SYDNEY GRAMMAR SCHOOL

2010 Half-Yearly Examination

FORM V MATHEMATICS

Friday 7th May 2010

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper

- Total marks — 108
- All six questions may be attempted.
- All six questions are of equal value.

Collection

- Write your name, class and master clearly on each leaflet.
- Hand in the six questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- Write your name on the question paper and place it inside your leaflet for Question One.

5P: BR 5Q: JMR

Checklist

- Writing leaflets: 6 per boy.
- Candidature — 35 boys

Examiner

JMR

QUESTION ONE (18 marks) Start a new leaflet.

(a) Write $\dfrac{\pi^2}{3}$ correct to 2 decimal places.

(b) Write $\cos 40^\circ 40'$ correct to 2 decimal places.

(c) Solve $\dfrac{2x}{3} = \dfrac{1}{2}$.

(d) Simplify $3x^2 \div 6x^3$.

(e) Factorise completely $2x^2 - 50$.

(f) Expand and simplify $6x - 2(x - 6)$.

(g) Evaluate $x^3 - 1$ when $x = 3$.

(h) (i) Find the gradient of the line $x + 2y - 3 = 0$.

(ii) Write down the gradient of the line which is perpendicular to $x + 2y - 3 = 0$.

(i) Expand and simplify $(2\sqrt{3} - 1)^2$.

(j) Write the exact value for:

(i) $\sin 60^\circ$

(ii) $\tan 120^\circ$

(k) Given $f(x) = x^2 + 4x - 3$, find:

(i) $f(1)$

(ii) $f(-a)$

(l) Find the natural domain of the function $f(x) = \dfrac{1}{4 - x}$.

(m) Evaluate $\dfrac{|x - 6|}{2}$ when $x = -8$.

(n) Factorise $x^3 - 27$.

Exam continues next page ...

<u>QUESTION TWO</u> (18 marks) Start a new leaflet.

(a) Express $0{\cdot}\dot{2}\dot{4}$ as a fraction in simplest form. You must show all working.

(b) (i) Sketch the graph of $x^2 + y^2 = 9$.

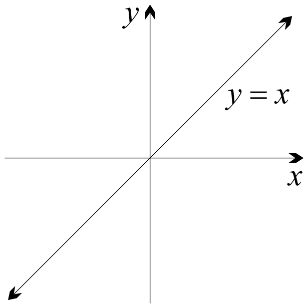
(ii) Use your graph to explain why $x^2 + y^2 = 9$ is not a function.

(c) Solve $|2x - 5| = 13$.

(d) Solve the inequation $|3x + 1| \leq 10$ and graph your solution on a number line.

(e) Classify each function as even, odd or neither.

(i)



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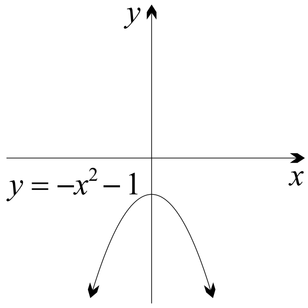

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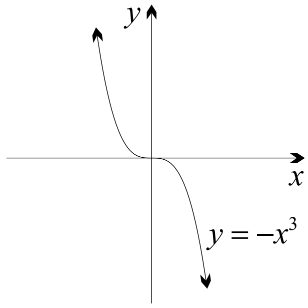
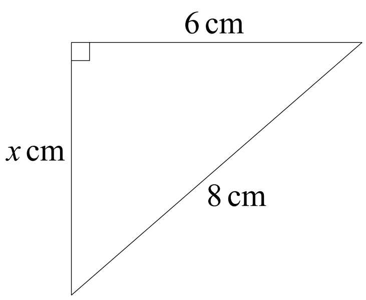


(f) Given the formula $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{c}$, find a if $b = -2$ and $c = -3$.

(g) Find the mid-point of the interval AB, given A is $(\frac{13}{2}, \frac{11}{2})$ and B is $(13, -11)$.

(h) Solve the inequation $x^2 + 4x \geq 5$.

(i)



Find the exact value of x in the diagram above.

Exam continues overleaf . . .

QUESTION THREE (18 marks) Start a new leaflet.

