 40 = 81$,
> Thus ± 7 or ± 11 are possible values.

(b) Describe the region in the graph. $\boxed{4}$



> **Solution:** Parabola: $y = ax(x-3)$, Dashed line: $y = x - 1$.
> $2 = a(1-3)$,
> $a = -1$.
> $\therefore\ y = x(3-x)$.
> So the region is $\{(x,\,y) : y \leqslant 3x - x^2\} \cap \{(x,\,y) : y > x - 1\}$.

QUESTION SEVEN

a) (i) $$A_n = PR^n - \frac{M(R^n - 1)}{R-1}.$$

$R = 1.014$
$n = 36.$
$P = 3000$
$A_{36} = 0.$

$$\therefore \quad PR^n = \frac{M(R^n - 1)}{R-1}.$$

$$M = \frac{PR^n(R-1)}{R^n - 1}.$$

$$= \frac{3000(1.014)^{36}(1.014-1)}{(1.014)^{36} - 1}.$$

$$= \$106.66$$

(ii). $$PR^n = \frac{MR^n - M}{R-1}.$$

$n = 23.63$
i.e. 24 repayments.

$$MR^n - PR^n(R-1) = M$$

$$R^n(M - PR + P) = M.$$

$$R^n = \frac{M}{M - PR + P}$$

$$n \log R = \log\left(\frac{M}{M - PR + P}\right).$$

$$n = \frac{\log\left(\frac{M}{M - PR + P}\right)}{\log R}.$$

(b) (i) $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$a = 1 \quad b = \cancel{0}1 \quad c = 0.$

$\cancel{\Rightarrow}$ $(x, y).$

$$d = \frac{x - y}{\sqrt{2}}.$$

(ii) $x = \dfrac{x - y}{\sqrt{2}}.$

$x\sqrt{2} = x - y.$

$y = x(1 - \sqrt{2}).$