SYDNEY GRAMMAR SCHOOL

2012 Half-Yearly Examination

FORM V
MATHEMATICS EXTENSION 1

Wednesday 16th May 2012

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.

Total — 99 Marks

- All questions may be attempted.

Section I – 11 Marks

- Questions 1–11 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 88 Marks

- Questions 12–19 are of equal value.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Collection

- Write your name, class and master clearly on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Twelve.
- Write your name and master on this question paper and submit it with your answers.

5A: DS	5B: TCW	5C: REP
5D: DNW	5E: LYL	5F: MLS
5G: SO	5H: BR	5I: SJE

Checklist

- SGS booklets — 8 per boy
- Multiple choice answer sheet
- Candidature — 134 boys

Examiner

DNW

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The value of $(3-\sqrt{2})^2$ is:

(A) 7 (B) 11 (C) $7-6\sqrt{2}$ (D) $11-6\sqrt{2}$

QUESTION TWO

The gradient of the line $4x-3y+2=0$ is:

(A) $-\dfrac{4}{3}$ (B) $-\dfrac{3}{4}$ (C) $\dfrac{4}{3}$ (D) $\dfrac{3}{4}$

QUESTION THREE

Which of the following are the correct two factors of x^3+64?

(A) $(x-4)(x^2+4x+16)$ (B) $(x+4)(x^2-4x+16)$

(C) $(x-4)(x^2+4x-16)$ (D) $(x+4)(x^2+4x+16)$

QUESTION FOUR

The domain of $y=\dfrac{1}{x^2-4}$ is:

(A) $x=2$ or $x=-2$ (B) $x\neq 2$ and $x\neq -2$

(C) $-2<x<2$ (D) $x<-2$ or $x>2$

QUESTION FIVE

The exact value of $\operatorname{cosec} 240^\circ$ is:

(A) $\dfrac{\sqrt{3}}{2}$ (B) $-\dfrac{\sqrt{3}}{2}$ (C) $\dfrac{2}{\sqrt{3}}$ (D) $-\dfrac{2}{\sqrt{3}}$

QUESTION SIX

The centre of the circle with equation $x^2-8x+y^2+2y=3$ is:

(A) $(4,-1)$ (B) $(8,-2)$ (C) $(-4,1)$ (D) $(-8,2)$

QUESTION SEVEN

The solution of $2x-x^2<0$ is:

(A) $0<x<2$ (B) $x<0$ or $x>2$

(C) $x>2$ (D) $x<0$

Exam continues overleaf ...

QUESTION EIGHT

The solution of $|x - 2| = -1$ is:

(A) $x = 1$

(B) $x = 3$

(C) $x = 3$ or 1

(D) there is no solution

QUESTION NINE

The limiting sum of the geometric series $6 - 3 + 1\frac{1}{2} - \ldots$ is:

(A) 4 (B) $4\frac{1}{2}$ (C) 5 (D) 12

QUESTION TEN

Let $A = (-1, -2)$ and $B = (5, 1)$. The point that divides AB in the ratio $2 : 1$ is:

(A) $(0, -1\frac{1}{2})$ (B) $(1, -1)$ (C) $(2, -\frac{1}{2})$ (D) $(3, 0)$

QUESTION ELEVEN

The derivative of $3x^3 + 2x$ is:

(A) $3x^2 + 2$ (B) $9x^2 + 2x$ (C) $9x^2 + 2$ (D) $3x^2$

End of Section I

Exam continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION TWELVE (11 marks) Use a separate writing booklet.

(a) Write down the exact value of:

(i) $\sec 60^\circ$

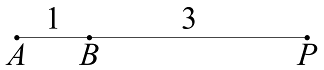
(ii) $\cot 30^\circ$

(iii) $\sin(-225^\circ)$

(b) Simplify $\sqrt{54} + \sqrt{24}$.

(c) Express $\dfrac{2}{3 - \sqrt{5}}$ in the form $a + b\sqrt{5}$, where a and b are rational.

(d)



Write down the ratio in which P divides the interval AB.

