Sydney Grammar School

2011 Half-Yearly Examination

FORM V
MATHEMATICS EXTENSION 1

Wednesday 11th May 2011

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper

- Total marks — 96
- All eight questions may be attempted.
- All eight questions are of equal value.

Collection

- Write your name, class and master clearly on each leaflet.
- Hand in the eight questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- Write your name on the question paper and place it inside your leaflet for Question One.

5A: BDD	5B: PKH	5C: FMW
5D: KWM	5E: SJE	5F: RCF
5G: MW	5H: SO	5I: MLS

Checklist

- Writing leaflets: 8 per boy.
- Candidature — 149 boys

Examiner

RCF

QUESTION ONE (12 marks) Start a new leaflet.

(a) Simplify:

(i) $\sqrt{48} - \sqrt{27}$

(ii) $(2\sqrt{5} + 5)^2$

(b) (i) Write down the exact value of $\cos 45°$.

(ii) Find the exact value of $\sec 135°$.

(c) Differentiate the following functions:

(i) $2x^3 - x^5$

(ii) x^{-3}

(iii) $\sqrt[3]{x}$

(d) Find the equation of the straight line perpendicular to the line $y = 3x - 5$ and passing through the point $(-1, 2)$. Give your answer in general form.

(e) Factorise $x^3 + 27$.

QUESTION TWO (12 marks) Start a new leaflet.

(a) Write down the domain of the function $y = \dfrac{1}{x-3}$.

(b) Consider the sequence $2, \frac{2}{3}, \frac{2}{9}, \ldots$

(i) Show the sequence is geometric.

(ii) State the values of a and r.

(iii) Find the limiting sum of the geometric series $2 + \frac{2}{3} + \frac{2}{9} + \cdots$.

(c) Given the points $A(-2, 5)$ and $B(1, -1)$, find the length of the interval AB. Give your answer in simplest exact form.

(d) Solve $|x - 4| = 5$.

(e) The area of $\triangle ABC$ is $14\,\text{cm}^2$. Given that $AB = 8\,\text{cm}$ and $AC = 7\,\text{cm}$, find all possible values of $\angle BAC$.

(f) Solve $(x + 3)(x - 4) \geq 0$.

Exam continues next page ...

QUESTION THREE (12 marks) Start a new leaflet.

(a) By rationalising the denominator and simplifying, find the values of a, b and c such that

$$\frac{3+\sqrt{12}}{2-\sqrt{3}} = a + b\sqrt{c}.$$

(b) Given that $\sin\theta = \frac{5}{7}$ and θ is obtuse, find in simplest form the exact values of:

(i) $\tan\theta$

(ii) $\sec\theta$

(c) Complete the square to find the centre and the radius of the circle with equation

$$x^2 + y^2 + 10y = 39.$$

(d) Show that $\dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \cos x \sin x} = 1 - \tan x.$

(e) Find the perpendicular distance from the point $(4, 2)$ to the line $3x + 4y - 5 = 0$.

QUESTION FOUR (12 marks) Start a new leaflet.

(a) Given $f(x) = \sqrt{x}\,\big(3x^2 - 2x + 1\big)$, find $f'(x)$.

(b) Given the points $A(-2, 5)$ and $B(7, -1)$, find the point P which divides the interval AB internally in the ratio $2 : 1$.

(c) (i) If $f(x) = x^2 + 3x$, find $f(x + h)$.

(ii) Hence differentiate $f(x)$ from first principles.

(d) The third term of an AP is 125 and the sixth term is 116.

(i) Find the first term a and the common difference d.

(ii) Find the value of the thirtieth term.

(iii) Find the minimum number of terms to be added for the sum to be negative.

Exam continues overleaf ...

QUESTION FIVE (12 marks) Start a new leaflet.

(a) (i) Sketch the graph of $y = \tan x$ for the restricted domain $0^\circ \le x \le 360^\circ$. Clearly indicate all intercepts and asymptotes.

(ii) Using your graph and knowledge of special angles, solve $\tan x = -\sqrt{3}$ for $0^\circ \le x \le 360^\circ$.

(b) Use the chain, product or quotient rules to differentiate the following functions:

(i) $\dfrac{5x+1}{5-x}$

(ii) $(x^3+3)(3x-2)^3$ (Give your final answer in factored form.)

(c) The lines $x - 2y + 3 = 0$ and $3x + y - 2 = 0$ intersect at point A. Without finding the co-ordinates of A, find the equation of the line that passes through A and $(2, 1)$. Give your answer in general form.

