BAULKHAM HILLS HS PERSEVERE

2017 Preliminary
Half Yearly Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 1 hour and 30 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks – 54

Section I Pages 2 – 4

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 5 – 8

44 marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 15 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer page in the writing booklet for Questions 1 – 10.

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1 How many numbers greater than 40000 can be formed with the digits 2, 3, 4, 5 and 6 if no digit is used more than once?

(A) 48 (B) 72 (C) 96 (D) 120

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2 In how many ways can a family of eight sit around a circular table if the two youngest family members want to sit together?

(A) $7!$ (B) $2\times 5!$ (C) $2\times 6!$ (D) $2\times 7!$

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3 What is the domain of the function $y=\dfrac{1}{\sqrt{4-x^2}}$?

(A) $x>2$ (B) $x<2$

(C) $-2<x<2$ (D) $x<-2$ or $x>2$

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4 If $2\sec\theta+3=0$, and $\cot\theta>0$, what is the exact value of $\sin\theta$?

(A) $-\dfrac{\sqrt{5}}{3}$ (B) $-\dfrac{\sqrt{5}}{2}$ (C) $\dfrac{\sqrt{5}}{3}$ (D) $\dfrac{\sqrt{5}}{2}$

5 How many value(s) of θ, for $0° \le \theta \le 360°$, satisfy the equation $\sin\theta\cos\theta = \sin\theta$?

(A) 2 (B) 3 (C) 4 (D) 5

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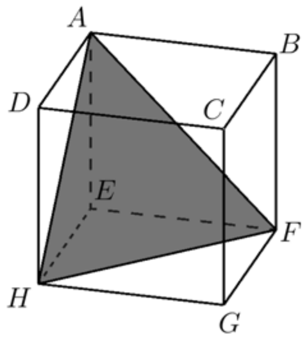
6 *ABCDEFGH* is a cube of side length *p*, what is the area of ΔAFH in terms of *p*?



(A) $\dfrac{p^2\sqrt{3}}{4}$ (B) $\dfrac{p^2\sqrt{3}}{2}$

(C) $\dfrac{p^2}{4}$ (D) $\dfrac{p^2}{2}$

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7 For $0° \le \alpha \le 90°$, the least value of $\dfrac{30}{3\sin^2\alpha + 2\sin^2(90° - \alpha)}$ is:

(A) 15 (B) 10 (C) 6 (D) 5

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8 When solving $\dfrac{x-1}{\sqrt{x}} > \dfrac{2}{x-1}$ within the natural domain, three students obtain the following inequalities:

Student 1: $(x-1)^2 > 2\sqrt{x}$

Student 2: $(x-1)^3 > 2(x-1)\sqrt{x}$

Student 3: $(x-1)^3\sqrt{x} > 2x(x-1)$

Which student(s) will obtain the correct solution to the original inequality?

(A) Student 1 only (B) Student 2 only

(C) Student 3 only (D) Both students 2 and 3

9 If $f\left(x+1\right)=x^{4}-2x+1$, what is the value of $f\left(0\right)$?

(A) 0 (B) 1 (C) 2 (D) 4

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10 Given that *n* and *r* are both positive integers such that $n \geq r$, which of the following is **NOT** always true?

(A) ${}^{n}P_{1}={}^{n}C_{1}$

(B) ${}^{n}P_{n}={}^{n}C_{n}$

(C) ${}^{n}P_{r}={}^{n}C_{r}\times r!$

(D) ${}^{n}P_{r}\geq {}^{n}C_{r}$

Section II

44 marks
Attempt Questions 11 – 14
Allow about 1 hour and 15 minutes for this section

Answer each question in the appropriate section of the writing booklet. Extra writing paper is available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (11 marks) Use the Question 11 section of the writing booklet.