(a) Consider the parabola $y = x^2 - 2x - 8$.

(i) Find the y-intercept.

(ii) Find the x-intercepts.

(iii) What is the equation of the axis of symmetry?

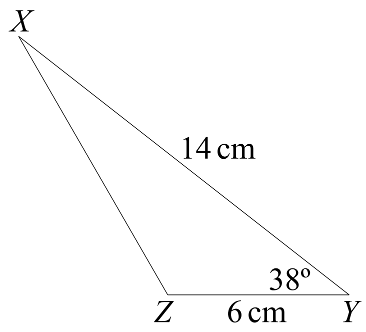
(iv) Find the coordinates of the vertex.

(v) Sketch the parabola, showing all of the above features.

(b) On the same number plane, sketch the graphs of $y = \sin x$ and $y = \cos x$ for the domain $0^\circ \le x \le 360^\circ$. Hence find the values of x for which $\sin x = \cos x$.

(c) If the line $ax - 2y + 13 = 0$ is parallel to the line $3x - y + 5 = 0$, find the value of a.

(d)



Find the area of $\triangle XYZ$ in the diagram above. Give your answer correct to 3 significant figures.

(e) (i) Sketch the graph of $y = |x| + 1$, showing all main features.

(ii) State the domain and range of $y = |x| + 1$.

(f) Determine whether the point $A(-8\frac{1}{2}, -15\frac{1}{2})$ lies on the line $7x - 3y + 13 = 0$.

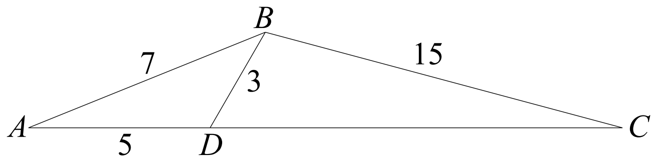
QUESTION FOUR (18 marks) Start a new leaflet.

(a) Find the acute angle θ, correct to the nearest minute, given that $\sec\theta = \dfrac{5}{2}$.

(b) Simplify $\tan\theta \sec\theta \cos^2\theta$.

(c) Solve $\sin x = -\dfrac{\sqrt{3}}{2}$, for $0^\circ \le x \le 360^\circ$.

(d)



In the diagram above, $\triangle ABC$ has dimensions $AB = 7$ cm and $BC = 15$ cm. The point D lies on AC such that $AD = 5$ cm and $BD = 3$ cm.

(i) Use the cosine rule to show that $\angle ADB = 120^\circ$.

(ii) Show that $\angle BCD = 10^\circ$, rounded to the nearest degree.

(iii) Find the length of DC, correct to the nearest millimetre.

(e) Find a and b, if $a + b\sqrt{2} = (3 + \sqrt{2})\,(2 - \sqrt{2})$.

(f) In $\triangle TUV$, $TU = 5$ cm, $TV = 9$ cm and $\angle TVU = 22^\circ$. Find the size of $\angle TUV$ correct to the nearest degree.

<u>QUESTION FIVE</u> (18 marks) Start a new leaflet.

(a) (i) On the same number plane, sketch the graphs of $y = 1 - x^2$ and $y = x - 5$, for $-4 \le x \le 4$.

(ii) Hence or otherwise find the coordinates of the points of intersection of the graphs.

(iii) Hence solve the inequation $1 - x^2 > x - 5$.

(b) On a number plane, sketch the region $y < 2^x$.

(c) Express $\dfrac{3 - \sqrt{2}}{2\sqrt{2} - 1}$ with a rational denominator.

(d) Solve the inequation $-18 \le 3(2x - 5) < 9$ and graph the solution on a number line.

(e) Given that $\tan\theta = \dfrac{4}{3}$ and $\sin\theta$ is negative, find $\cos\theta$ without evaluating θ.

(f) Solve simultaneously:

$$\begin{aligned} x^2 - y^2 &= 25 \\ x^2 + y^2 &= 43 \end{aligned}$$

(g) Find the centre and radius of the circle $x^2 + y^2 - 16x + 56 = 0$.