QUESTION SIX (12 marks) Start a new leaflet.

(a) Find the values of x for which the geometric series $(2-x) + (2-x)^2 + (2-x)^3 + \cdots$ has a limiting sum.

(b) Solve the following equations over the domain $0^\circ \le \theta \le 360^\circ$.

(i) $\operatorname{cosec}\theta = -\sqrt{2}$

(ii) $2\sin^2\theta + \cos\theta = 1$

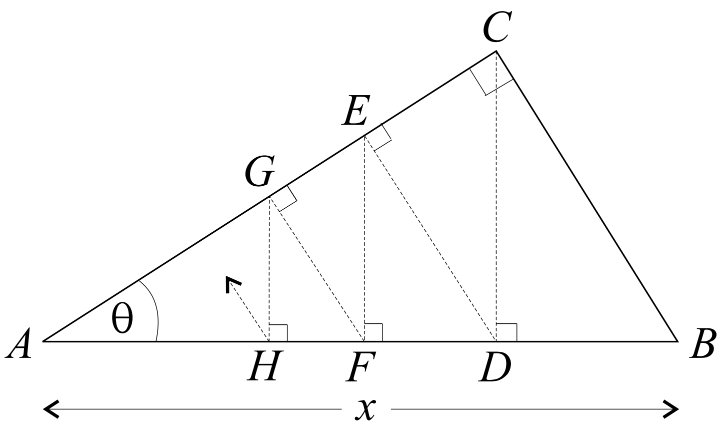
(iii) $\cos(2\theta + 60^\circ) = \frac{1}{\sqrt{2}}$

QUESTION SEVEN (12 marks) Start a new leaflet.

(a) Sketch the graph of the function $y = 2^{-x} - 4$, clearly indicating any asymptotes and intercepts with the axes.

(b) Solve $\dfrac{3x}{x-2} \geq 1$.

(c)



In the diagram above, let side $AB = x$ and $\angle BAC = \theta$

(i) Write down an expression for the side BC.

(ii) Find expressions for CD and EF in terms of x and θ.

(iii) Hence find the limiting sum of the series $CD + EF + GH + \cdots$.

QUESTION EIGHT (12 marks) Start a new leaflet.

(a) Consider the points $A(2, 1)$ and $B(-2, 4)$ in the number plane.

(i) Write down the equation of a line with gradient m that passes through A. Give your answer in general form.

(ii) Find the equations, in general form, of the two lines ℓ_1 and ℓ_2 that pass through A and are 3 units from B.

(iii) Find the equations of the two lines that bisect the angles at A between ℓ_1 and ℓ_2.

(b) Consider the function $f(x) = \dfrac{x+4}{2x^2-8}$.

(i) Determine whether $f(x)$ is even, odd or neither.

(ii) Find all intercepts and asymptotes of the graph of $y = f(x)$.

(iii) Find the gradient function $f'(x)$, and hence find the x co-ordinates of the two points where the graph has zero gradient.

(iv) Sketch the graph of the function $y = f(x)$.

END OF EXAMINATION

Qn 1 HY 2011 - FIFTH FORM EXTENSION ONE (RCF)

a) (i) $\sqrt{48}-\sqrt{27}$
$= 4\sqrt{3}-3\sqrt{3}$
$= \sqrt{3}$ ✓

(ii) $(2\sqrt{5}+5)^2 = (2\sqrt{5}+5)(2\sqrt{5}+5)$
$= 20+10\sqrt{5}+10\sqrt{5}+25$
$= 45+20\sqrt{5}$ ✓
$\left\{ \text{OR } 5(9+4\sqrt{5}) \right\}$

b) (i) $\cos 45° = \frac{1}{\sqrt{2}}$ $\left\{\text{OR } \frac{\sqrt{2}}{2}\right\}$ ✓

(ii) $\sec 135° = \frac{1}{\cos 135°} = \frac{-1}{\cos 45°} = (-\sqrt{2})$ ✓

c) (i) $y = 2x^3 - x^5$
$\frac{dy}{dx} = 6x^2 - 5x^4$ ✓

(ii) $y = x^{-3}$
$\frac{dy}{dx} = -3x^{-4}$ ✓
$\left\{\text{OR } \frac{-3}{x^4}\right\}$

