Year 11 Extension I Half Yearly Examination 2009

Question 1 (15 Marks) **Marks**

(a) Find $\frac{dy}{dx}$ if:

(i) $y = x^2 \sin 3x$. **2**

(ii) $y = \sqrt{1 - e^{4x}}$. **2**

(iii) $y = \frac{x^3}{4 - x^2}$. **3**

(b) Find the exact value of $\sin 28° \cos 32° + \cos 28° \sin 32°$. **2**

(c) Sketch $y = x + |1 - x|$. **2**

(d) Prove that the tangent to $y = \frac{x^2}{\log_e x}$, at the point where $x = e$, passes through the origin. **4**

Question 2 (15 Marks) (START A NEW PAGE) **Marks**

(a) (i) Write down the expansion of $\tan(A + B)$ **1**

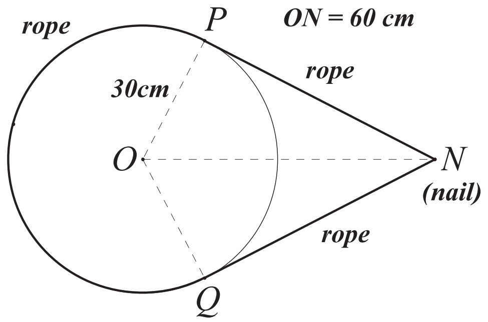
(ii) Hence, using suitable values for *A* and *B*, prove that $\tan 75° = 2 + \sqrt{3}$. **3**

(b) Solve for *x*: $\sqrt{10 - x} = x + 2$. **3**

(c) Evaluate $\lim_{\theta \to 0} \frac{3\theta}{\tan 2\theta}$. **1**

(d) Show that the second derivative of $\log_e(1 + \cos\theta)$ is $\frac{-1}{1 + \cos\theta}$. **3**

(e) A rope passes around a circle of radius 30cm and also around a nail (see diagram). If the nail is 60cm from the centre of the circle, find the exact length of the rope. **3**



Question 3 (15 Marks) (START A NEW PAGE) **Marks**

(a) (i) The point P divides the interval joining points $A(-1,7)$ and $B(4,2)$ in the ratio $k:l$. Write down the coordinates of point P in terms of k and l. **1**

(ii) Hence, or otherwise, find the ratio in which the line $y = 2x + 3$ divides the interval joining $A(-1,7)$ and $B(4,2)$. **2**

(b) $ABCDE$ is a regular pentagon.

(i) Write down the size of $\angle ABC$. **1**

(ii) Prove that $\angle BAC = 36°$. **2**

(iii) If AC meets BD at P, display this information on a diagram and prove that $APDE$ is a rhombus. **3**

(c) A cylinder, open at one end, has a fixed volume of $125\pi\ m^3$.

(i) If the height of the cylinder is h metres and its base radius is r metres, prove that the external surface area, $S\ m^2$, is given by **2**

$$S = \pi r^2 + \frac{250\pi}{r}.$$

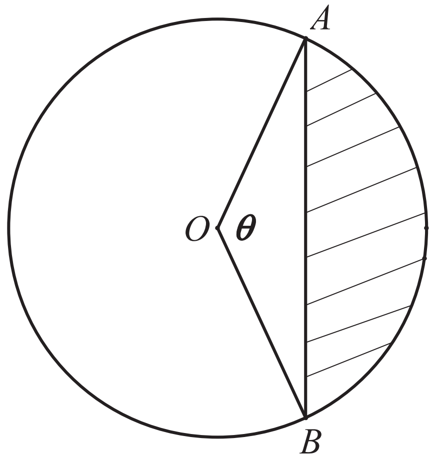
(ii) Hence find the dimensions of the cylinder with least surface area. **4**

QUESTION 4 – Turn over for Question 4

Question 4 (15 Marks) (START A NEW PAGE) **Marks**

(a) Solve for n: $5^{2n} - 3.5^{n} - 28 = 0$. **3**

(b) (i) A chord AB divides a circle into two segments so that the area of the major segment is twice the area of the minor segment (shaded area). If the chord of the minor segment subtends an angle of θ at the centre of the circle, prove that $3\sin\theta = 3\theta - 2\pi$. **2**



(ii) Using a graph, show that the solution of the equation $3\sin\theta = 3\theta - 2\pi$ lies within the range of values $\frac{2\pi}{3} < \theta < \pi$. **2**

(c) Given the curve with equation $y = 2x + \frac{8}{x^2}$.

