SYDNEY GRAMMAR SCHOOL

2010 Half-Yearly Examination

FORM V
MATHEMATICS EXTENSION 1

Friday 7th May 2010

General Instructions
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper
- Total marks — 96
- All eight questions may be attempted.
- All eight questions are of equal value.

Collection
- Write your name, class and master clearly on each leaflet.
- Hand in the eight questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- Write your name on the question paper and place it inside your leaflet for Question One.

5A: DNW	5B: DS	5C: TCW
5D: MLS	5E: RCF	5F: PKH
5G: KWM	5H: REP	5I: SJE

Checklist
- Writing leaflets: 8 per boy.
- Candidature — 149 boys

Examiner
KWM

QUESTION ONE (12 marks) Start a new leaflet.

(a) Write down the exact value of $\cos 150°$.

(b) Solve $|x+1| = 3$.

(c) Consider the straight line $\sqrt{3}\,x - y + 2 = 0$.

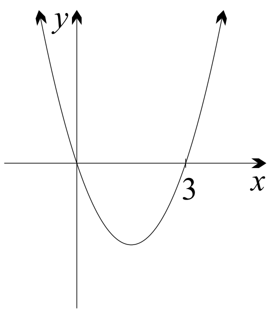
(i) Write down the gradient of the line.

(ii) Find the angle of inclination of the line.

(d) Write down the value of $8^{-\frac{2}{3}}$.

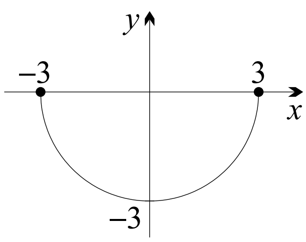
(e) Factorise $x^3 - 8$.

(f)



The diagram above shows the curve $y = x(x-3)$.
Use the curve to solve the inequation $x(x-3) \geq 0$.

(g)



The diagram above shows the graph of the relation $y = -\sqrt{9 - x^2}$.

(i) Write down the domain of the relation.

(ii) Explain why the relation is a function.

(h) Calculate the limiting sum of the geometric series $3 + 1 + \frac{1}{3} + \ldots$.

Exam continues next page ...

QUESTION TWO (12 marks) Start a new leaflet.

(a) The first term of an arithmetic progression is -1 and the third term term is 9.

(i) Find the common difference.

(ii) Find the fiftieth term.

(iii) Find the sum of the first fifty terms.

(b) By rationalising the denominator, express $\dfrac{3}{2+\sqrt{5}}$ in the form $a + b\sqrt{5}$.

(c) The function $f(x)$ is defined by $f(x) = x^2 - x$.

(i) Find $f(x+h)$.

(ii) Use the definition $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find $f'(x)$.

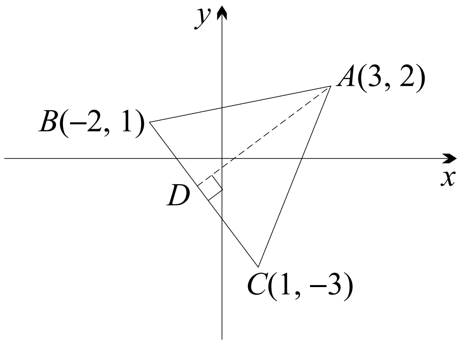
(iii) Find the equation of the tangent to the curve $y = x^2 - x$ at the point $P(1, 0)$.

(d) Solve for x

$$2\log_a 2 + \log_a 3 = \log_a x.$$

QUESTION THREE (12 marks) Start a new leaflet.

(a)



In the diagram above $\triangle ABC$ has vertices $A(3,2)$, $B(-2,1)$ and $C(1,-3)$. AD is an altitude of the triangle.

(i) Find the length of the side BC.

(ii) Find the gradient of BC.

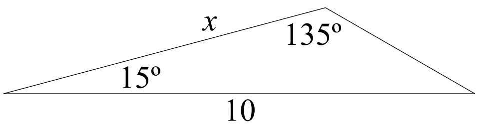
(iii) Show that the equation of BC is $4x + 3y + 5 = 0$.

(iv) Use the perpendicular distance formula to find the length of the altitude AD.