(e) Find the fifteenth term of the arithmetic sequence $5, 8, 11, \ldots$.

(f) Differentiate:

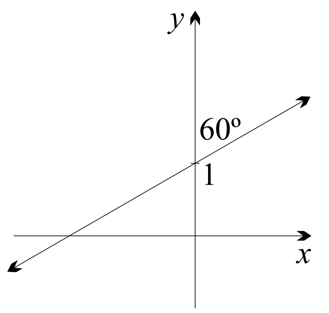
(i) $g(x) = x(x^2 + 1)$

(ii) $h(x) = \sqrt{x}$

(iii) $f(x) = \dfrac{3}{x^2}$

QUESTION THIRTEEN (11 marks) Use a separate writing booklet.

(a)



Determine the equation of the line shown in the diagram above.

Exam continues overleaf ...

QUESTION THIRTEEN (Continued)

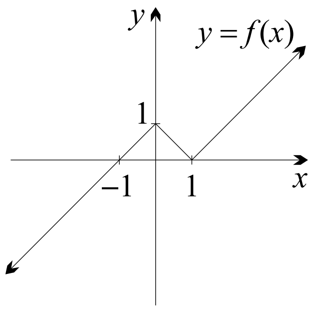
(b) Use a suitable series to express $0{\cdot}\dot{2}\dot{1}$ as a fraction in simplest form.

(c) Solve $\dfrac{3}{x-1} \le 2$.

(d) Given that $\sin\theta = -\frac{3}{7}$ and that $\tan\theta < 0$, find the exact value of $\cos\theta$.

QUESTION FOURTEEN (11 marks) Use a separate writing booklet.

(a)



The graph of $y = f(x)$ is shown above. Sketch the following on separate axes, showing all key features.

(i) $y = f(-x)$

(ii) $y = f(x+1)$

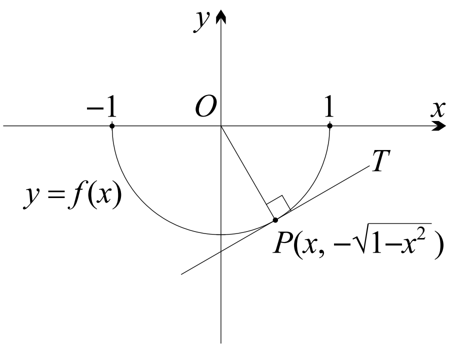
(b) Simplify $\dfrac{\tan\theta}{\sec\theta}$.

(c) Solve $\tan 2\theta = \dfrac{1}{\sqrt{3}}$, for $0^\circ \le \theta < 360^\circ$.

(d) (i) Sketch $y = (x+2)(x-1)$, showing the intercepts with the axes.

(ii) Hence sketch $y = \Big|(x+2)(x-1)\Big|$ on a separate set of axes.

(e)



The graph of $y = f(x)$ is shown above, where $f(x) = -\sqrt{1-x^2}$. Use the fact that the tangent to a circle is perpendicular to the radius at the point of contact to find the derivative of $f(x) = -\sqrt{1-x^2}$.

Exam continues next page ...

QUESTION FIFTEEN (11 marks) Use a separate writing booklet.

(a) Prove the identity:

$$\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$$

(b) In music, the frequencies $f_1, f_2, f_3 \ldots$ of the notes in a chromatic scale form a geometric progression. It is known that the frequency of A below middle C is $f_1 = 220\,\text{Hz}$. The frequency of A above middle C is $f_{13} = 440\,\text{Hz}$. Find f_9, the frequency of F above middle C. Round your answer to one decimal place.

(c) (i) Find $\dfrac{dy}{dx}$ when $y = (x^2 - 1)^4$.

(ii) Find $\dfrac{dy}{dx}$ when $y = x^2(x-3)^5$. Give your answer in factored form.