(i) Find the coordinates of the intercepts with the axes. **1**

(ii) Write down the equations of all asymptotes. **2**

(iii) Find the coordinates of all stationary points and determine their nature. **3**

(iv) Sketch the curve $y = 2x + \frac{8}{x^2}$, showing the above features. **2**

THIS IS THE END OF THE EXAMINATION

Year 11 Extension I Half Yearly Examination 2009 - Solutions

Question 1 (15 Marks) **Marks**

1(a) (i) **2**

$$y = x^2 \sin 3x$$
$$\frac{dy}{dx} = (2x)(\sin 3x) + (x^2)(3\cos 3x)$$
$$= 2x\sin 3x + 3x^2 \cos 3x$$

(ii) **2**

$$y = \sqrt{1-e^{4x}}$$
$$= \left(1-e^{4x}\right)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}\left(1-e^{4x}\right)^{-\frac{1}{2}} \times \left(-4e^{4x}\right)$$
$$= \frac{-2e^{4x}}{\sqrt{1-e^{4x}}}$$

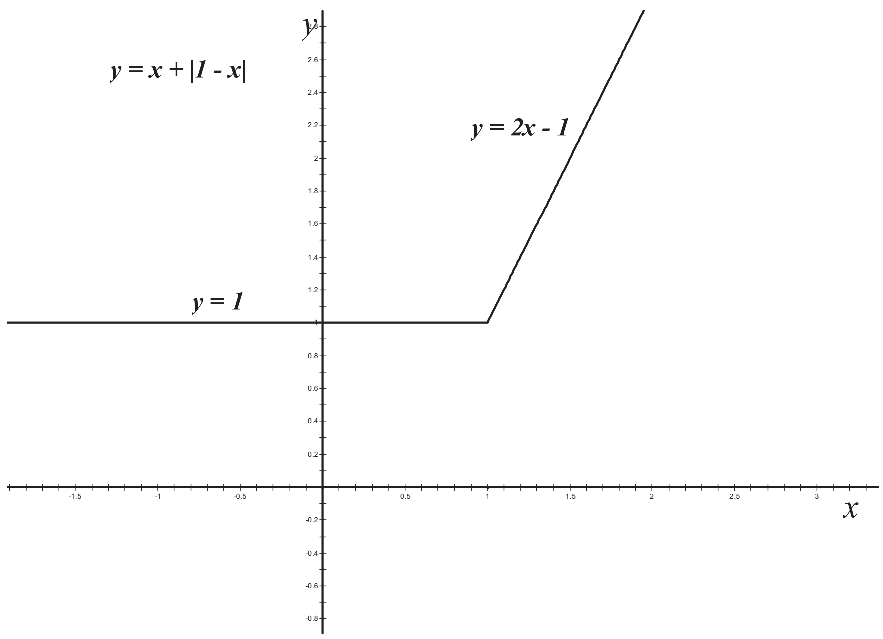
(iii) **3**

$$y = \frac{x^3}{4-x^2}$$
$$\frac{dy}{dx} = \frac{(4-x^2)(3x^2)-(-2x)(x^3)}{(4-x^2)^2}$$
$$= \frac{12x^2 - 3x^4 + 2x^4}{(4-x^2)^2}$$
$$= \frac{12x^2 - x^4}{(4-x^2)^2}$$

1(b) **2**

$$\sin 28^\circ \cos 32^\circ + \cos 28^\circ \sin 32^\circ = \sin(28^\circ + 32^\circ)$$
$$= \sin 60^\circ$$
$$= \frac{\sqrt{3}}{2}$$