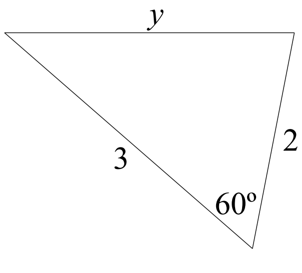
(v) Hence or otherwise, calculate the area of $\triangle ABC$.

(b) Find the exact value of the pronumeral in each diagram below.

(i)



(ii)



(c) The sum of the first n terms of a sequence is given by $S_n = n^2 - n$.

(i) Find S_8 and S_9.

(ii) Hence find the ninth term of the sequence.

Exam continues next page . . .

QUESTION FOUR (12 marks) Start a new leaflet.

(a) For each of the following functions find $\dfrac{dy}{dx}$.

(i) $y = x^3 + 2x - 3$

(ii) $y = (3x + 2)^4$

(iii) $y = \sqrt{x}$

(iv) $y = \dfrac{3x - 2}{x}$

(b) Use the quotient rule to differentiate $\dfrac{2x}{x^2 + 1}$.

(c) Use the product rule to differentiate $(2x + 1)^2(x^2 + 1)$ and express the derivative in factored form.

(d) (i) Fully factorise $x^3 - x^2 - 2x$.

(ii) Sketch the curve $y = x^3 - x^2 - 2x$, indicating all intercepts with the axes.

(iii) Hence or otherwise, solve the inequation $x^3 - x^2 - 2x < 0$.

QUESTION FIVE (12 marks) Start a new leaflet.

(a) Given $\tan\theta = -\frac{5}{12}$ and $\cos\theta > 0$, find the exact value of $\sin\theta$.

(b) Sketch the graph of the function $y = 1 - |x - 2|$.

(c) Find the equation of the straight line that passes through the point of intersection of the lines $2x - y + 1 = 0$ and $x + y - 2 = 0$ and also passes through the point $(2, -1)$. Do not find the point of intersection of the two lines and leave your answer in general form.

(d) A function $h(x)$ is defined by $h(x) = \dfrac{2}{x^2 + 1}$.

(i) Evaluate $h(0)$.

(ii) Show that $h(x)$ is an even function.

(iii) What value does $h(x)$ approach as $x \to \infty$?

(iv) Sketch the function $y = h(x)$ and state its range.

Exam continues overleaf ...

QUESTION SIX (12 marks) Start a new leaflet.

(a) (i) Solve $\sin x + \sqrt{3}\cos x = 0$, for $0^\circ \le x \le 360^\circ$.

(ii) Solve $\sec^2 x + \tan x = 3$, for $0^\circ \le x \le 360^\circ$.
(Give your solutions correct to the nearest minute.)

(b) (i) Write down the radius and centre of the circle $(x+2)^2 + (y-3)^2 = 25$.

(ii) Show that the line $4x - 3y - 8 = 0$ is a tangent to the circle $(x+2)^2 + (y-3)^2 = 25$.

(c) Prove the identity $\dfrac{\cos\alpha}{1+\sin\alpha} + \dfrac{1+\sin\alpha}{\cos\alpha} = 2\sec\alpha$.

QUESTION SEVEN (12 marks) Start a new leaflet.

(a) The third and sixth terms of a geometric sequence are $\dfrac{1}{27}$ and $\dfrac{1}{729}$ respectively.

(i) Find the first term and the common ratio.

(ii) Find the sum of the first seven terms. (Express your answer as a fraction in simplest terms.)

(iii) How many terms of the sequence exceed $\dfrac{1}{1\,000\,000}$?

(b) (i) Show that $x = 60^\circ$ is a solution to the equation $\sin x = \frac{1}{2}\tan x$.