(a) Solve $\dfrac{x}{x-4} \le 5$. **3**

(b) The letters A, E, I, O and U are vowels. In how many ways can the letters of the word **MATHEMATICS** be arranged in a line if:

(i) there are no restrictions? **1**

(ii) the vowels are all together? **1**

(iii) both the letter **M**'s must be immediately followed by the letter **A**? **1**

(c)

(i) Solve $x^4 - 4x^2 + 3 = 0$. **1**

(ii) Hence, or otherwise, solve $\tan^4\theta - 4\tan^2\theta + 3 = 0$ for $0° \le \theta \le 360°$. **2**

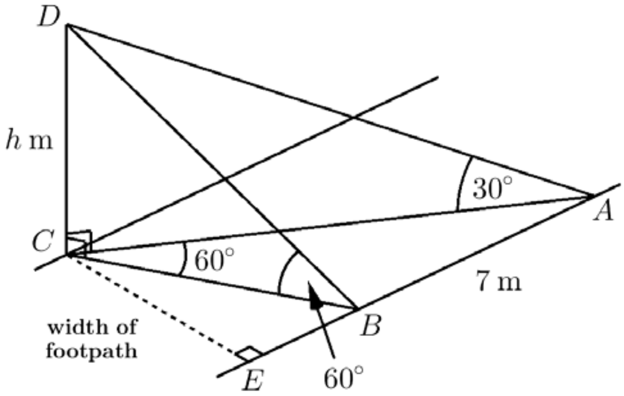
(d) Sketch the region on the Cartesian plane satisfying the inequality $y < -\sqrt{x-1}$. **2**

End of Question 11

Question 12 (11 marks) Use the Question 12 section of the writing booklet.

(a) A footpath on horizontal ground has two parallel edges. CD is a vertical flagpole of height h metres which stands with its base, C, on one edge of the footpath. A and B are two points on the other edge of the footpath such that $AB = 7$ m and $\angle ACB = 60°$.

From A and B, the angles of elevation of the top of the flagpole, D, are $30°$ and $60°$ respectively.



(i) Show that $BC = \dfrac{h}{\sqrt{3}}$. **1**

(ii) Find a similar expression for AC. **1**

(iii) Hence, find the exact height of the flagpole. **2**

(iv) Find the exact width of the footpath. **3**

(b) From a group of 6 men and 8 women, a committee of 5 people is to be chosen, how many ways can this committee be formed if:

(i) there are no restrictions? **1**

(ii) the committee consists of 2 men and 3 women? **1**

(iii) the committee must have at least 1 woman? **1**

(iv) the entire committee is of the same gender? **1**

End of Question 12

Question 13 (11 marks) Use the Question 13 section of the writing booklet.

(a) Consider the function $f(x) = x^2 - |2x| - 3$.

(i) Show that $f(x)$ is an even function. **1**

(ii) Hence, or otherwise, sketch the graph of $y = f(x)$, showing all intercepts. **2**

(iii) What is the range of $y = f(x)$? **1**

(b)

(i) On the same set of axes, sketch the graphs of $y = |2x-4|$ and $y = |x+1|$, showing all intercepts. **2**

(ii) Hence, or otherwise, solve the inequality $|2x-4| \leq |x+1|$. **2**

(c)

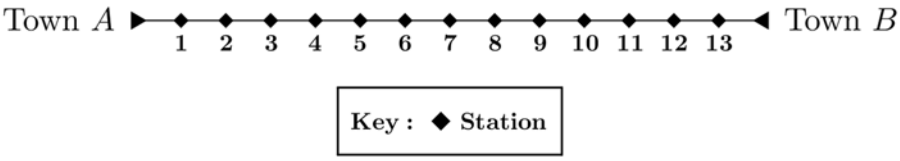
(i) Fully factorise $64k^6 - 1$ as a difference of two squares and as a difference of two cubes. **2**

(ii) Hence, or otherwise, factorise $16k^4 + 4k^2 + 1$. **1**

End of Question 13

Question 14 (11 marks) Use the Question 14 section of the writing booklet.

(a) A train is leaving Town A, heading towards Town B without turning around. There are 13 train stations between the two towns:



(i) In how many ways can the train stop at 4 of the 13 stations? **1**

(ii) In how many ways can the train stop at 4 of the 13 stations if the train does not stop at consecutive stations? **1**

(b) Show that $\dfrac{(\sin^2\alpha-\cos^2\alpha)(1-\sin\alpha\cos\alpha)}{\cos\alpha(\sec\alpha-\operatorname{cosec}\alpha)(\sin^3\alpha+\cos^3\alpha)}=\sin\alpha$. **3**

(c) It can be shown that:

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$ (DO NOT PROVE THIS.)