(d) Let $f(x)$ be the function:

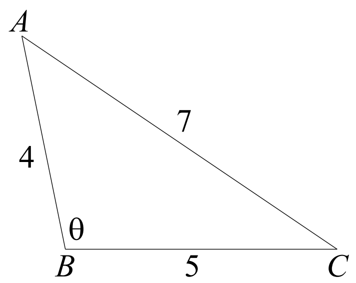
$$f(x) = \begin{cases} ax^2 & \text{when } x \le 3 \\ \sqrt{25 - x^2} & \text{when } 3 < x \le 5 \end{cases}$$

Given that $f(x)$ is continuous at $x = 3$, find the value of a.

QUESTION SIXTEEN (11 marks) Use a separate writing booklet.

(a) What is the least number of terms of the series $50 + 43 + 36 + \ldots$ that must be taken so that the sum is negative?

(b)



The diagram above shows $\triangle ABC$ where $AB = 4$, $BC = 5$, $AC = 7$ and $\angle ABC = \theta$.

(i) Find the value of $\cos\theta$.

(ii) Hence find the exact area of $\triangle ABC$.

(c) Differentiate $f(x) = \left(x + \dfrac{1}{x}\right)^2$.

(d) Determine the perpendicular distance between the lines

$$3x - y + 1 = 0 \qquad \text{and} \qquad 3x - y + 3 = 0.$$

Exam continues overleaf ...

QUESTION SEVENTEEN (11 marks) Use a separate writing booklet.

(a) How many positive integer powers of 3 are less than 10^{12} ?

(b) Use the quotient rule to show that the derivative of $y = \dfrac{x}{x^2+1}$ is $\dfrac{dy}{dx} = \dfrac{1-x^2}{(x^2+1)^2}$.

(c) (i) Determine the coordinates of the points where the line $y = x + 2$ intersects the parabola $y = x^2$.

(ii) Sketch the intersection of the regions $y < x+2$ and $y \geq x^2$. Pay careful attention to any corners.

(d) The lines ℓ_1 and ℓ_2 intersect at P. The equations of the lines are:

$$\begin{aligned} \ell_1 &: \quad x + 2y + 3 = 0 \\ \ell_2 &: \quad 3x + y - 2 = 0 \end{aligned}$$

Determine the equation of the line PQ where $Q = (-1, 2)$, without finding the coordinates of P.

QUESTION EIGHTEEN (11 marks) Use a separate writing booklet.

(a) Let $y = f(x)$ where $f(x) = \dfrac{x}{x^2-1}$.

(i) (α) Show that $f(x)$ is an odd function.

(β) Solve $f(x) = 1$.

(γ) Hence, or otherwise, solve $f(x) = -1$.

(ii) Find any x-intercepts of $y = f(x)$.

(iii) Find the equations of any asymptotes.

(iv) Sketch the graph of $y = f(x)$ for $-4 \leq x \leq 4$, showing the features you have found.

(b) Let $h(x) = \dfrac{1}{2}\left(1 + \dfrac{x}{|x|}\right)$, where $x \neq 0$.

Sketch $y = h(x)$ by considering separately $x < 0$ and $x > 0$.

Exam continues next page ...

QUESTION NINETEEN (11 marks) Use a separate writing booklet.

(a) Let $f(x) = x^3 - 2x^2 + 1$. The function $g(x)$ is the result of shifting $f(x)$ to the right by k units. Given that the coefficient of x^2 in $g(x)$ is zero, find the value of k.

(b) Consider the parabola with equation $y = -x^2 + 4x - 3$.

(i) Find the equation of the tangent to this parabola at the point where $x = p$.

(ii) Given that the tangent in part (i) passes through the origin, determine p.

(c) Consider the sequence defined by:

$$x_1 = 3$$
$$x_n = \frac{(x_{n-1})^2 + 6}{2x_{n-1}} \quad \text{for } n \geq 2\,.$$

In the following you may assume that every term of this sequence is positive.

(i) Write down the first three terms, simplifying any fractions.

(ii) Show that $x_n - \sqrt{6} \geq 0$ for $n \geq 2$.

(iii) Let $\epsilon_n = x_n - \sqrt{6}$.

(α) Write down the exact value of ϵ_1.