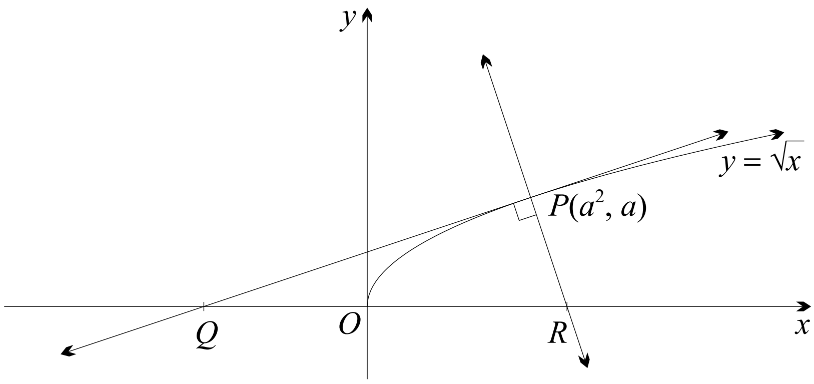
(ii) On the same set of axes sketch the graphs of the functions
$y = \sin x$ and $y = \frac{1}{2}\tan x$, for $-180^\circ \le x \le 180^\circ$.

(iii) Use your graphs to solve $\sin x \le \frac{1}{2}\tan x$, for $-90^\circ < x < 90^\circ$.

Exam continues next page ...

QUESTION EIGHT (12 marks) Start a new leaflet.

(a)



In the diagram above the tangent and normal to the curve $y = \sqrt{x}$ at a variable point $P(a^2, a)$, where $a > 0$, intersect the x-axis at the points Q and R respectively.

(i) Find the equation of the tangent at the point $P(a^2, a)$.

(ii) Find the co-ordinates of Q and R.

(iii) Hence show that the difference in the distances of the points Q and R from the origin is constant.

(b) The two curves $y = x^2 + ax + b$ and $y = cx - x^2$ share a common tangent at the point $(1, 0)$. Find the values of the constants a, b and c.

(c) Find the equation of the line which bisects the acute angle between the lines $\ell_1 : 3x - 6y - 10 = 0$ and $\ell_2 : 2x - y - 4 = 0$.

END OF EXAMINATION

QUESTION 1

(a) $\cos 150^\circ = -\cos 30^\circ$

$= -\frac{\sqrt{3}}{2}$ ✓

(b) $|x+1| = 3$

$x+1=3$ or $x+1=-3$

$x=2$ ✓ or $x=-4$ ✓

(c) $\sqrt{3}x - y + 2 = 0$

$y = \sqrt{3}x + 2$

(i) gradient $= \sqrt{3}$

(ii) $\tan\alpha = \sqrt{3}$ ✓

$\alpha = 60^\circ$

angle of inclination is 60° ✓

(d) $8^{-\frac{2}{3}} = (2)^{-2}$

$= \frac{1}{4}$ ✓

(e) $x^3 - 8 = (x-2)(x^2+2x+4)$ ✓

(f) $x(x-3) \geq 0$

$x \leq 0, \quad x \geq 3$ ✓

(g)

(i) Domain: $-3 \leq x \leq 3$ ✓

(ii) the graph satisfies the vertical line test. ✓

(h) $3 + 1 + \frac{1}{3} + \cdots$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{3}{1-\frac{1}{3}}$ ✓

$S_\infty = \frac{3}{\frac{2}{3}}$

$S_\infty = \frac{9}{2}$

$S_\infty = 4\frac{1}{2}$ ✓

12

QUESTION 2

(a)

$$\left.\begin{aligned} a &= -1 \\ a + 2d &= 9 \end{aligned}\right\} \quad \begin{matrix} ① \\ ② \end{matrix}$$