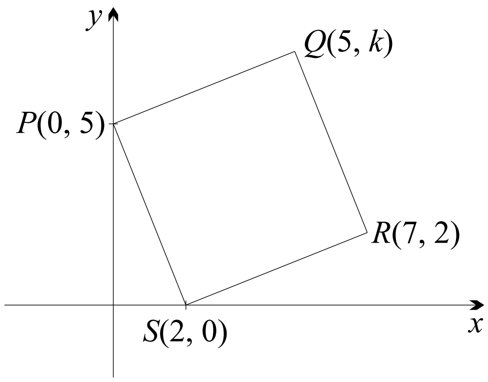
Exam continues next page ...

QUESTION SIX (18 marks) Start a new leaflet.

(a) Solve $\cos^2 x + \cos x = 0$, for $0^\circ \leq x \leq 360^\circ$.

(b) Simplify $\dfrac{a^2 - b^2}{\dfrac{b}{a} - \dfrac{a}{b}}$.

(c)



In the diagram above $PQRS$ is a square.

(i) Find the gradient of PR.

(ii) Show that the equation of QS is $7x - 3y - 14 = 0$.

(iii) Find k, the y-coordinate of Q.

(iv) Find the length of PR.

(v) Hence or otherwise find the area of $PQRS$.

(d) Factorise completely $t^4 + t^2 - 2$.

(e) Two people are 10 km apart. They start walking at the same time, each with a constant speed. If they walk directly towards each other, they will meet in one hour. If they walk in the same direction, with the slower walker leading the way, they will meet in 5 hours. By letting the speed of the slower walker be x km/h and the speed of the faster walker be y km/h, set up two equations and solve them simultaneously to find the speed of each person.

END OF EXAMINATION

SOLUTION : FORM V HALF-YEARLY 2010

1. (a) 3.29

(b) 0.76

(c) $\frac{2x}{3} = \frac{1}{2}$

$4x = 3$

$x = \frac{3}{4}$

(d) $3x^2 \div 6x^3 = \frac{3x^2}{6x^3}$

$= \frac{1}{2x}$

(e) $2x^2 - 50 = 2(x^2 - 25)$

$= 2(x+5)(x-5)$

(f) $6x - 2(x-6) = 6x - 2x + 12$

$= 4x + 12$

(g) $x^3 - 1 = 3^3 - 1$

$= 27 - 1$

$= 26$

(h) (i) $x + 2y - 3 = 0$

$2y = -x + 3$

$y = -\frac{1}{2}x + 3$

$m_1 = -\frac{1}{2}$

(ii) $m_2 = 2$

(i) $(2\sqrt{3} - 1)^2 = 12 - 4\sqrt{3} + 1$

$= 13 - 4\sqrt{3}$

(j)(i) $\sin 60° = \frac{\sqrt{3}}{2}$

(ii) $\tan 120° = -\tan 60°$

$= -\sqrt{3}$

(k) (i) $f(x) = x^2 + 4x - 3$

$f(1) = 1^2 + 4 - 3$

$= 2$

(ii) $f(-a) = (-a)^2 + 4(-a) - 3$

$= a^2 - 4a - 3$

(l) Natural domain:

$x \neq 4$

(m) $\frac{|x-6|}{2} = \frac{|-8-6|}{2}$

$= \frac{|-14|}{2}$

$= \frac{14}{2}$

$= 7$

(n) $x^3 - 27 = (x-3)(x^2 + 3x + 9)$

(18 marks)

2. (a) Let $x = 0.242424\ldots$

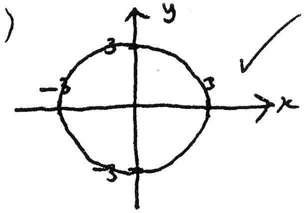
$100x = 24.242424\ldots$

$99x = 24$

$x = \frac{24}{99}$

$x = \frac{8}{33}$

(b) (i)



(ii) Graph fails the vertical line test.

