SYDNEY GRAMMAR SCHOOL

2014 Half-Yearly Examination

FORM V

MATHEMATICS 3 UNIT

Monday 19th May 2014

General Instructions

- Writing time — 1 hour 30 minutes
- Write using black or blue pen.
- Board-approved calculators and templates may be used.

Total — 80 Marks

- All questions may be attempted.

Section I – 8 Marks

- Questions 1–8 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 72 Marks

- Questions 9–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your name, class and master on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Nine.
- Write your name and master on this question paper and submit it with your answers.

5A: BDD	5B: MLS	5C: LYL	5D: LRP
5E: PKH	5F: BR	5G: SG	

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 125 boys

Examiner

BDD

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Given $f(x) = \dfrac{1}{3x}$, what is the value of $f'(2)$?

(A) $-\dfrac{1}{12}$

(B) $-\dfrac{1}{6}$

(C) $\dfrac{1}{3}$

(D) $-\dfrac{3}{4}$

QUESTION TWO

Which of the following statements is NOT correct?

(A) $1 + \tan^2\theta = \sec^2\theta$

(B) $\cos(90^\circ - \theta) = \sin\theta$

(C) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(D) $\sec\theta = \dfrac{1}{\sin\theta}$

QUESTION THREE

The recurring decimal 0·002020202... may be regarded as the limiting sum of a GP with common ratio:

(A) 0·001

(B) 0·002

(C) 0·01

(D) 0·02

QUESTION FOUR

Which of the following is a correct limit statement about $y = \dfrac{1}{x-2}$?

(A) $y \to \infty$ as $x \to 2^-$.

(B) $y \to \infty$ as $x \to -2$.

(C) $y \to -\infty$ as $x \to 2^-$.

(D) $y \to -\infty$ as $x \to -2$.

QUESTION FIVE

The expression $\dfrac{\log_3 x^2}{\log_3 x}$ simplifies to:

(A) x

(B) $\log_3 x$

(C) $\log_3(x^2 - x)$

(D) 2

Exam continues next page ...

QUESTION SIX

A line passing through the point of intersection of $2x + 3y - 1 = 0$ and $4x - 9y + 3 = 0$ may be written in the form $2x + 3y - 1 + k(4x - 9y + 3) = 0$ for some real number k.

For what value of k does this line pass through the point $(2, 1)$?

(A) $k = 2$ (B) $k = -2$

(C) $k = -3$ (D) $k = 3$

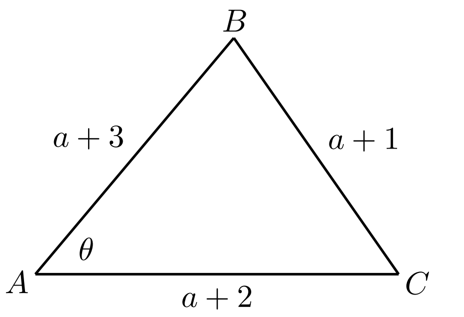
QUESTION SEVEN

The common ratio of the geometric sequence $\sqrt{2} - 1$, $6\sqrt{2} - 3$, $45\sqrt{2} + 9$, ... is:

(A) $3 + \sqrt{2}$ (B) $9 + 3\sqrt{2}$

(C) $5\sqrt{2} - 2$ (D) $\dfrac{3 - \sqrt{2}}{9}$

QUESTION EIGHT



The value of $\cos\theta$ in the triangle above, for $a > -1$, is:

(A) $\dfrac{a + 2}{a + 3}$ (B) $\dfrac{a + 6}{2a + 6}$

(C) $\dfrac{a^2 + 12a + 14}{2(a + 3)(a + 2)}$ (D) $\dfrac{a^2 + 12}{2a^2 + 10a + 12}$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) Simplify $\dfrac{x^2 - 3x - 10}{x + 2}$. **1**

(b) Write down the natural domain of the function $f(x) = \sqrt{x^2 - 4}$. **1**

(c) Differentiate:

(i) $y = 3x^2 + 2x + 4$ **1**

(ii) $y = 6\sqrt{x}$ **1**

(d) Find the sum of the first twelve terms of the sequence 3, 7, 11, ... **2**

(e) Given the geometric sequence $1536, 768, 384, \ldots$

(i) find the common ratio, **1**

(ii) find the tenth term. **1**

(f) Determine whether the function $f(x) = 2^x - 2^{-x}$ is even, odd or neither. **1**

(g) Consider the function $f(x) = 5x^2 + 4x$.