(iii) $y = x^{\frac{1}{3}}$ ✓
$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$ ✓
$\left\{\text{OR } \frac{1}{3\sqrt[3]{x^2}}\right\}$

d) $y = 3x-5$ $m_1 = 3$ ∴ Perp Grad $m_2 = \left(-\frac{1}{3}\right)$ thru $(-1,2)$
$y-2 = -\frac{1}{3}(x-(-1))$ ✓
$3y-6 = -x-1$
$x+3y-5=0$ ✓

e) $x^3+27 = (x+3)(x^2-3x+9)$ ✓ ✓

$\frac{}{12}$

Qu 2

a) $x-3 \neq 0$

$\therefore x \in \mathbb{R}, x \neq 3$ ✓

b) $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}$

$\frac{t_3}{t_2} = \frac{2/9}{2/3} = \frac{1}{3}$ $\quad \frac{t_2}{t_1} = \frac{2/3}{2} = \frac{1}{3}$ ✓

$\therefore$ GP common ratio $r = \frac{1}{3}$

(ii) $a = 2$, $r = \frac{1}{3}$ ✓

(iii) $S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{2/3} = 3$ ✓, since $|r|<1$

c) $A(-2,5)$ $B(1,-1)$

$AB = \sqrt{(5--1)^2 + (-2-1)^2}$ ✓

$= \sqrt{6^2 + 3^2}$

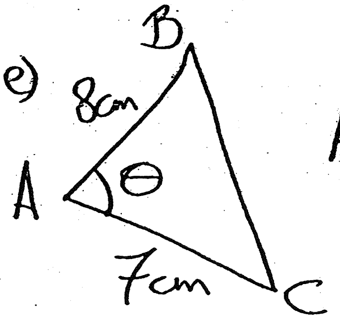
$= \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$ u ✓

d) $|x-4| = 5$

ie $x-4 = 5$ or $x-4 = (-5)$

$x = 9$ ✓ or $x = (-1)$ ✓

e)



Area $= 14\text{cm}^2$

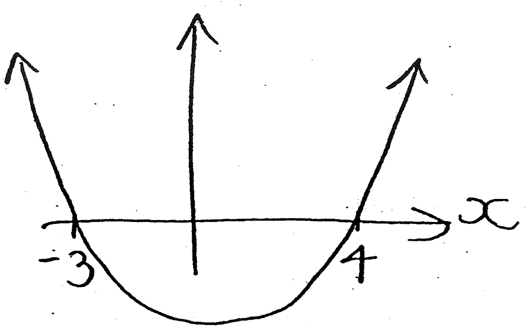
Area $= \frac{1}{2}bc\sin A$

$\therefore 14 = \frac{1}{2} \times 7 \times 8 \times \sin\theta$ ✓

$\frac{1}{2} = \sin\theta$

$\therefore \theta = 30°$ or $150°$ ✓

e) $(x+3)(x-4) \geq 0$



✓

$\therefore x \geq 4$, or $x \leq -3$ ✓

12

Qn 3

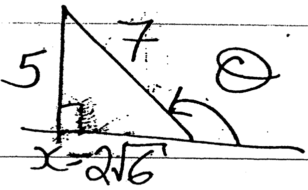
a) $\frac{3+\sqrt{12}}{2-\sqrt{3}} = \frac{(3+\sqrt{12})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$ ✓

$= \frac{6+2\sqrt{12}+3\sqrt{3}+6}{4-3}$

$= 12+7\sqrt{3}$ ✓ $= a+b\sqrt{c}$

$\therefore a=12,\ b=7,\ c=3$ ✓

b) $\sin\theta = 5/7$



$\therefore x = \sqrt{49-25}$

$= \sqrt{24}$

$= 2\sqrt{6}$ ✓

obtuse ⇒ 2^{nd} quad ✓

(i) $\tan\theta = -\frac{5}{2\sqrt{6}}$ ✓

(ii) $\sec\theta = \frac{1}{\cos\theta} = -\frac{7}{2\sqrt{6}}$ ✓ $= \left(-\frac{7\sqrt{6}}{12}\right)$

c) $x^2+y^2+10y=39$

$x^2+y^2+10y+25=64$

$x^2+(y+5)^2=64$

Circle: centre $(0,-5)$
radius 8u

d) $\frac{\cos^2x-\sin^2x}{\cos^2x+\cos x\sin x} = \frac{(\cos x+\sin x)(\cos x-\sin x)}{\cos x(\cos x+\sin x)}$ ✓

$= \frac{\cos x-\sin x}{\cos x}$ ✓

$= 1-\tan x$

$=$ RHS ■

e) $(4,2)$ $3x+4y-5=0$

$p = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$= \frac{|3\times4+4\times2-5|}{\sqrt{3^2+4^2}}$