1(c) **2**



4

1(d) $y = \dfrac{x^2}{\log_e x}$

$$\frac{dy}{dx} = \frac{(\ln x)(2x) - \left(\frac{1}{x}\right)(x^2)}{(\ln x)^2}$$

$$= \frac{2x\ln x - x}{(\ln x)^2}$$

$$= \frac{x(2\ln x - 1)}{(\ln x)^2}$$

when $x = e$, $\dfrac{dy}{dx} = \dfrac{e(2\ln e - 1)}{(\ln e)^2}$

$= e$

when $x = e$, $y = \dfrac{e^2}{\ln e}$

$= e^2$

Tangent is : $y - e^2 = e(x - e)$

$y = ex$

when $x = 0$, $y = e(0)$

$= 0$

$\therefore$ origin lies on the tangent

Question 2 (15 Marks) (START A NEW PAGE) **Marks**

2(a) (i) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ **1**

(ii) $\tan 75° = \tan(45+30)°$ **3**

$$= \frac{\tan 45° + \tan 30°}{1-\tan 45° \tan 30°}$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times = \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{(\sqrt{3}+1)^2}{3-1}$$

$$= \frac{4+2\sqrt{3}}{2}$$

$$= 2+\sqrt{3}$$

2(b) $\sqrt{10-x}=x+2$ **3**

$10-x=(x+2)^2$

$10-x=x^2+4x+4$

$x^2+5x-6=0$

$(x+6)(x-1)=0$

$x=-6 \text{ or } 1$

but $10-x\geq 0$ and $x+2\geq 0$

$\therefore \quad -2\leq x\leq 10$

$\therefore \quad x=1$

2(c) $\displaystyle\lim_{\theta\to 0}\frac{3\theta}{\tan 2\theta}=\lim_{\theta\to 0}\left(\frac{2\theta}{\tan 2\theta}\times\frac{3}{2}\right)$ **1**

$$=1\times\frac{3}{2}$$

$$=\frac{3}{2}$$

2(d) $f(x)=\log_e(1+\cos\theta)$ **3**

$$f'(x)=\frac{-\sin\theta}{1+\cos\theta}$$

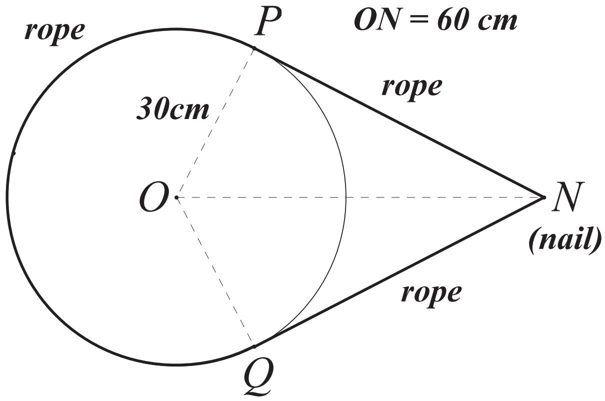
$$f''(x)=-\left\{\frac{(1+\cos\theta)(\cos\theta)-(-\sin\theta)(\sin\theta)}{(1+\cos\theta)^2}\right\}$$

$$=-\left\{\frac{\cos\theta+\cos^2\theta+\sin^2\theta}{(1+\cos\theta)^2}\right\}$$

$$=-\left\{\frac{\cos\theta+1}{(1+\cos\theta)^2}\right\}$$

$$=\frac{-1}{1+\cos\theta}$$

2(e)



Let $\angle PON=\alpha$

$\angle OPN=90°$ (tangent is perpendicular to radius at point of contact)

$$\cos\alpha=\frac{30}{60}$$

$$\alpha=\frac{\pi}{3}$$

$$\angle POQ=\frac{2\pi}{3}$$

reflex $\angle POQ=\frac{4\pi}{3}$

major arc $PQ=30\times\frac{4\pi}{3}\ cm$

$=40\pi\ cm$

$OP^2+PN^2=ON^2$ (Pythagoras' Theorem)