(i) For any ΔABC, explain why $\tan A+\tan B+\tan C=\tan A\tan B\tan C$. **2**

(ii) It is given, for ΔXYZ , that $\dfrac{\tan X}{5}=\dfrac{\tan Y}{6}=\dfrac{\tan Z}{7}=k$ for some constant k, **3**

show that $k=\sqrt{\dfrac{3}{35}}$.

(iii) Hence calculate the size of the smallest angle in ΔXYZ correct to the nearest minute. **1**

End of Paper

YEAR 11 HALF YEARLY EXAMINATION 2017
MATHEMATICS EXTENSION 1
MARKING GUIDELINES

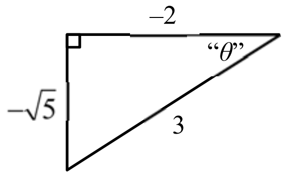
Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	C
3	C
4	A
5	B

Question	Answer
6	B
7	B
8	D
9	D
10	B

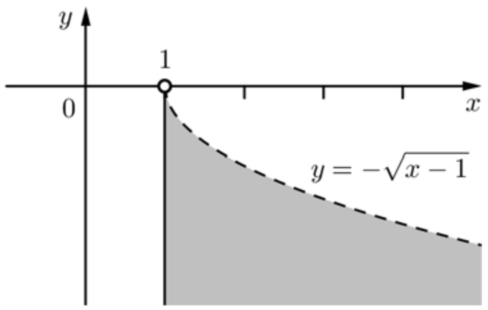
Questions 1 – 10

Sample solution	
1.	Number over $40000 = 3 \times 4!$ $= 72$ $\therefore$ 72 numbers are greater than 40000.
2.	2 ways to arrange the two youngest family members 6! ways to arrange the remaining six family members as well as the group of two around a circular table.
3.	$4 - x^2 > 0$ $4 > x^2$ $\therefore -2 < x < 2$
4.	$\cot\theta > 0$ if $\tan\theta > 0$ $2\sec\theta + 3 = 0$ $\sec\theta = \frac{-3}{2}$ $\cos\theta = \frac{2}{-3}$ $\therefore \cos\theta < 0$ and $\tan\theta > 0$ (i.e. θ is in quadrant 3) $\sin\theta = -\frac{\sqrt{5}}{3}$ 
5.	$\sin\theta\cos\theta = \sin\theta$ $\sin\theta(\cos\theta - 1) = 0$ $\therefore \sin\theta = 0$ or $\cos\theta = 1$ For $0° \le \theta \le 360°$, $\theta = 0°, 180°, 360°$
6.	$AH = AF = FH$ (equal diagonals of the square faces) ie. ΔAFH is equilateral. By Pythagoras' theorem, $AH = p\sqrt{2}$. Using the sine rule for area, $A_{\Delta AFH} = \frac{1}{2} \times p\sqrt{2} \times p\sqrt{2} \times \sin 60°$ $= \frac{p^2\sqrt{3}}{2}$

7.	$$\frac{30}{3\sin^2\alpha+2\sin^2(90°-\alpha)}=\frac{30}{3\sin^2\alpha+2\cos^2\alpha}$$ $$=\frac{30}{\sin^2\alpha+2}$$ Least value occurs when denomiator is largest, i.e. $\sin^2\alpha=1$. Least value of $\frac{30}{\sin^2\alpha+2}=\frac{30}{1+2}$ $=10$
8.	Student 1 multiplied both sides by $(x-1)$, which could be negative, the inequality sign may needed to have been flipped. Student 2 multiplied both sides by $\sqrt{x}(x-1)^2$, which is positive, the inequality is ok. Student 3 multiplied both sides by $(\sqrt{x})^2(x-1)^2$, which is positive, the extra $\sqrt{x}$ does not introduce any extra solutions in the natural domain, the inequality is ok.
9.	Let $x=-1$: $f(0)=(-1)^4-2\times(-1)+1$ $=1+2+1$ $=4$
10.	The number of ways of choosing 1 item from a set of n items does not depend on order, so ${}^nP_1={}^nC_1$. By definition, $${}^nC_r=\frac{n!}{r!(n-r)!}$$ $$=\frac{1}{r!}{}^nP_r$$ ${}^nC_r\times r!={}^nP_r$ The number of ways of choosing r items from a set of n items where order is important is always more than the number of ways of choosing r items from a set of n items where order is not important, i.e. ${}^nP_r\geq{}^nC_r$, with equality when $r=1$. By elimination, ${}^nP_n\neq{}^nC_n$ is not always true. (Note: For the given conditions, this statement is only true for $n=1$.)