$= \frac{|12+8-5|}{5}$

$= 3$ units

12

Qu 4

a) $y = \sqrt{x}\,(3x^2 - 2x + 1)$

$y = 3x^{5/2} - 2x^{3/2} + x^{1/2}$ ✓

$\frac{dy}{dx} = \frac{15}{2}x^{3/2} - 3x^{1/2} + \frac{1}{2}x^{-1/2}$ ✓

$\left\{ \text{or } \frac{15(\sqrt{x})^3 - 6\sqrt{x} + 1/\sqrt{x}}{2} \right\}$

$\left\{ \text{or } \frac{15x^2 - 6x + 1}{2\sqrt{x}} \right\}$

b) $A(-2, 5)$ $B(7, -1)$ $k : l = 2 : 1$

P: $x_P = \frac{l x_A + k x_B}{k + l}$

$= \frac{1 \times (-2) + 2 \times 7}{2 + 1}$

$= \frac{12}{3}$

$= 4$ ✓

$y_P = \frac{1 \times 5 + 2 \times (-1)}{2 + 1}$

$= \frac{3}{3}$

$= 1$ ✓

$\therefore P(4, 1)$

c) (i) $f(x) = x^2 + 3x$

$f(x+h) = (x+h)^2 + 3(x+h)$ ✓

$= x^2 + 2xh + h^2 + 3x + 3h$

(ii) $f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{(x+h) - x} \right)$

$= \lim_{h \to 0} \left(\frac{2xh + h^2 + 3h}{h} \right)$ ✓

$= \lim_{h \to 0} (2x + 3 + h)$

$= 2x + 3$

d)(i) AP $t_3 = a+2d = 125$ ①
$t_6 = a+5d = 116$ ②

② − ① ∴ $3d = -9$
$d = (-3)$

sub into ① $a - 6 = 125$
$a = 131$

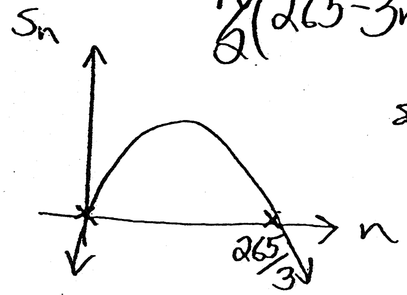
∴ First term, $a = 131$
Common difference, $d = (-3)$

(ii) $t_{30} = a + 29d$
$= 131 + 29(-3)$
$= 131 - 87$
$= 44$

(iii) $S_n = \frac{n}{2}(2a + (n-1)d)$
$S_n = \frac{n}{2}(262 + (n-1)(-3))$
$= \frac{n}{2}(265 - 3n)$

Require $S_n < 0$

∴ $\frac{n}{2}(265-3n) < 0$



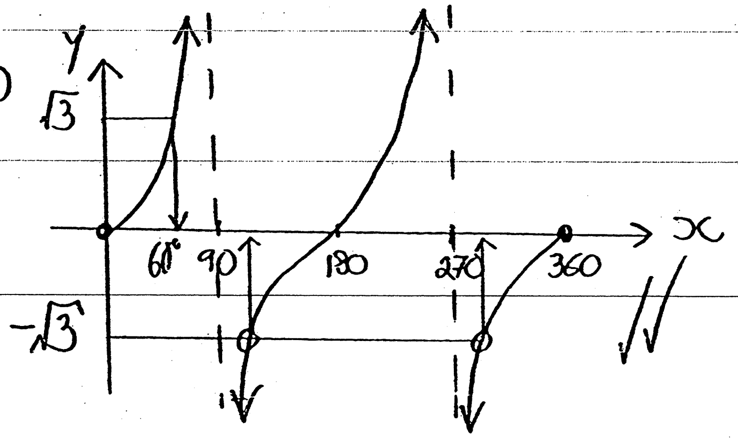
since $n > 0$
$n \in \mathbb{Z}$

$n > \frac{265}{3}$
$n = 89$ need S_{89}.