$$PN=\sqrt{60^2-30^2}$$

$$=30\sqrt{3}$$

Length of rope = $\left(40\pi+60\sqrt{3}\right)$ cm

Question 3 (15 Marks) (START A NEW PAGE) **Marks**

3(a) (i) $P\left(\frac{-l+4k}{k+l}, \frac{7l+2k}{k+l}\right)$ **1**

(ii) now point P lies on the line $y = 2x+3$ **2**

$\therefore \quad \frac{7l+2k}{k+l} = 2\left(\frac{-l+4k}{k+l}\right)+3$

$7l+2k = -2l+8k+3k+3l$

$6l = 9k$

$\frac{k}{l} = \frac{6}{9}$

$k : l = 2 : 3$

3(b) (i) $\angle ABC = 108°$. **1**

(ii) $AB = BC$ (all sides of a regular pentagon are equal) **2**

$\angle BAC = \angle BCA = x°$ (equal angles are opposite equal sides in ΔABC)

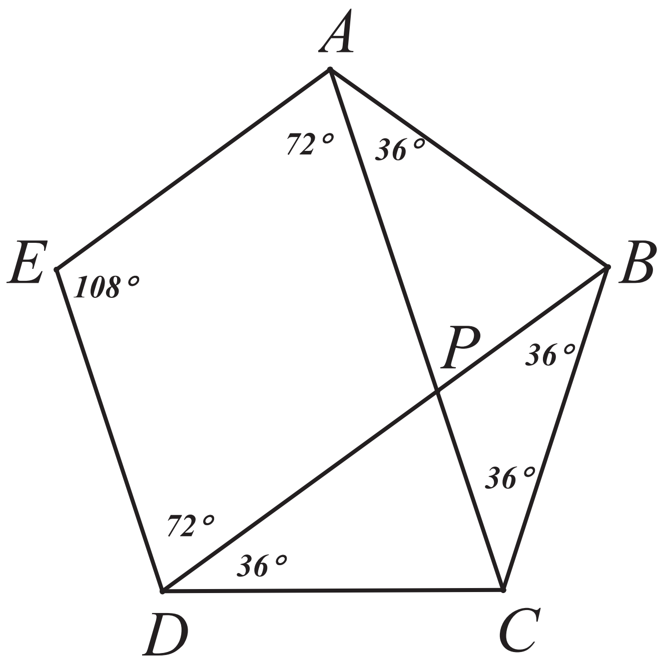
$108 + 2x = 180$ (angle sum of $\Delta ABC = 180°$)

$2x = 72$

$x = 36$

$\angle BAC = 36°$

(iii) **3**



$\angle EAB = \angle ABC = \angle BCD = \angle CDE = \angle DEA = 108°$ $\left(\begin{array}{l}\text{all angles of a regular} \\ \text{pentgon} = 108°\end{array}\right)$

$\angle EAC + 36° = 108°$

$\angle EAC = 72°$

$\angle AED + \angle EAC = 108° + 72°$

$= 180°$

$\therefore AP \parallel ED$ (cointerior angles are supplementary)

Similarly using ΔBCD, $AE \parallel PD$

$\therefore APDE$ is a parallelogram (opposite sides are parallel)

also $AE = ED$ (all sides of a regular pentagon are equal)

$\therefore APDE$ is a rhombus (parallelogram with a pair of equal adjacent sides)

3(c) (i) **2**

$V = \pi r^2 h$

$\pi r^2 h = 125\pi$

$h = \frac{125}{r^2}$

$S = \pi r^2 + 2\pi r h$

$= \pi r^2 + 2\pi r\left(\frac{125}{r^2}\right)$

$S = \pi r^2 + \frac{250\pi}{r}$

(ii) **4**

$S = \pi r^2 + \frac{250\pi}{r}$

$\frac{dS}{dr} = 2\pi r - \frac{250\pi}{r^2}$

For stat. pt. $\frac{dS}{dr} = 0$

$2\pi r - \frac{250\pi}{r^2} = 0$

$r^3 = 125$

$r = 5$

$\frac{d^2S}{dr^2} = 2\pi + \frac{500\pi}{r^3}$

when $r = 5$, $\frac{d^2S}{dr^2} = 2\pi + \frac{500\pi}{5^3}$

> 0

Concave up, therefore local min. tp.