(β) Show that $\dfrac{\epsilon_n}{\epsilon_{n-1}} < \dfrac{1}{2}$ for $n \geq 2$.

(γ) Hence determine $\lim\limits_{n \to \infty} x_n$.

End of Section II

END OF EXAMINATION

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Wednesday 16th May 2012

- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

NAME: ....

CLASS: ....MASTER: ....

Question One

A ○ B ○ C ○ D ○

Question Two

A ○ B ○ C ○ D ○

Question Three

A ○ B ○ C ○ D ○

Question Four

A ○ B ○ C ○ D ○

Question Five

A ○ B ○ C ○ D ○

Question Six

A ○ B ○ C ○ D ○

Question Seven

A ○ B ○ C ○ D ○

Question Eight

A ○ B ○ C ○ D ○

Question Nine

A ○ B ○ C ○ D ○

Question Ten

A ○ B ○ C ○ D ○

Question Eleven

A ○ B ○ C ○ D ○

Q 1 (D) **Q 2** (C) **Q 3** (B) **Q 4** (B)
Q 5 (D) **Q 6** (A) **Q 7** (B) **Q 8** (D)
Q 9 (A) **Q 10** (D) **Q 11** (C)

QUESTION TWELVE (11 marks)

(a) (i) $\sec 60^\circ = 2$ ✓

(ii) $\cot 30^\circ = \sqrt{3}$ ✓

(iii) $\sin(-225^\circ) = \dfrac{1}{\sqrt{2}}$ ✓

(b)
$$\begin{aligned}\sqrt{54} + \sqrt{24} &= 3\sqrt{6} + 2\sqrt{6} \\ &= 5\sqrt{6}\end{aligned}$$ ✓

(c)
$$\frac{2}{3-\sqrt{5}} = \frac{2}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$$ ✓
$$= \frac{3+\sqrt{5}}{2} \quad \text{or} \quad \tfrac{3}{2} + \tfrac{1}{2}\sqrt{5}$$ ✓

(d) $-4:3$ or $4:-3$ or externally $4:3$ ✓

(e)
$$\begin{aligned}T_{15} &= 5 + 14 \times 3 \\ &= 47\end{aligned}$$ ✓

(f) (i) $g'(x) = 3x^2 + 1$ ✓

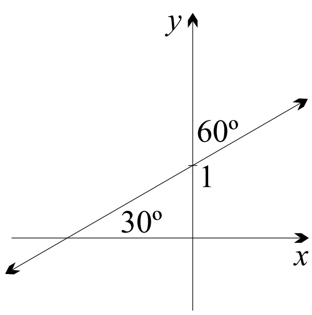
(ii) $h'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}$ or $\dfrac{1}{2\sqrt{x}}$ ✓

(iii) $f'(x) = -6x^{-3}$ or $-\dfrac{6}{x^3}$ ✓

Total for Question 12: 11 Marks

QUESTION THIRTEEN (11 marks)

(a)
$$\begin{aligned}\text{gradient} &= \tan 30^\circ \\ &= \tfrac{1}{\sqrt{3}}\end{aligned}$$ ✓
so $y = \tfrac{1}{\sqrt{3}}x + 1$ ✓



(b) $$0{\cdot}\dot{2}\dot{1} = 0{\cdot}21 + 0{\cdot}0021 + 0{\cdot}000021 + \dots$$ √

$$= \frac{0{\cdot}21}{1 - \frac{1}{100}}$$ √

$$= \frac{21}{99}$$

$$= \frac{7}{33}$$ √

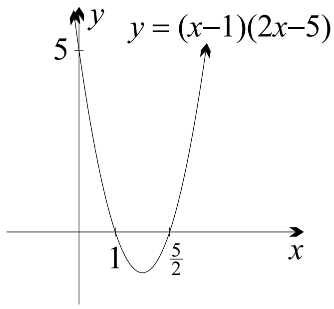
(c) $$\frac{3}{x-1} \le 2 \quad ; \quad x \ne 1$$