/12

Qn5

a) (i)



(ii) $\tan x = (-\sqrt{3})$

$\Rightarrow x = 90 + 30$

$= 120^\circ$

OR $x = 270 + 30$

$= 300^\circ$

b) (i) $y = \dfrac{5x+1}{5-x}$ Quotient Rule ; Let $u = 5x+1$ $v = 5-x$

$\dfrac{du}{dx} = 5$ $\dfrac{dv}{dx} = (-1)$

$$\frac{dy}{dx} = \frac{(5-x)\times 5 - (5x+1)(-1)}{(5-x)^2}$$

$$= \frac{25 - 5x + 5x + 1}{(5-x)^2}$$

$$= \frac{26}{(5-x)^2}$$

(ii) $y = (x^3+3)(3x-2)^3$ Product Rule : Let $u = (x^3+3)$, $v = (3x-2)^3$

$\dfrac{du}{dx} = 3x^2$ $\dfrac{dv}{dx} = 3(3x-2)^2$

$= 9(3x-2)^2$

$$\frac{dy}{dx} = (3x-2)^3 \times 3x^2 + (x^3+3)\,9(3x-2)^2$$

$$= 3(3x-2)^2\left[x^2(3x-2) + 3(x^3+3)\right]$$

$$= 3(3x-2)^2\left[3x^3 - 2x^2 + 3x^3 + 9\right] = 3(3x-2)^2(6x^3 - 2x^2 + 9)$$

c) $l_1 : x - 2y + 3 = 0$

$l_2 : 3x + y - 2 = 0$

Concurrent lines $x - 2y + 3 + k(3x + y - 2) = 0$

$(1+3k)x + (k-2)y + (3-2k) = 0$

Passes thru' $(2,1)$ $\therefore$ $2(1+3k) + (k-2) + (3-2k) = 0$

$2 + 6k + k - 2 + 3 - 2k = 0$

$5k + 3 = 0$

$k = \left(-\frac{3}{5}\right)$

$\therefore$ Line is $-\frac{4}{5}x - \frac{13y}{5} + \frac{21}{5} = 0$

$4x + 13y - 21 = 0$

12

Qn6

a) $(2-x)+(2-x)^2+(2-x)^3+\ldots$

GP $a=(2-x)$

$r=(2-x)$

for limiting sum $|r|<1$

$-1<2-x<1$ ✓

$-3<-x<-1$ ✓

$3>x>1$ ✓

ie $1<x<3$

b) (i) $\operatorname{cosec}\theta=-\sqrt{2}$

$\sin\theta=-\frac{1}{\sqrt{2}}$ ✓ $\therefore$ 3rd & 4th quadrants

$\theta=225°, 315°$ ✓

(ii) $2\sin^2\theta+\cos\theta=1$ ✓

$2(1-\cos^2\theta)+\cos\theta-1=0$ ✓

$1-2\cos^2\theta+\cos\theta=0$

$2\cos^2\theta-\cos\theta-1=0$ ✓

$(2\cos\theta+1)(\cos\theta-1)=0$

$\cos\theta=\left(-\frac{1}{2}\right)$ OR 1

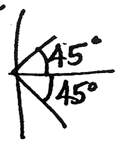
$\theta=120°, 240°$ ✓, $0°, 360°$ ✓

(iii) $\cos(2\theta+60°)=\frac{1}{\sqrt{2}}$ ✓ Change Domain

$0°\leq\theta\leq360°$

$60°\leq2\theta+60°\leq780°$ ✓

$2\theta+60°=315°, 405°, 675°, 765°$ ✓



$2\theta=255°, 345°, 615°, 705°$

$\theta=127\frac{1}{2}°, 172\frac{1}{2}°, 307\frac{1}{2}°, 352\frac{1}{2}°$ ✓

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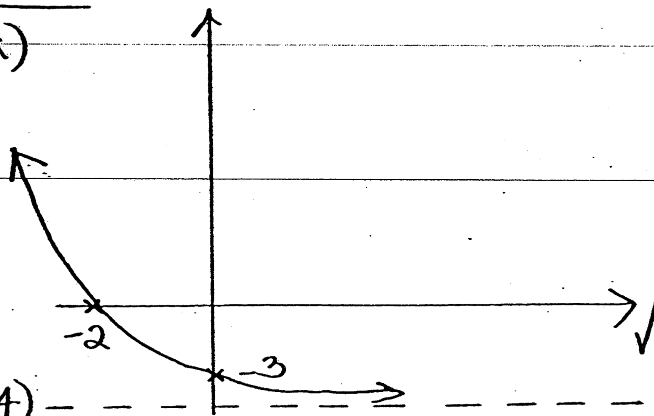
Qn 7

a)



$y = (-4)$

Asymptote

$y = 2^x$

Reflect in y axis

$y = 2^{-x}$

Translate down 4 u

$y = 2^{-x} - 4$

$x = 0$

$y = 2^{-0} - 4$

$= 1 - 4$

$= -3$

$(0, -3)$.