(i) $2d = 10$

$d = 5$ ✓

(ii) $t_n = a + (n-1)d$

$t_{50} = -1 + 49 \times 5$

$t_{50} = 244$ ✓

(iii) $S_n = \frac{n}{2}(a + l)$

$S_{50} = 25(-1 + 244)$

$S_{50} = 6075$ ✓

(b) $\frac{3}{2+\sqrt{5}} \times \frac{2-\sqrt{5}}{2-\sqrt{5}}$ ✓

$= \frac{6 - 3\sqrt{5}}{4 - 5}$

$= -6 + 3\sqrt{5}$ ✓

(c) $f(x) = x^2 - x$

(i) $f(x+h) = (x+h)^2 - (x+h)$

$= x^2 + 2xh + h^2 - x - h$ ✓

(ii) $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h}$

$= \lim\limits_{h \to 0} \frac{2xh + h^2 - h}{h}$ ✓

$= \lim\limits_{h \to 0} \frac{h(2x + h - 1)}{h}$

$= \lim\limits_{h \to 0} 2x - 1 + h$

$= 2x - 1$ ✓

(iii) $f'(x) = 2x - 1$

$f'(1) = 1$

gradient $= 1$ ✓ $P(1, 0)$

$y - y_1 = m(x - x_1)$

$y = 1(x - 1)$

tangent: $y = x - 1$ ✓

(d) $2\log_a 2 + \log_a 3 = \log_a x$

$\log_a 4 + \log_a 3 = \log_a x$ ✓

$\log_a 12 = \log_a x$

$\underline{x = 12}$ ✓

12

QUESTION 3

(a) $A(3,2)$, $B(-2,1)$, $C(1,-3)$

(i) $\overline{BC} = \sqrt{(-2-1)^2 + (1--3)^2}$

$\overline{BC} = \sqrt{9+16}$

$= 5$ units ✓

(ii) gradient $BC = \dfrac{1--3}{-2-1}$

$= -\dfrac{4}{3}$ ✓

(iii) $m = -\dfrac{4}{3}$ $\quad B(-2,1)$

$y - y_1 = m(x - x_1)$

$y - 1 = -\dfrac{4}{3}(x+2)$

$3y - 3 = -4x - 8$

$4x + 3y + 5 = 0$ ✓

(iv) $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$d = \dfrac{|12 + 6 + 5|}{\sqrt{16+9}}$ ✓

$d = \dfrac{23}{5}$

$= 4\frac{3}{5}$ units ✓

(v) Area $\Delta ABC = \frac{1}{2}bh$

$= \frac{1}{2}\overline{BC} \times \overline{AD}$

$= \frac{1}{2} \times 5 \times \frac{23}{5}$

$= \frac{23}{2}$ ✓

$= 11\frac{1}{2}$ sq units.

(b)

(i) $\dfrac{x}{\sin 30°} = \dfrac{10}{\sin 135°}$ ✓

$\dfrac{x}{\frac{1}{2}} = \dfrac{10}{\frac{1}{\sqrt{2}}}$

$2x = 10\sqrt{2}$ ✓

$x = 5\sqrt{2}$ units.

(ii)

$y^2 = 4 + 9 - 2 \times 3 \times \cos 60°$ ✓

$y^2 = 4 + 9 - 6$

$y = \sqrt{7}$ units. ✓

(c)

$S_n = n^2 - n$

(i) $S_8 = 64 - 8$ $\qquad S_9 = 81 - 9$

$= 56$ ✓ $\qquad S_9 = 72$

(ii) $t_9 = S_9 - S_8$

$= 72 - 56$

$t_9 = 16$ ✓

12

QUESTION 4

(a)