Since function is continuous for $r > 0$ and there is only one stat. pt. then local min. is absolute min.

$\therefore$ least surface area when radius $= 5m$ and height $= 5m$

Question 4 (15 Marks) (START A NEW PAGE) **Marks**

4(a) **3**

$5^{2n} - 3.5^n - 28 = 0$

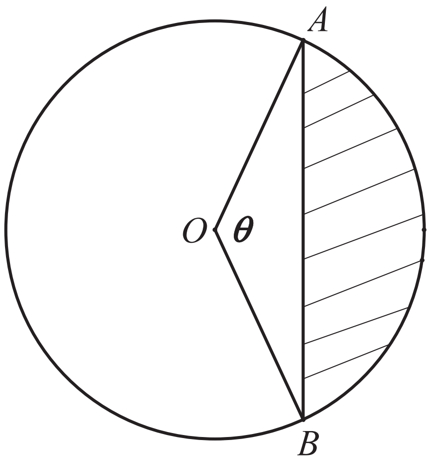
$(5^n - 7)(5^n + 4) = 0$

$5^n = 7$ or $5^n = -4$

$x = \log_5 7 \quad (5^n > 0)$

2

4(b) (i)



Let shaded area = A square units and radius = r units

then $A = \frac{1}{2}r^2(\theta - \sin\theta)$

also area of major segment $= \pi r^2 - A$

$\therefore \pi r^2 - A = 2A$

$3A = \pi r^2$

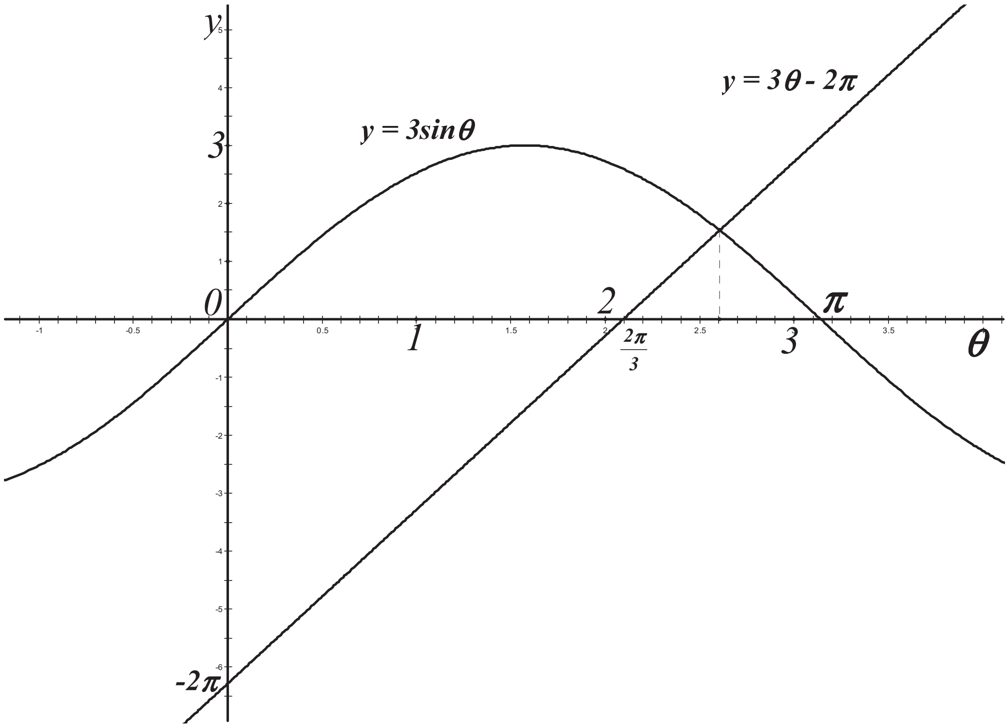
$\therefore 3 \times \frac{1}{2}r^2(\theta - \sin\theta) = \pi r^2$