$y = 0$

$2^{-x} = 4$

$\frac{1}{2^x} = 4$ Both Ints

$\therefore 2^x = \frac{1}{4}$

$x = -2$

$(-2, 0)$

b) $\frac{3x}{x-2} \geq 1 \quad \times (x-2)^2$ Dom: $x \neq 2$

$3x(x-2) \geq (x-2)^2$

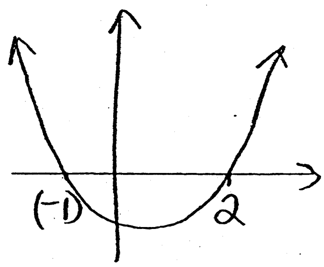
$3x(x-2) - (x-2)^2 \geq 0$

$(x-2)[3x-(x-2)] \geq 0$

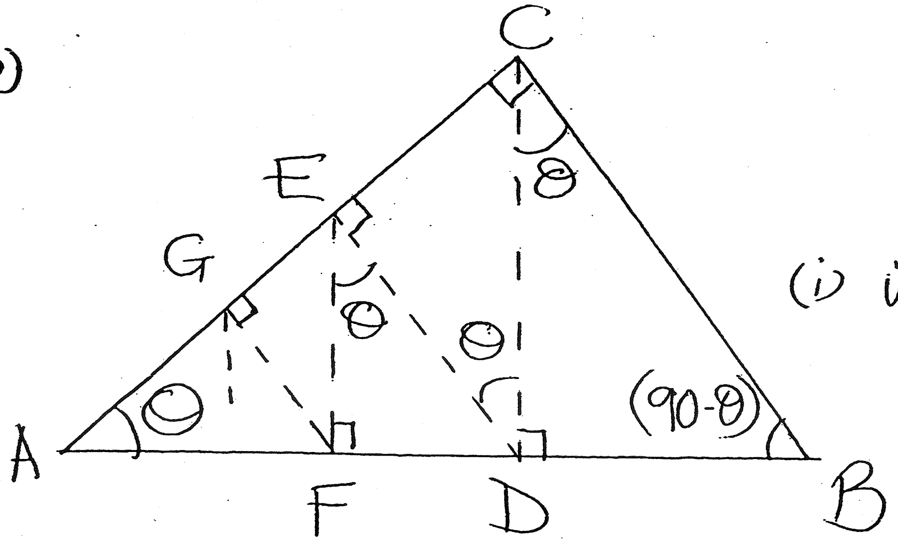
$(x-2)(2x+2) \geq 0$

$(x-2)(x+1) > 0$

$x \leq (-1)$ OR $x > 2$ NB: $(x \neq 2)$



b)



$\angle CBA = 90 - \theta$ (Angle Sum of $\triangle ABC$)

(i) in $\triangle ABC$

$\sin\theta = \frac{BC}{AB}$

$BC = x\sin\theta$

(ii) in $\triangle CBD$

$\sin(90-\theta) = \frac{CD}{BC}$

$(x\sin\theta)\cos\theta = CD$

$\angle BCD = \theta$ (Angle Sum of $\triangle BCD$)

in $\triangle DEC$ $\angle CDE = \theta$ (Alternate angles $BC \parallel DE$)

$\therefore \cos\theta = \frac{DE}{CD}$

$DE = x\sin\theta\cos^2\theta$

in $\triangle DEF$ $\angle FED = \theta$ (Alternate Angles $CD \parallel EF$)

$\therefore \cos\theta = \frac{EF}{DE}$

$\underline{EF = x\sin\theta\cos^3\theta}$ ✓

(iii) $CD + EF + GH + \ldots$

$x\sin\theta\cos\theta + x\sin\theta\cos^3\theta + x\sin\theta\cos^5\theta + \ldots$