(i) $y = x^3 + 2x - 3$

$\frac{dy}{dx} = 3x^2 + 2$

(ii) $y = (3x+2)^4$

$\frac{dy}{dx} = 4(3x+2)^3 \times 3$

$= 12(3x+2)^3$

(iii) $y = \sqrt{x}$

$y = x^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$

$= \frac{1}{2\sqrt{x}}$

(iv) $y = \frac{3x-2}{x}$

$y = 3 - 2x^{-1}$

$\frac{dy}{dx} = \frac{2}{x^2}$

(b) $f(x) = \frac{2x}{x^2+1}$

$f'(x) = \frac{(x^2+1)2 - 2x \times 2x}{(x^2+1)^2}$

$= \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2}$

$= \frac{2-2x^2}{(x^2+1)^2}$

$= \frac{2(1-x)(1+x)}{(x^2+1)^2}$

(c) $f(x) = (2x+1)^2(x^2+1)$

$f'(x) = 2(2x+1)(x^2+1)2 + (2x+1)^2 \times 2x$

$= 2(2x+1)\{2(x^2+1) + x(2x+1)\}$

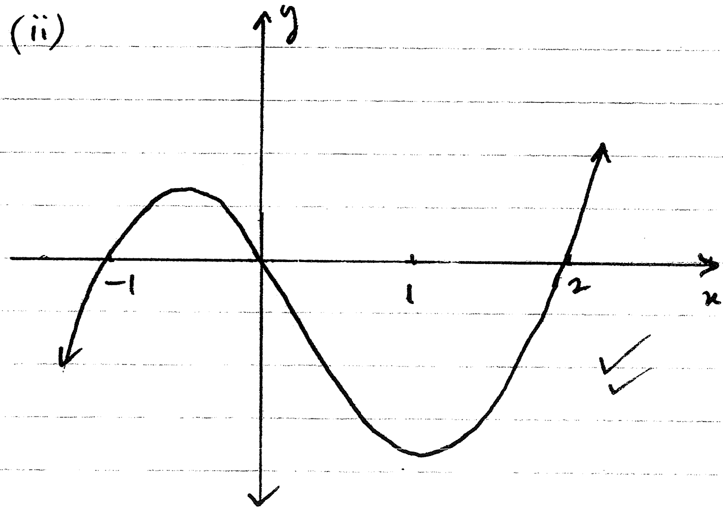
$= 2(2x+1)(4x^2+x+2)$

(d) (i) $x^3 - x^2 - 2x$

$= x(x^2 - x - 2)$

$= x(x+1)(x-2)$

(ii)



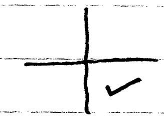
(iii) $x(x+1)(x-2) < 0$

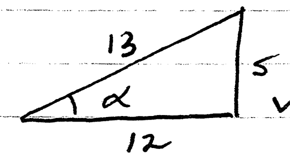
$x < -1$ OR $0 < x < 2$

12

QUESTION 5

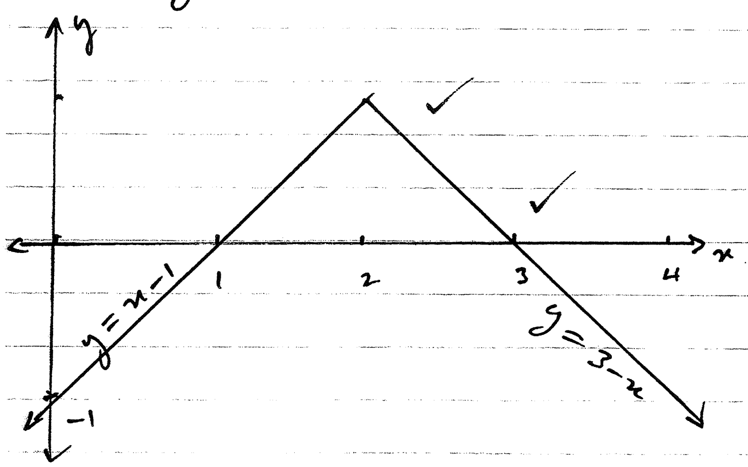
(a) $\tan\theta = -\frac{5}{12}$





$\sin\theta = -\frac{5}{13}$

(b) $y = 1 - |x-2|$



(c)

$2x - y + 1 + k(x+y-2) = 0$

$(2,-1)$: $4+1+1+k(2-1-2) = 0$

$6 - k = 0$

$\underline{k = 6}$

$2x - y + 1 + 6(x+y-2) = 0$

$\underline{8x + 5y - 11 = 0}$

(d) $h(x) = \frac{2}{x^2+1}$

(i) $h(0) = 2$

(ii) $h(-x) = \frac{2}{(-x)^2+1}$

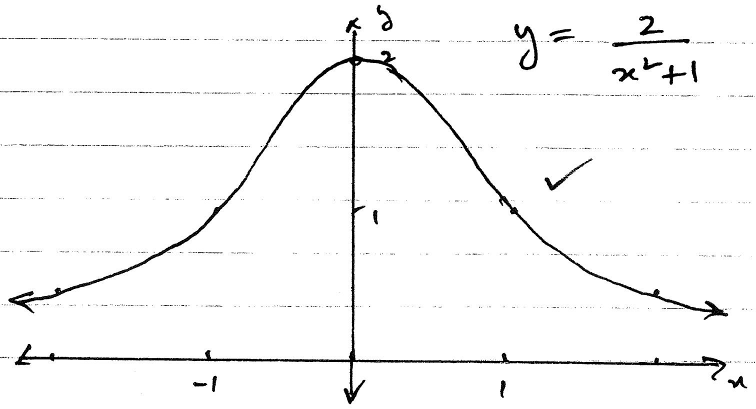
$= \frac{2}{x^2+1}$

$= h(x)$

$\therefore$ $h(x)$ is an even function.