$3(\theta - \sin\theta) = 2\pi$

$3\theta - 3\sin\theta = 2\pi$

$3\sin\theta = 3\theta - 2\pi$

(ii) **2**



$$\therefore \quad \frac{2\pi}{3} < \theta < \pi$$

4(c) (i) **1**

$y = 2x + \dfrac{8}{x^2}, \quad (x \neq 0)$

when $y = 0$, $\quad 2x + \dfrac{8}{x^2} = 0$

$x^3 = -4$

$x = -\sqrt[3]{4}$

intercept with x-axis is $\left(-\sqrt[3]{4}, 0\right)$

No intercepts with y-axis

(ii) Vertical asymptote: $x = 0$ **2**

Oblique asymptote: $y = 2x$

(iii) **3**

$y = 2x + \dfrac{8}{x^2}, \quad (x \neq 0)$

$\dfrac{dy}{dx} = 2 - \dfrac{16}{x^3}$

For stat. pts $\dfrac{dy}{dx} = 0$

$2 - \dfrac{16}{x^3} = 0$

$x^3 = 8$

$x = 2$

when $x = 2$, $y = 4 + \dfrac{8}{4}$

$= 6$

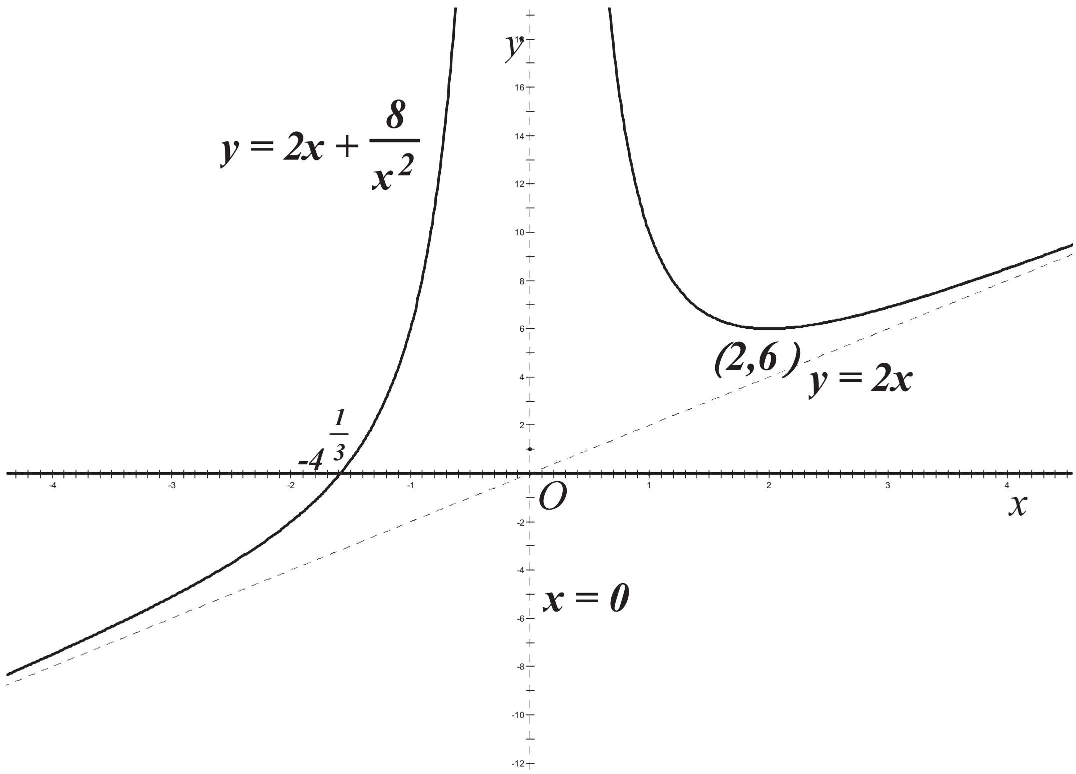
$$\frac{d^2y}{dx^2} = \frac{48}{x^4}$$

when $x = 2$, $\frac{d^2y}{dx^2} = 4 + \frac{48}{16}$

> 0

$\therefore$ concave up, local min. tp. at $(2,6)$

(iv) **2**



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