(c) $|2x - 5| = 13$

Either $2x - 5 = 13$ or $2x - 5 = -13$

$2x = 18$ $\quad$ $2x = -8$

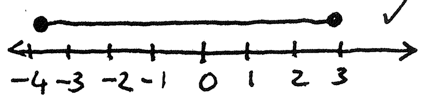
$x = 9$ or $x = -4$

(d) $|3x + 1| \leq 10$

$\therefore -10 \leq 3x + 1 \leq 10$

$-11 \leq 3x \leq 9$

$-\frac{11}{3} \leq x \leq 3$



(e) (i) odd – has point symmetry

(ii) even – symmetrical about y axis

(iii) odd – has point symmetry.

or Algebraic method (i) $f(-x) = -f(x)$

(ii) $f(-x) = f(x)$

(iii) $f(-x) = -f(x)$

(f) $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$

$\frac{1}{a} - \frac{1}{2} = -\frac{1}{3}$

$\frac{1}{a} = \frac{1}{6}$

$a = 6$

(g) $M = \left(\frac{\frac{13}{2} + 13}{2}, \frac{\frac{11}{2} - 11}{2} \right)$

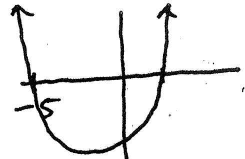
$= \left(\frac{39}{4}, \frac{-11}{4} \right)$

(h) $x^2 + 4x \geq 5$

$x^2 + 4x - 5 \geq 0$

Sketch: $y = x^2 + 4x - 5$

x intercepts $-5, 1$



Solution: $x \leq -5$ or $x \geq 1$

(i) $x^2 + 6^2 = 8^2$

$x^2 + 36 = 64$

$x^2 = 28$

$x = \sqrt{28}$

$x = 2\sqrt{7}$

(18 marks)

3. (a)(i) $(0,-8)$ ✓

(ii) $x^2-2x-8=0$

$(x-4)(x+2)=0$

$x = 4$ or -2 ✓

x-intercepts: $(4,0)$ and $(-2,0)$

(iii) $x = \frac{-b}{2a}$

$= \frac{2}{2}$

$x = 1$ ✓

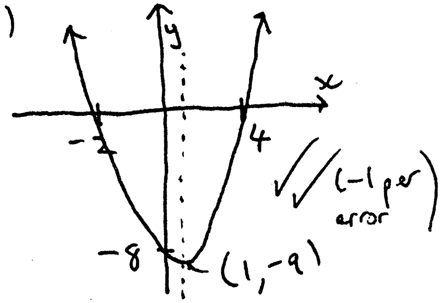
(iv) When $x=1$

$y = (1)^2-2(1)-8$

$= -9$

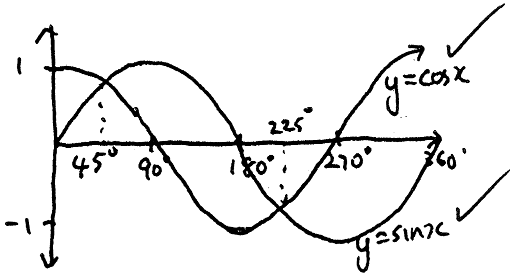
Vertex is $(1,-9)$ ✓

(v)



✓✓ (−1 per error)

(b)



✓✓

$\sin x = \cos x$ when $x = 45°$ or $225°$ ✓

(c) $3x-y+5=0$

$y = 3x+5$

so $m = 3$ ✓

$ax-2y+13=0$

$2y = ax+13$

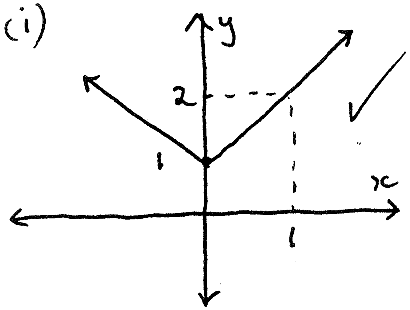
$y = \frac{a}{2}x + \frac{13}{2}$

$\frac{a}{2} = 3 \Rightarrow$ [illegible] ✓

(d) $A = \frac{1}{2} \times 14 \times 6 \times \sin 38°$ ✓

$= 25.9 \text{ cm}^2$ (3 sig. figs) ✓

(e)(i)



✓

(ii) Domain: $x \in R$ ✓

Range: $y \geq 1$ ✓

(f) LHS $= 7 \times (-8\frac{1}{2}) - 3(-15\frac{1}{2}) + 13$

$= -59\frac{1}{2} + 46\frac{1}{2} + 13$ ✓

$= 0$

$=$ RHS.