(i) Simplify the expression $f(x + h) - f(x)$. **2**

(ii) Hence use the formula **1**

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

to differentiate $f(x) = 5x^2 + 4x$ from first principles.

Exam continues next page ...

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\sum_{k=2}^{4} k^2$. **1**

(b) Let A and B be points with coordinates $A(-9, 7)$ and $B(6, 3)$. Find the coordinates of the point $P(x, y)$ that divides AB in the ratio $4 : 1$. **2**

(c) Suppose T divides FG externally in the ratio $3 : 5$. In what ratio does F divide TG? **1**

(d) Solve:

(i) $2\sin^2\theta - \sin\theta - 1 = 0$, for $0^\circ \le \theta \le 360^\circ$. **3**

(ii) $\cos 2\theta = \dfrac{\sqrt{3}}{2}$, for $0^\circ \le \theta \le 360^\circ$. **3**

(e) Consider the limiting sum $45 + 15 + 5 + \cdots$.

(i) Give a reason why the limiting sum is known to exist. **1**

(ii) Find the limiting sum. **1**

QUESTION ELEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Find the equation of the tangent to $y = x^2 - 3x + 2$ at $x = 4$. **3**

(b) Differentiate:

(i) $y = (2x - 1)(x + 3)$ **2**

(ii) $y = (7 - 4x)^9$ **2**

(c) Consider the graphs of $y = x^2$ and $y = \frac{1}{2}x + 3$.

(i) Determine the two points of intersection of the graphs. **2**

(ii) Shade the region where $y > x^2$ and $y \le \frac{1}{2}x + 3$. **3**

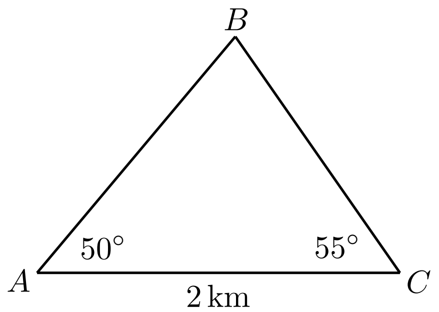
Be careful to mark the corners and boundaries of the region correctly.

QUESTION TWELVE (12 marks) Use a separate writing booklet. **Marks**

(a) Solve the inequation $\dfrac{2x}{x-3} > 4$. **3**

(b) A geometric sequence has fifth term 18 and ninth term 1458. Find the common ratio and the first term. **3**

(c)



From two points A and C on a straight horizontal road a balloon at B directly above the road is observed to have angles of elevation of 50° and 55° respectively.

(i) Show that $BC \doteqdot 1{\cdot}59\,\text{km}$. **2**

(ii) Find the height of the balloon above the road. Give your answer correct to 3 significant figures. **1**

(d) The line ℓ has equation $12x - 5y + c = 0$ for some constant c.