Section II

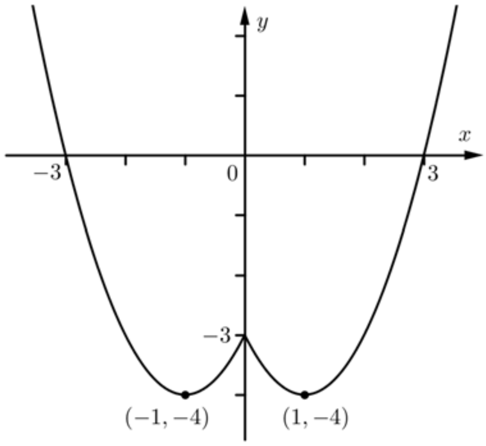
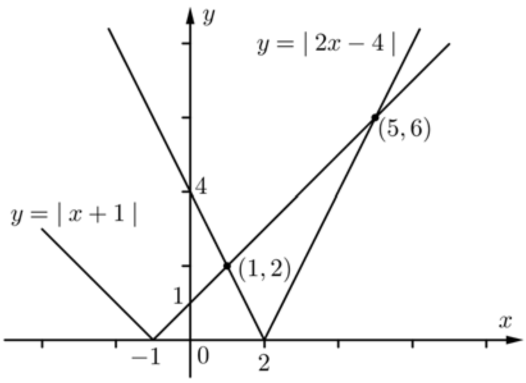
Question 11

	Sample solution	Suggested marking criteria
(a)	Natural domain for $\frac{x}{x-4} : x \neq 4$ $\frac{x}{x-4} \leq 5$ $x(x-4) \leq 5(x-4)^2$ $0 \leq (x-4)[5(x-4)-x]$ $0 \leq 4(x-4)(x-5)$ $\therefore x < 4$ or $x \geq 5$	• 3 – correct solution • 2 – correctly identifies the two critical points $x = 4$ and $x = 5$ • 1 – attempts to solve the inequation using a suitable method – recognises a restriction in the domain
(b)	(i) $\frac{11!}{2!2!2!} = 4\ 989\ 600$	• 1 – correct answer, or equivalent numerical expression
	(ii) There are $\frac{4!}{2!} = 12$ ways of arranging the vowels. Treating the vowels as a single entity, the number of ways to arrange the remaining 7 letters and the single group of vowels is: $\frac{8!}{2!2!} = 10\ 080$ Total number of arrangements $= 10\ 080 \times 12$ $= 120\ 960$	• 1 – correct answer, or equivalent numerical expression
	(iii) Treating **MA** as a single entity, the number of ways to arrange **(MA)THE(MA)TICS** in a line is: $\frac{9!}{2!2!} = 90\ 720$	• 1 – correct answer, or equivalent numerical expression
(c)	(i) Let $u = x^2$: $u^2 - 4u + 3 = 0$ $(u-3)(u-1) = 0$ $u = 1$ or $u = 3$ $x^2 = 1$ $\quad$ $x^2 = 3$ $x = \pm 1$ $\quad$ $x = \pm\sqrt{3}$	• 1 – correct solution
	(ii) Using the results from (i), and letting $x = \tan\theta$: $\tan\theta = \pm 1$ or $\tan\theta = \pm\sqrt{3}$ $\theta = 45°, 135°, 225°, 315°$ $\quad$ $\theta = 60°, 120°, 240°, 300°$ $\therefore \theta = 45°, 60°, 120°, 135°, 225°, 240°, 300°, 315°$	• 2 – correct solution • 1 – correct attempt at solving an appropriate trigonometric equation
(d)		• 2 – correct region and boundaries • 1 – significant progress towards the correct region and boundaries