(iii) as $x \to \infty$, $h(x) \to 0$.

(iv)



Range: $0 < y \leq 2$

12

QUESTION 6

(a)

(i) $\sin x + \sqrt{3}\cos x = 0$

$\frac{\sin x}{\cos x} + \sqrt{3} = 0$

$\tan x = -\sqrt{3}$ ✓

✓ ✓

$x = 120°$ or $x = 300°$ ✓

(ii) $\sec^2 x + \tan x = 3$

$(1+\tan^2 x) + \tan x = 3$

$\tan^2 x + \tan x - 2 = 0$ ✓

$(\tan x - 1)(\tan x + 2) = 0$ ✓

$\tan x = 1$ $\quad$ $\tan x = -2$

$x = 45°, 225°$ ✓ $x = 116°34', 296°34'$

(b) (i) $(x+2)^2 + (y-3)^2 = 25$

radius = 5 units ✓

centre $(-2, 3)$

(ii) The distance from the line $4x - 3y - 8 = 0$ to the centre of the circle $(-2,3)$

$$d = \frac{|-8-9-8|}{\sqrt{16+9}}$$ ✓

$$d = \frac{25}{5}$$

$d =$ 5 units. ✓

Since the perpendicular distance from the line to the centre equals the radius of the circle, the line intersects the circle at one point only. ✓
Hence the line is a tangent.

(c)

$$LHS = \frac{\cos\alpha}{1+\sin\alpha} + \frac{1+\sin\alpha}{\cos\alpha}$$

$$= \frac{\cos^2\alpha + (1+\sin\alpha)^2}{\cos\alpha(1+\sin\alpha)}$$ ✓

$$= \frac{\cos^2\alpha + \sin^2\alpha + 2\sin\alpha + 1}{\cos\alpha(1+\sin\alpha)}$$

$$= \frac{2 + 2\sin\alpha}{\cos\alpha(1+\sin\alpha)}$$ ✓

$$= \frac{2(1+\sin\alpha)}{\cos\alpha(1+\sin\alpha)}$$ ✓

$$= \frac{2}{\cos\alpha}$$

$= 2\sec\alpha$

$=$ RHS

12

QUESTION 7

(a)

(i) $ar^2 = \frac{1}{27}$ ——— ①

$ar^5 = \frac{1}{729}$ ——— ②

$\frac{②}{①}$ $\frac{ar^5}{ar^2} = \frac{1}{729} \div \frac{1}{27}$ ✓

$r^3 = \frac{27}{729}$

$r = \frac{3}{9}$ ✓

$r = \frac{1}{3}$. $a = \frac{1}{3}$

(ii) $S_n = \frac{a(1-r^n)}{1-r}$

$S_7 = \frac{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^7\right)}{1-\frac{1}{3}}$ ✓

$S_7 = \frac{1}{2}\left(1 - \frac{1}{2187}\right)$

$= \frac{1}{2} \times \frac{2186}{2187}$

$= \frac{1093}{2187}$ ✓

(iii) $t_n > 10^{-6}$

$ar^{n-1} > 10^{-6}$

$\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} > 10^{-6}$

$\frac{1}{3^n} > \frac{1}{1000\,000}$ ✓

$3^n < 1000\,000$

$n \log 3 < \log 1000\,000$

$n \log 3 < 6$ ✓

$n < \frac{6}{\log 3}$

$n < 12.57\ldots$

$n = 12$ ✓

12 terms of the sequence exceed 10^{-6}.

(b)