so $3(x-1) \le 2(x-1)^2$ √

or $0 \le 2(x-1)^2 - 3(x-1)$

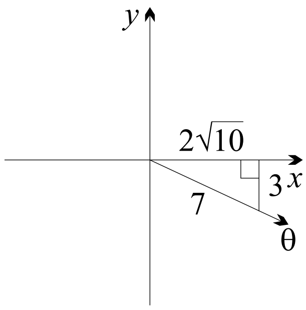
thus $0 \le (x-1)(2x-5) \quad ; \quad x \ne 1$ √

hence $x < 1$ or $x \ge \frac{5}{2}$ √



(d) Both $\sin\theta$ and $\tan\theta$ are negative so θ is in the 4th quadrant, and $\cos\theta > 0$. √

√

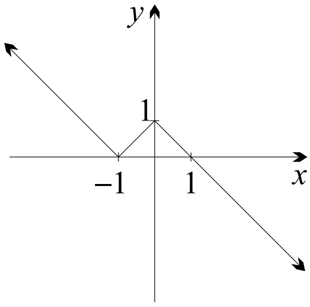


so $$\cos\theta = \frac{2\sqrt{10}}{7}$$ √

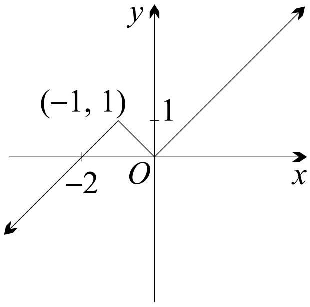
Total for Question 13: 11 Marks

QUESTION FOURTEEN (11 marks)

(a) (i) √



(ii)



$\boxed{\surd}$

(b) $$\frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cos\theta \times \dfrac{1}{\cos\theta}}$$
$$= \sin\theta$$ $\boxed{\surd}$

(c) $$\tan 2\theta = \frac{1}{\sqrt{3}} \quad 0^\circ \le \theta < 360^\circ$$

Let $u = 2\theta$ then

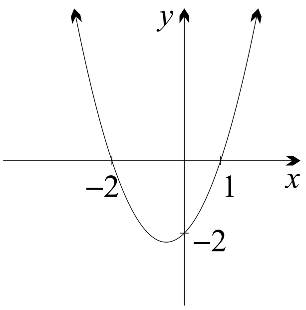
$$\tan u = \frac{1}{\sqrt{3}} \quad 0^\circ \le \theta < 720^\circ$$ $\boxed{\surd}$

The related angle is 30° so

$$u = 30^\circ, 210^\circ, 390^\circ, 570^\circ$$ $\boxed{\surd}$

thus $$\theta = 15^\circ, 105^\circ, 195^\circ, 285^\circ$$ $\boxed{\surd}$

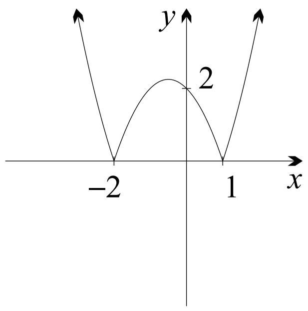
(d) (i)



x-int $\boxed{\surd}$

y-int $\boxed{\surd}$

(ii)



$\boxed{\surd}$

(e) $$\text{grad } OP = -\frac{\sqrt{1-x^2}}{x}$$ $\boxed{\surd}$

so $$f'(x) = \frac{x}{\sqrt{1-x^2}}$$ $\boxed{\surd}$

Total for Question 14: 11 Marks

QUESTION FIFTEEN (11 marks)

(a)
$$\begin{aligned}\text{LHS} &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} &&\checkmark\\ &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\\ &= \frac{1}{\cos\theta\sin\theta} \qquad \text{(by Pythagoras)} &&\checkmark\\ &= \text{RHS}\end{aligned}$$

[Do not penalise the lack of a reason above.]