GP $a = x\sin\theta\cos\theta$ ✓

$r = \cos^2\theta$

$S_\infty = \frac{a}{1-r} = \frac{x\sin\theta\cos\theta}{1-\cos^2\theta}$

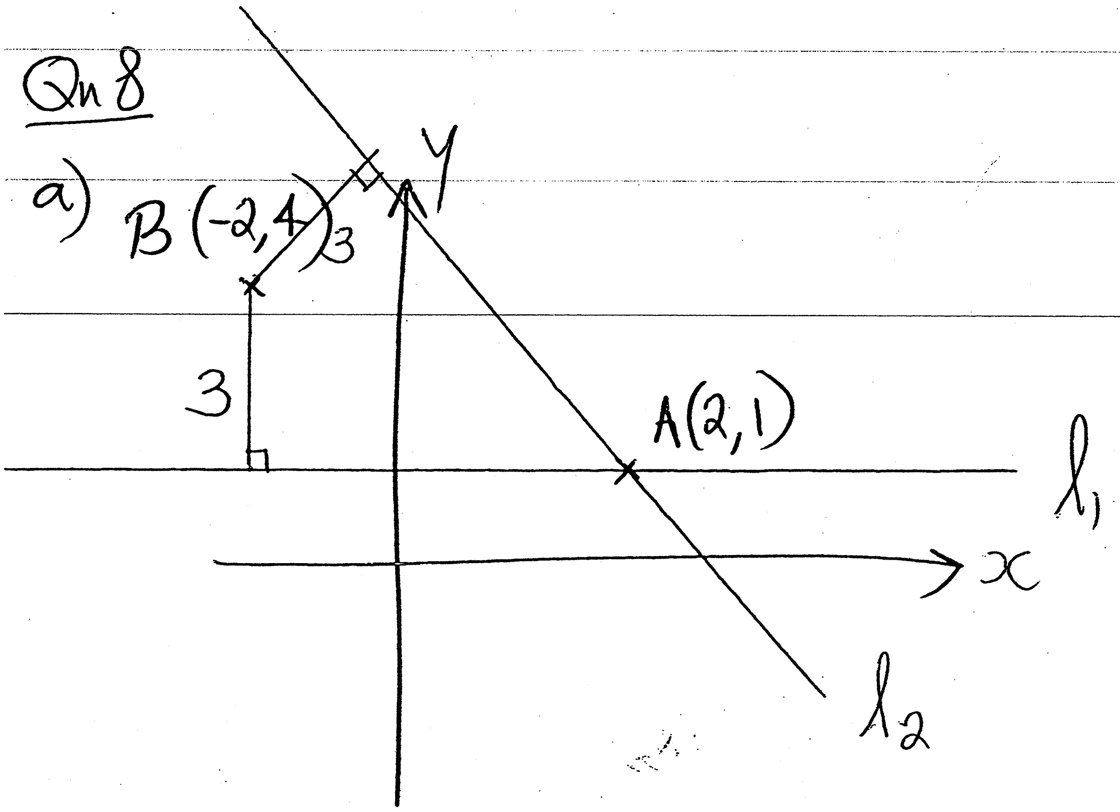
$= \frac{x\sin\theta\cos\theta}{\sin^2\theta}$ ✓

$\underline{= x\cot\theta}$

T2

Qn 8

a)



(i) $y-1=m(x-2)$

$y-1=mx-2m$

$0=mx-y+(1-2m)$ ✓ (OR Gradient-Intercept accepted)

(ii) Perp Distance from (-2,4) to $mx-y+(1-2m)$ is 3

$$\frac{|-2m-4+1-2m|}{\sqrt{m^2+(-1)^2}}=3. \checkmark$$

$$|-3-4m|=3\sqrt{m^2+1}$$

(sq) (sq)

$$(3+4m)^2=9m^2+9$$

$$9+24m+16m^2=9m^2+9$$

$$24m+7m^2=0$$

$$m(24+7m)=0 \checkmark$$

$$m=0 \text{ or } m=\left(-\frac{24}{7}\right) \checkmark$$

$l_1: y=1$ $\quad l_2: 0=-\frac{24}{7}x-y+\left(1+\frac{48}{7}\right)$

$0=-24x-7y+55$

$24x+7y-55=0$ ✓ Both

One angle bisector is line AB.

$m_{AB} = \frac{4-1}{-2-2} = \left(\frac{-3}{4}\right)$

Eqn AB $\quad y-1 = -\frac{3}{4}(x-2)$

$4y-4 = -3x+6$

$3x+4y-10=0$

Perp Line $m_2 = \frac{4}{3}$

Eqn othe angle bisector

$y-1 = \frac{4}{3}(x-2)$

$3y-3 = 4x-8$

$4x-3y-5=0$

Angle bisectors are $3x+4y-10=0$ ✓
and $4x-3y-5=0$.