Question 12

Sample solution			Suggested marking criteria
(a)	(i)	In ΔDBC , $\tan 60° = \dfrac{h}{BC}$ $BC = \dfrac{h}{\tan 60°}$ $= \dfrac{h}{\sqrt{3}}$	• 1 – correct solution
	(ii)	In ΔDAC , $\tan 30° = \dfrac{h}{AC}$ $AC = \dfrac{h}{\tan 30°}$ $= h\sqrt{3}$	• 1 – correct solution
	(iii)	In ΔABC , $AB^2 = BC^2 + AC^2 - 2 \times BC \times AC \times \cos 60°$ $49 = \left(\dfrac{h}{\sqrt{3}}\right)^2 + \left(h\sqrt{3}\right)^2 - \cancel{2} \times \dfrac{h}{\cancel{\sqrt{3}}} \times h\cancel{\sqrt{3}} \times \dfrac{1}{\cancel{2}}$ $49 = \dfrac{h^2}{3} + 3h^2 - h^2$ $49 = \dfrac{7h^2}{3}$ $21 = h^2$ $h = \sqrt{21}$ $\left(\text{since } h > 0\right)$	• 2 – correct solution • 1 – uses cosine rule, showing appropriate substitution – finds AC in terms of h
	(iv)	By sine area rule, $A_{\Delta ABC} = \dfrac{1}{2} \times \dfrac{h}{\cancel{\sqrt{3}}} \times h\cancel{\sqrt{3}} \times \sin 60°$ $= \dfrac{h^2}{2} \times \dfrac{\sqrt{3}}{2}$ $= \dfrac{21\sqrt{3}}{4}$ $A_{\Delta ABC} = \dfrac{1}{2} \times 7 \times CE$ $\dfrac{21\sqrt{3}}{4} = \dfrac{7}{2} \times CE$ $\dfrac{3\sqrt{3}}{2} = CE$	• 3 – correct solution • 2 – finds the area of ΔABC – significant progress towards solution using $\cos \angle CBA = \frac{1}{2\sqrt{7}}$ or $\sin \angle CAB = \frac{\sqrt{21}}{14}$, or equivalent merit • 1 – attempts to use the sine area rule to find the area of ΔABC
(b)	(i)	${}^{14}C_5 = 2002$	• 1 – correct answer, or equivalent numerical expression
	(ii)	${}^{6}C_2 \times {}^{8}C_3 = 840$	• 1 – correct answer, or equivalent numerical expression
	(iii)	There are ${}^{6}C_5 = 6$ committees that consist of men only. Therefore, there are $2002 - 6 = 1996$ committees with at least 1 woman.	• 1 – correct answer, or equivalent numerical expression
	(iv)	There are ${}^{6}C_5 = 6$ committees that consist of men only. There are ${}^{8}C_5 = 56$ committees that consist of women only. Therefore, there are $56 + 6 = 62$ committees that are entirely made up of members of the same gender.	• 1 – correct answer, or equivalent numerical expression