So, A does lie on the line. ✓

(18 marks)

4.(a) $\sec\theta = \frac{5}{2}$

$\therefore \cos\theta = \frac{2}{5}$ ✓

$\theta \doteqdot 66°25'$ (nearest minute) ✓

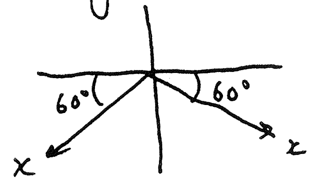
(b) $\tan\theta\sec\theta\cos^2\theta = \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{1}$ ✓✓

$= \sin\theta$ ✓

(c) $\sin x = -\frac{\sqrt{3}}{2}$

Sine is negative in 3rd/4th quadrant ✓

Related angle is $60°$ ✓



$x = 240°$ or $300°$ ✓

(d)(i) $\cos\angle ADB = \frac{5^2+3^2-7^2}{2 \times 5 \times 3}$ ✓

$\angle ADB = 120°$

(ii) $\angle BDC = 60°$ (Angles on a line) ✓

$\frac{\sin C}{3} = \frac{\sin 60°}{15}$

$\sin C = \frac{3 \times \sin 60°}{15}$

$\angle C = 9.97°$ ✓

$\doteqdot 10°$ (nearest degree)

(iii) $\angle DBC \doteqdot 110°$ (Angle Sum ΔBCD) ✓

$\frac{CD}{\sin 110°} = \frac{15}{\sin 60°}$

$CD = \frac{15 \times \sin 110°}{\sin 60°}$

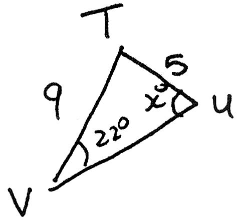
$\doteqdot 16.3$ cm (nearest mm) ✓

(e) $(3+\sqrt{2})(2-\sqrt{2}) = 6-3\sqrt{2}+2\sqrt{2}-2$ ✓

$= 4-\sqrt{2}$

So $a=4$ and $b=-1$ ✓

(f)



$\frac{\sin x°}{9} = \frac{\sin 22°}{5}$ ✓

$\sin x° = \frac{9 \times \sin 22°}{5}$

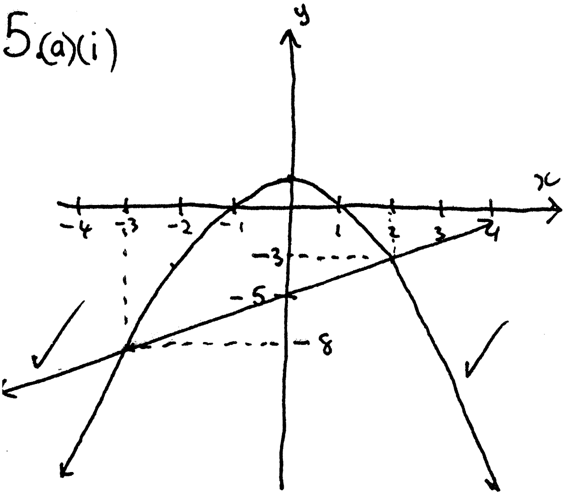
$x° = 42°$ or $(180-42)°$

Both angles are possible

$x° = 42°$ or $138°$ ✓✓

(18 marks)

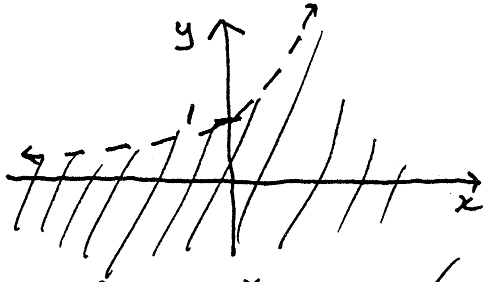
5 (a)(i)



(ii) $(-3,-8)$ and $(2,-3)$ ✓

(iii) $-3 \leq x \leq 2$. ✓

(b)