(i) $\sin x = \frac{1}{2}\tan x$

$x = 60°$: LHS $= \sin 60°$

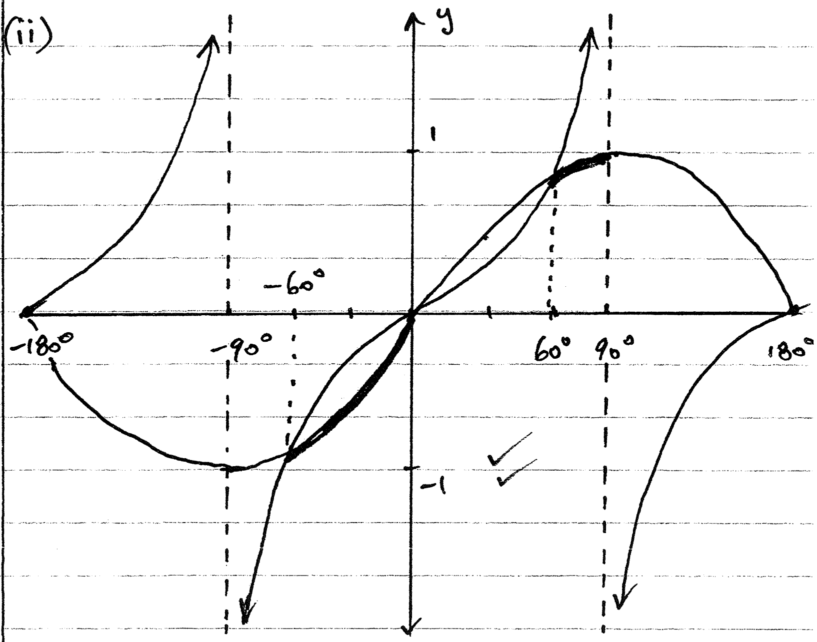
$= \frac{\sqrt{3}}{2}$ ✓

RHS $= \frac{1}{2}\tan 60°$

$= \frac{1}{2} \times \sqrt{3}$

$= \frac{\sqrt{3}}{2}$ $\therefore$ 60° is a solutⁿ.

(ii)



(iii) $\sin x \le \frac{1}{2}\tan x$ $\quad -90° < x < 90°$

$60° \le x < 90°$ or $-60° \le x \le 0°$ ✓ ✓

12

QUESTION 8

(a) (i) $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

at $P(a^2, a)$ gradient $= \dfrac{1}{2a}$

$$y - y_1 = m(x - x_1)$$
$$y - a = \frac{1}{2a}(x - a^2)$$

$$2ay - 2a^2 = x - a^2$$
$$\underline{x - 2ay + a^2 = 0} \text{ tangent.}$$

(ii) Q: put $y = 0$.
$x = -a^2$

$Q(-a^2, 0)$

equation of the normal.

$$y - a = -2a(x - a^2)$$

put $y = 0$

$$-a = -2a(x - a^2)$$
$$\frac{1}{2} = x - a^2$$
$$x = a^2 + \frac{1}{2}.$$

$R\left(a^2 + \frac{1}{2}, 0\right)$

(iii) $\overline{OR} - \overline{OQ}$

$$|a^2 + \tfrac{1}{2}| - |-a^2|$$
$$= a^2 + \frac{1}{2} - a^2$$
$$= \frac{1}{2} \;//$$

(b) $\begin{cases} ① \; y = x^2 + ax + b \\ ② \; y = cx - x^2 \end{cases}$

$(1, 0)$ lies on both curves

① $1 + a + b = 0$

② $c - 1 = 0$

$\therefore \underline{c = 1}$

$$\frac{dy}{dx} = 2x + a$$
$$\frac{dy}{dx} = c - 2x$$

at $x = 1$ the gradients are equal.

$$a + 2 = c - 2$$
$$a + 2 = 1 - 2$$
$$a + 2 = -1$$
$$\underline{a = -3}$$

① $1 - 3 + b = 0$

$\underline{b = 2}$

(c) Let $P(x, y)$ be any point on the angle bisector, d_1 the distance from P to l_1 and d_2 the distance from P to l_2.

$$\frac{|3x - 6y - 10|}{\sqrt{45}} = \frac{|2x - y - 4|}{\sqrt{5}}$$
$$|3x - 6y - 10| = 3|2x - y - 4|$$

$$3x - 6y - 10 = 6x - 3y - 12$$
$$\underline{3x + 3y - 2 = 0} \quad \text{OR}$$

$$3x - 6y - 10 = -6x + 3y + 12$$
$$\underline{9x - 9y - 22 = 0}$$

The acute angle bisector must have a gradient that lies between l_1: $m_1 = 1/2$ and l_2: $m_2 = 2$.

$\therefore$ $9x - 9y - 22 = 0$ is the acute angle bisector of l_1 and l_2.

12