(b)
$$a = 220$$
and $ar^{12} = 440$ ✓

so $r = \sqrt[12]{2}$ (positive only since all frequencies are positive) ✓

[Do not penalise a negative answer so long as the positive is also given.]

$$f_9 = 220 \times 2^{\frac{2}{3}}$$
$$\doteqdot 349{\cdot}2 \quad \text{(correct to 1 decimal place)}$$ ✓

(c) (i)
$$y = (x^2 - 1)^4$$
Let $u = x^2 - 1$

so $u' = 2x$

and $y = u^4$. ✓

Thus
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 4u^3 \times u'\\ &= 8x(x^2-1)^3 &&\checkmark\end{aligned}$$

[Accept an answer with no working only if it is completely correct.]

(ii)
$$\begin{aligned}y &= x^2(x-3)^5\\ &= uv\end{aligned}$$
where $u = x^2$ so $u' = 2x$

and $v = (x-3)^5$ so $v' = 5(x-3)^4$. ✓

$$\begin{aligned}y' &= uv' + u'v\\ &= x^2 5(x-3)^4 + 2x(x-3)^5 &&\checkmark\\ &= x(x-3)^4(7x-6) &&\checkmark\end{aligned}$$

(d) For continuity at $x = 3$ we require $\lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x)$.

Now $\quad f(3) = 9a$

and $\quad \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{25 - x^2}$

$\quad = 4$

Thus $\quad 9a = 4$

or $\quad a = \frac{4}{9}$ ✓

Total for Question 15: 11 Marks

QUESTION SIXTEEN (11 marks)

(a) For the arithmetic sequence $a = 50$ and $d = -7$, so:

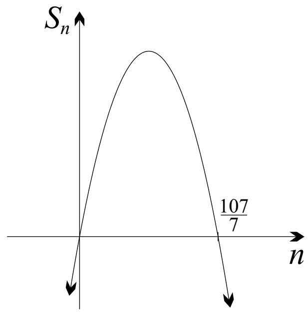
$S_n = \frac{1}{2}n(107 - 7n)$ ✓

Thus we need to solve

$\frac{1}{2}n(107 - 7n) < 0$ ✓

hence $\quad n > \frac{107}{7} \quad$ or $\quad 15\frac{2}{7}$

Thus $\quad n = 16$ ✓



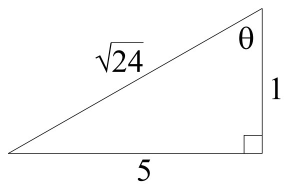
(b) (i) Now $\quad 7^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \cos\theta \quad$ (by the cosine rule)

so $\quad \cos\theta = -\frac{1}{5}$ ✓

(ii) $\quad$ Area $= \frac{1}{2} \times 4 \times 5 \times \sin\theta$ ✓

$= \frac{1}{2} \times 4 \times 5 \times \frac{\sqrt{24}}{5}$

$= 4\sqrt{6}$ square units ✓



(c) Expanding $\quad f(x) = x^2 + 2 + x^{-2}$ ✓

so $\quad f'(x) = 2x - 2x^{-3} \quad$ or $\quad 2x - \dfrac{2}{x^3}$ ✓

(d) The point $(0, 3)$ is on the second line (y-intercept), ✓
so the distance d to the first line is:

$$d = \frac{\left|3 \times 0 - 1 \times 3 + 1\right|}{\sqrt{3^2 + (-1)^2}}$$ ✓

$$= \frac{\sqrt{10}}{5}$$ ✓

Total for Question 16: 11 Marks

QUESTION SEVENTEEN (11 marks)

(a) That is, find the largest n such that

$$3^n < 10^{12}$$ ✓

Taking logs of both sides

$$n \log_{10} 3 < 12$$

or $$n < \frac{12}{\log_{10} 3}$$

$$< 25{\cdot}15\ldots$$

thus $n = 25$ ✓

(b) $$y = \frac{x}{x^2+1}$$

$$= \frac{u}{v}$$

where $u = x$ so $u' = 1$

and $v = x^2 + 1$ so $v' = 2x$ ✓

Then $$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$= \frac{1 \times (x^2+1) - x \times 2x}{(x^2+1)^2}$$ ✓