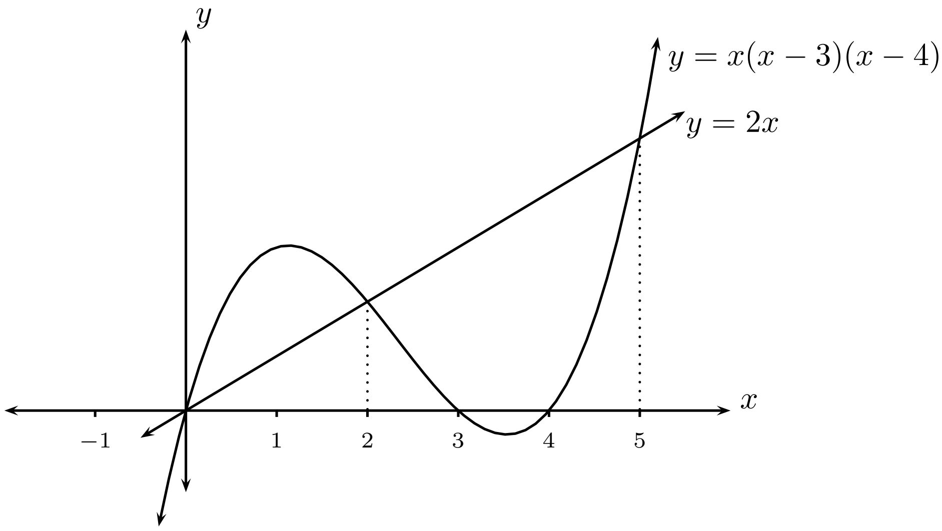
(i) Write down a simplified expression for the perpendicular distance from ℓ to a point $P(x_0, y_0)$. **1**

(ii) The line is known to be a tangent to the circle $(x-2)^2 + (y-3)^2 = 1$. Find the possible values of c. **2**

Exam continues next page . . .

QUESTION THIRTEEN (12 marks) Use a separate writing booklet. **Marks**

(a)



Use the graph above to solve the inequation $x(x-3)(x-4) > 2x$. **1**

(b) Prove that $\dfrac{1}{1-\cos\theta} + \dfrac{1}{1+\cos\theta} = 2\,\mathrm{cosec}^2\,\theta$. **2**

(c) Use mathematical induction to prove that for all positive integers n, **3**

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}\,.$$

(d) Consider the graph of $y = \dfrac{(x-4)^2}{4x(x-2)}$.

(i) Find any intercepts with the x and y axes. **1**

(ii) Write down the equation(s) of any vertical asymptote(s). **1**

(iii) Find the equation of the horizontal asymptote. **1**

(iv) Copy and fill in the following table: **1**

x	-1	1	3
y			

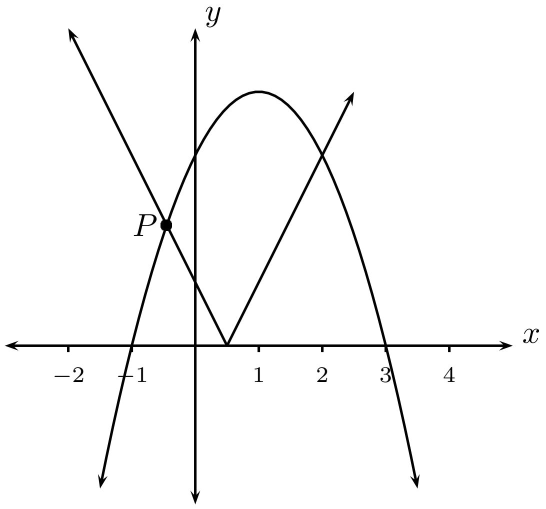
(v) Use parts (i)-(iv) to sketch the curve. **2**

Exam continues overleaf ...

QUESTION FOURTEEN (12 marks) Use a separate writing booklet. **Marks**

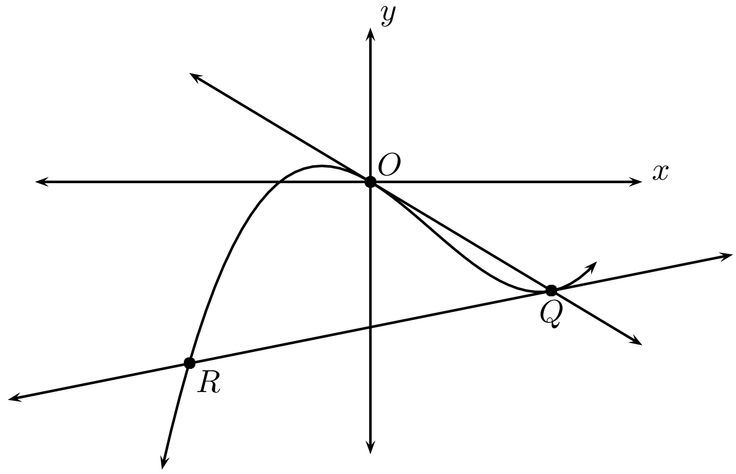
(a) For what values of x is the tangent to $y = \left(\frac{8}{x} - x^2\right)^5$ horizontal? **2**

(b)



The diagram above shows the graphs of $y = 3x + 3 - x^2$ and $y = |2x - 1|$. **3**
Find the exact value of the x-coordinate of the point P where the graphs intersect in the second quadrant.