1 mark for $y=2^x$ graph ✓
1 mark for dotted boundary ✓
1 mark for correct region. ✓

(c) $\dfrac{3-\sqrt{2}}{2\sqrt{2}-1} \times \dfrac{2\sqrt{2}+1}{2\sqrt{2}+1} = \dfrac{6\sqrt{2}+3-4-\sqrt{2}}{8-1}$ ✓

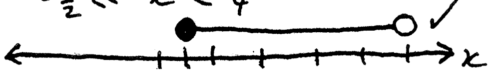
$= \dfrac{5\sqrt{2}-1}{7}$ ✓

(d) $-18 \leq 3(2x-5) < 9$ ✓

$-6 \leq 2x-5 < 3$

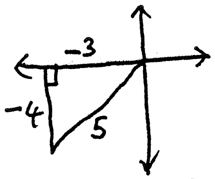
$-1 \leq 2x < 8$

$-\frac{1}{2} \leq x < 4$ ✓



✓

(e) tan is positive and sin is negative in 3rd quadrant ✓



$\cos\theta = -\dfrac{3}{5}$ ✓

(f) $x^2 - y^2 = 25$

$x^2 + y^2 = 43$

$2x^2 = 68$

$x^2 = 34$

$x = \pm\sqrt{34}$ ✓

$34 - y^2 = 25$

$y^2 = 9$

$y = \pm 3$ ✓

(g) $x^2+y^2-16x+56=0$

$x^2-16x+64+y^2 = 8$

$(x-8)^2+y^2 = 8$ ✓

Centre $=(8,0)$
Radius $=\sqrt{8}$
$= 2\sqrt{2}$ ✓

(18 marks)

6 (a) $\cos^2 x + \cos x = 0$

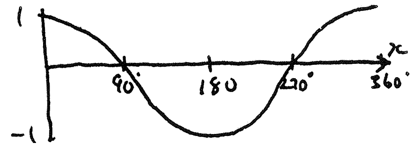
Let $\cos x = u$

$u^2+u=0$ ✓

$u(u+1)=0$

$u=0$ or -1

$\cos x = 0$ or $\cos x = -1$, $0° \leq x \leq 360°$ ✓



$x = 90°, 180°$ or $270°$ ✓

(b) $\dfrac{a^2-b^2}{\frac{b}{a}-\frac{a}{b}} = \dfrac{a^2-b^2}{\frac{b^2-a^2}{ab}}$

$= \dfrac{a^2-b^2}{1} \times \dfrac{ab}{b^2-a^2}$ ✓

$= \dfrac{-1(b^2-a^2)}{1} \times \dfrac{ab}{b^2-a^2}$

$= -ab$ ✓

(c)(i) $m = \dfrac{2-5}{7-0}$

$= -\dfrac{3}{7}$ ✓

(ii) $m = \dfrac{7}{3}$ ✓

$y - 0 = \dfrac{7}{3}(x-2)$

$3y = 7x - 14$ ✓

$7x - 3y - 14 = 0$

(iii) When $x = 5$ ✓

$35 - 3y - 14 = 0$

$3y = 21$

$y = 7$

So, $k = 7$. ✓

(iv) $d = \sqrt{(2-5)^2+(7-0)^2}$

$= \sqrt{9+49}$ ✓

$= \sqrt{58}$ units ✓

(v) Area of square $= \frac{1}{2}x^2$ ✓
where x is the diagonal.

Area: PQRS $= \frac{1}{2} \times \sqrt{58}^2$

$= 29$ units^2 ✓

(d) $t^4+t^2-2 = (t^2+2)(t^2-1)$ ✓

$= (t^2+2)(t+1)(t-1)$ ✓

(e) Speed of slower walker $= x$ km/h
Speed of faster walker $= y$ km/h

In one hour:

$x + y = 10$ ✓

In five hours:

$5x + 10 = 5y$ ✓

Substitute $x = 10 - y$

$5(10-y)+10 = 5y$

$50 - 5y + 10 = 5y$

$60 = 10y$

$y = 6$

So $x = 4$

The speed of the slower walker was 4 km/h
The faster walker walked at 6 km/h. ✓

(18 marks)