$$= \frac{1-x^2}{(x^2+1)^2}$$

(c) (i) That is, solve

$$x^2 = x + 2$$

so $x^2 - x - 2 = 0$

or $(x-2)(x+1) = 0$

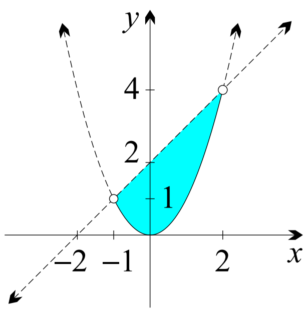
thus $x = -1$ or 2 ✓

and $y = 1$ or 4

That is: $(-1, 1)$ and $(2, 4)$.

[Do not penalise failure to find y-values.]

(ii)



The regions is below the line,
above the parabola,
and excludes the corners and line.

✓✓✓

(d) The general equation of the line through P is

$$(x + 2y + 3) + k(3x + y - 2) = 0$$ ☑

So at $Q(-1, 2)$ this gives

$$6 - 3k = 0$$

so $$k = 2$$ ☑

Thus the equation of PQ is

$$(x + 2y + 3) + 2(3x + y - 2) = 0$$

or $$7x + 4y - 1 = 0$$ ☑

Total for Question 17: 11 Marks

QUESTION EIGHTEEN (11 marks)

(a) For $y = f(x)$ where $f(x) = \dfrac{x}{x^2 - 1}$:

(i) (α)

$$\begin{aligned} f(-x) &= \frac{(-x)}{(-x)^2 - 1} \\ &= -\frac{x}{x^2 - 1} \\ &= -f(x) \end{aligned}$$ ☑

(β) That is $$\frac{x}{x^2 - 1} = 1$$

so $$x^2 - x - 1 = 0$$

or $$x = \frac{1 \pm \sqrt{5}}{2}$$ ☑

(γ) Since $f(x)$ is odd, the values are $x = -\dfrac{1 \pm \sqrt{5}}{2}$. ☑

(ii) $f(0) = 0$. There are no other intercepts. ☑

(iii)

$$\begin{aligned} \lim_{x \to \infty} f(x) &= \lim_{x \to \infty} \frac{\frac{1}{x}}{1 - \frac{1}{x^2}} \\ &= 0 \end{aligned}$$

hence also, by the oddness of $f(x)$,

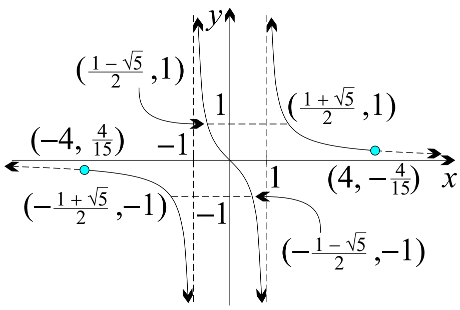
$$\lim_{x \to -\infty} f(x) = 0\,.$$

So the x-axis is an asymptote (or $y = 0$) ☑

There are vertical asymptotes at $x = -1$ or 1 ☑

where the denominator is zero and the numerator is not zero.

(iv)



odd function ✓

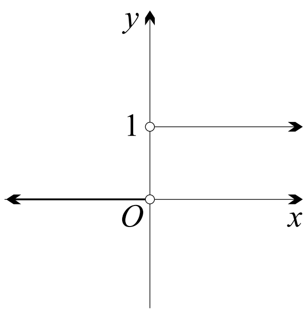
shape ✓

[Do not penalise failure to show coordinates where $y = \pm 1$.]