) $f(x) = \frac{x+4}{2x^2-8}$

$= \frac{x+4}{2(x+2)(x-2)}$ $\qquad$ Dom: $x \neq 2$ or (-2)

(i) $f(-x) = \frac{-x+4}{2(-x)^2-8}$

$= \frac{-x+4}{2x^2-8}$ $\quad \neq f(x)$
$\quad \neq -f(x)$ $\quad \therefore$ No Symmetry ✓

(ii) ~~y-intercept~~ $x=0$ $\quad f(0) = \frac{4}{-8} = \left(-\frac{1}{2}\right)$ $\quad (0, -\frac{1}{2})$

~~x-intercept~~ $y=0$ $\quad x+4=0 \quad x=(-4) \quad (-4,0)$ ✓

Vertical Asymptotes $\quad x=2$ and $x=(-2)$

Horizontal Asymptote $\quad f(x) = \frac{\frac{1}{x}+\frac{4}{x^2}}{2-\frac{8}{x^2}}$ ✓

as $x \to \pm\infty \quad f(x) \to 0$

$\therefore y=0$

(iii) $f(x) = \frac{x+4}{2x^2-8}$ Quotient Rule $u = x+4$ $v = 2x^2-8$

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = 4x$

$f'(x) = \frac{(2x^2-8) - 4x(x+4)}{(2x^2-8)^2}$ ✓

$= \frac{-8-2x^2-16x}{4(x^2-4)^2}$

$f'(x) = \frac{-(4+8x+x^2)}{2(x^2-4)^2}$

Horizontal Gradient $f'(x) = 0$ $\therefore x^2+8x+4 = 0$

$\Delta = 8^2 - 4\times4$

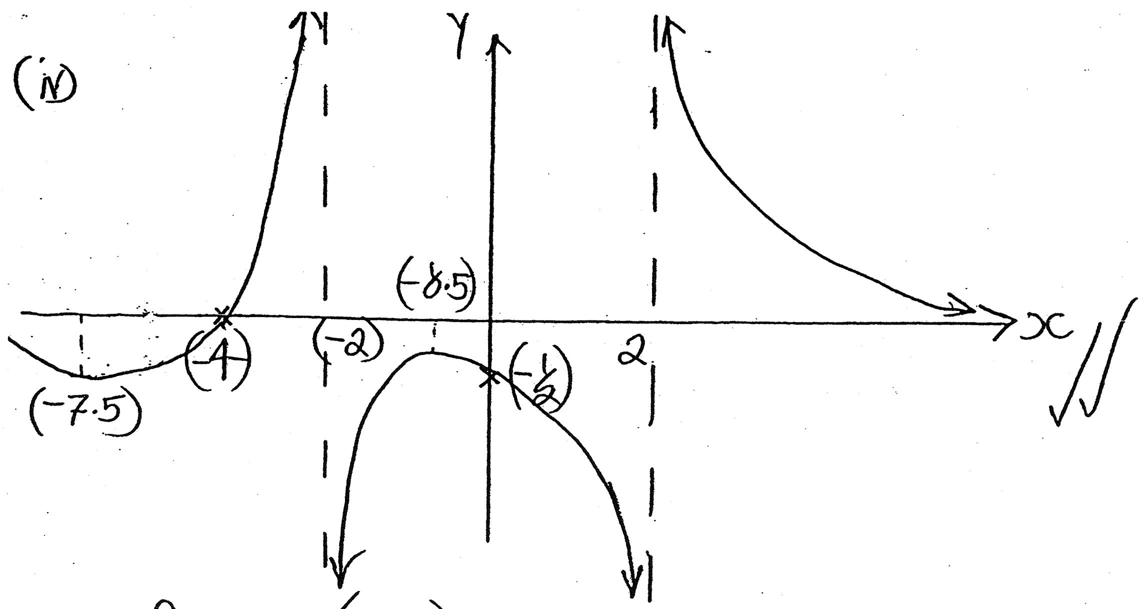
$= 64-16$

$= 48$

Zero Gradient at $x = -4+2\sqrt{3} \doteqdot -0.5$ $x = \frac{-8\pm\sqrt{48}}{2}$ ✓

and $x = -4-2\sqrt{3} \doteqdot -7.5$ $= -4\pm2\sqrt{3}$

(iv)



$f(x) = \frac{(x+4)}{2(x+2)(x-2)}$

as $x \to 2^-$ $f(x) \to \frac{6}{2\times4\times(-\varepsilon)} \to -\infty$. (Not Reqd)

as $x \to 2^+$ $f(x) \to \frac{6}{2\times4\times\varepsilon} \to \infty$

as $x \to (-2)^-$ $f(x) \to \frac{2}{2\times(-4)\times(-\varepsilon)} \to \infty$

$x \to (-2)^+$ $f(x) \to \frac{2}{2\times(-4)\times\varepsilon} \to -\infty$

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