Question 13

Sample solution			Suggested marking criteria
(a)	(i)	$f(-x)=(-x)^2-\lvert -2x\rvert-3$ $=x^2-\lvert 2x\rvert-3$ (since $\lvert -A\rvert=\lvert A\rvert$ for all A) $=f(x)$ Therefore, $f(x)$ is an even function.	• 1 – correct solution
	(ii)		• 2 – correct solution • 1 – correctly sketches one branch of $y=f(x)$
	(iii)	The vertex of the parabola $y=x^2-2x-3$ is $(1,-4)$. By symmetry of the even function, the range of $f(x)=x^2-\lvert 2x\rvert-3$ is $y\geq -4$.	• 1 – correct solution
(b)	(i)		• 2 – correct graphs • 1 – one correct graph
	(ii)	$1\leq x\leq 5$	• 2 – correct solution • 1 – correctly identifies two critical points
(c)	(i)	By difference of two cubes, $64k^6-1=\left(4k^2-1\right)\left(16k^4+4k^2+1\right)$ $=(2k+1)(2k-1)\left(16k^4+4k^2+1\right)$ By difference of two squares, $64k^6-1=\left(8k^3+1\right)\left(8k^3-1\right)$ $=(2k+1)\left(4k^2-2k+1\right)(2k-1)\left(4k^2+2k+1\right)$	• 2 – correct solution • 1 – correct expression for one of the factorisations
	(ii)	$16k^4+4k^2+1=\left(4k^2-2k+1\right)\left(4k^2+2k+1\right)$	• 1 – correct solution

Question 14

Sample solution			Suggested marking criteria
(a)	(i)	${}^{13}C_4 = 715$	• 1 – correct answer, or equivalent numerical expression
	(ii)	Let S represent a "stop" and P represent a "pass". The problem is equivalent to arranging 9 P's and 4 S's so that no S is adjacent to one another. With 9 P's, there are 10 spaces that the 4 S's could go: $\_P\_P\_P\_P\_P\_P\_P\_P\_P\_$ Number of ways $= {}^{10}C_4$ $= 210$	• 1 – correct answer, or equivalent numerical expression
(b)	(i)	$\text{LHS} = \dfrac{\left(\sin^2\alpha - \cos^2\alpha\right)(1 - \sin\alpha\cos\alpha)}{\cos\alpha(\sec\alpha - \operatorname{cosec}\alpha)\left(\sin^3\alpha + \cos^3\alpha\right)}$ $= \dfrac{(\sin\alpha + \cos\alpha)(\sin\alpha - \cos\alpha)(1 - \sin\alpha\cos\alpha)}{\cos\alpha(\sec\alpha - \operatorname{cosec}\alpha)(\sin\alpha + \cos\alpha)\left(\sin^2\alpha - \sin\alpha\cos\alpha + \cos^2\alpha\right)}$ $= \dfrac{\sin\alpha - \cos\alpha}{\cos\alpha\left(\dfrac{1}{\cos\alpha} - \dfrac{1}{\sin\alpha}\right)}$ $= \dfrac{\sin\alpha - \cos\alpha}{\cos\alpha\left(\dfrac{\sin\alpha - \cos\alpha}{\sin\alpha\cos\alpha}\right)}$ $= \sin\alpha$ $= \text{RHS}$	• 3 – correct solution • 2 – significant progress towards a valid solution • 1 – correctly factorises the difference of two squares – correctly factorises the sum of two cubes – correctly simplifies $\cos\alpha(\sec\alpha - \operatorname{cosec}\alpha)$
(c)	(i)	In any ΔABC, $A + B + C = 180°$ $\therefore \tan(A + B + C) = 0$ $\dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = 0$ $\tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$ $\tan A + \tan B + \tan C = \tan A \tan B \tan C$	• 2 – correct solutions • 1 – recognising $\tan(A + B + C) = 0$
	(ii)	$\dfrac{\tan X}{5} = \dfrac{\tan Y}{6} = \dfrac{\tan Z}{7} = k$ Therefore, $\tan X = 5k$, $\tan Y = 6k$ and $\tan Z = 7k$. $\tan X + \tan Y + \tan Z = \tan X \tan Y \tan Z$ $5k + 6k + 7k = 5k \times 6k \times 7k$ $18k = 210k^3$ $0 = 6k\left(35k^2 - 3\right)$ $k \neq 0$ as this implies $\tan X = \tan Y = \tan Z = 0$, which yields a degenerate triangle. $k \not< 0$ as X, Y and Z cannot be all obtuse, therefore $k = \sqrt{\dfrac{3}{35}}$	• 3 – correct solution • 2 – justifies why k is positive • 1 – attempts to solve for k using an appropriate method
	(iii)	$\tan X = 5\sqrt{\dfrac{3}{35}}$ $X = 55°40'$	• 1 – correct solution