(b) $h(x) = \frac{1}{2}\left(1 + \frac{x}{|x|}\right)$

$$= \begin{cases} \frac{1}{2}\left(1 + \frac{x}{-x}\right) & \text{when } x < 0 \\ \frac{1}{2}\left(1 + \frac{x}{x}\right) & \text{when } x > 0 \end{cases}$$ ✓

$$= \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x > 0 \end{cases}$$ ✓



✓

Total for Question 18: 11 Marks

QUESTION NINETEEN (11 marks)

(a)
$$\begin{aligned} g(x) &= f(x-k) \\ &= (x-k)^3 - 2(x-k)^2 + 1 \\ &= x^3 - (3k+2)x^2 + (3k^2+4k)x + (1-2k^2-k^3) \end{aligned}$$ ✓

so $3k + 2 = 0$

thus $k = -\frac{2}{3}$ ✓

(b) For $y = -x^2 + 4x - 3$:

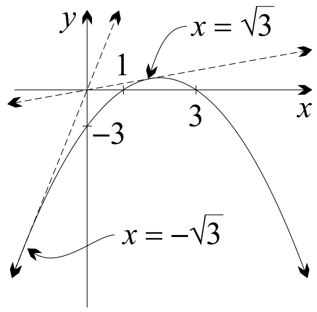
(i) $y' = -2x + 4$

so at $x = p$ $\quad y' = 4 - 2p$ ✓

and $y = -p^2 + 4p - 3$

Thus the equation of the tangent is:

$$y + p^2 - 4p + 3 = (4 - 2p)(x - p)$$ ✓



(ii) The point $(0, 0)$ is on the tangent so:

$$p^2 - 4p + 3 = 2p^2 - 4p$$ ✓

so $p^2 = 3$ ✓

or $p = -\sqrt{3}$ or $\sqrt{3}$ ✓

(c) (i)

$$x_1 = 3$$

$$x_2 = \frac{9+6}{6} = \frac{5}{2}$$

$$x_3 = \frac{\frac{25}{4}+6}{5} = \frac{49}{20}$$ ☑

(ii)

$$\begin{aligned} \text{LHS} &= x_n - \sqrt{6} \\ &= \frac{(x_{n-1})^2 + 6}{2x_{n-1}} - \sqrt{6} \qquad \text{for } n \geq 2 \\ &= \frac{(x_{n-1})^2 - 2x_{n-1}\sqrt{6} + 6}{2x_{n-1}} \\ &= \frac{(x_{n-1} - \sqrt{6})^2}{2x_{n-1}} \end{aligned}$$ ☑

Now the numerator is positive or zero and the denominator is positive hence LHS ≥ 0 .

(iii) (α) $\epsilon_1 = 3 - \sqrt{6}$ ☑

(β)

$$\begin{aligned} \frac{\epsilon_n}{\epsilon_{n-1}} &= \frac{x_n - \sqrt{6}}{x_{n-1} - \sqrt{6}} \\ &= \frac{(x_{n-1} - \sqrt{6})^2}{2x_{n-1}} \times \frac{1}{(x_{n-1} - \sqrt{6})} \\ &= \frac{x_{n-1} - \sqrt{6}}{2x_{n-1}} \\ &< \frac{x_{n-1}}{2x_{n-1}} \qquad \text{(increased numerator)} \\ &< \frac{1}{2}\,. \end{aligned}$$ ☑

(γ) Since $\dfrac{\epsilon_n}{\epsilon_{n-1}} < \dfrac{1}{2}$ and $\epsilon_n \geq 0$,

every term of the sequence ϵ_n is no larger than the corresponding term of the geometric progression

$$(3 - \sqrt{6}), \tfrac{1}{2}(3 - \sqrt{6}), \tfrac{1}{4}(3 - \sqrt{6}), \ldots .$$

That is $\quad 0 \leq \epsilon_n \leq (3 - \sqrt{6})(\tfrac{1}{2})^n$

so $\quad 0 \leq \lim_{n\to\infty} \epsilon_n \leq \lim_{n\to\infty} (3 - \sqrt{6})(\tfrac{1}{2})^n$

and $\quad 0 \leq \lim_{n\to\infty} \epsilon_n \leq 0$.

Thus $\quad \lim_{n\to\infty} \epsilon_n = 0$

so $\quad \lim_{n\to\infty} x_n - \sqrt{6} = 0$ ☑

hence $\quad \lim_{n\to\infty} x_n = \sqrt{6}$.

Total for Question 19: 11 Marks

DNW