(c) Consider a cubic $y = ax^3 + bx^2 + cx$, where $a, b \neq 0$, passing through the origin O. One such cubic is shown in the diagram below.



(i) Find the equation of the tangent at O for $y = ax^3 + bx^2 + cx$. **1**

(ii) Find the coordinates of the point Q where the tangent in part (i) meets the cubic again. **2**

(iii) Show that the y-intercept of the tangent at Q is independent of c. **2**

(iv) Suppose the tangent at Q meets the cubic again at R. Show that the x-coordinates of R, O and Q form an arithmetic sequence. **2**

End of Section II

END OF EXAMINATION

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

NAME: ....

CLASS: ....MASTER: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

SECTION I - Multiple Choice

QUESTION ONE

$$\begin{aligned} f(x) &= \frac{1}{3x} \\ &= \frac{1}{3}x^{-1} \\ \text{Hence } f'(x) &= \frac{1}{3} \times (-1)x^{-2} \\ &= -\frac{1}{3x^2} \\ f'(2) &= -\frac{1}{3 \times (2)^2} \\ &= -\frac{1}{12} \end{aligned}$$

The correct answer is $\boxed{\text{A}}$

QUESTION TWO

$\sec\theta = \dfrac{1}{\cos\theta}$, hence the incorrect statement is $\boxed{\text{D}}$.

QUESTION THREE

$$\begin{aligned} 0{\cdot}002020202\ldots &= 0{\cdot}002 + 0{\cdot}00002 + 0{\cdot}0000002 + \ldots + 0{\cdot}000000002 + \cdots \\ &= 0{\cdot}002 + 0{\cdot}002 \div 100 + 0{\cdot}002 \div 10000 + 0{\cdot}002 \div 1000000 + \cdots \end{aligned}$$

Each sucessive term is divided by 100, hence the common ratio is $\dfrac{1}{100} = 0{\cdot}01$.

The correct answer is $\boxed{\text{C}}$.

QUESTION FOUR

The function has an asymptote at $x = 2$, so we want a limit statement for $x \to 2$, that is A or C.
If x approaches 2 from below, it is less than 2, so $x - 2$ is negative.
Hence the correct answer is $\boxed{\text{C}}$.

QUESTION FIVE

$$\begin{aligned} \frac{\log_3 x^2}{\log_3 x} &= \frac{2 \times \log_3 x}{\log_3 x} \\ &= 2 \end{aligned}$$

Hence the correct answer is $\boxed{\text{D}}$.

QUESTION SIX

If we substitute the point $(2,1)$ into the expression $2x+3y-1+k(4x-9y+3)=0$, then we get:

$$\begin{aligned} 2(2)+3(1)-1+k(4(2)-9(1)+3) &= 0 \\ 6+2k &= 0 \\ k &= -3 \end{aligned}$$

Hence the correct answer is $\boxed{\text{C}}$.

QUESTION SEVEN

$$\begin{aligned} r &= \frac{6\sqrt{2}-3}{\sqrt{2}-1} \\ &= \frac{(6\sqrt{2}-3)(\sqrt{2}+1)}{\sqrt{2}^2-1^2} \\ &= \frac{12+6\sqrt{2}-3\sqrt{2}-3}{1} \\ &= 9+3\sqrt{2} \end{aligned}$$

Hence $\boxed{\text{B}}$.

QUESTION EIGHT

By the cosine rule,

$$\begin{aligned} \cos\theta &= \frac{AB^2+AC^2-BC^2}{2\times AB\times AC} \\ &= \frac{(a+3)^2+(a+2)^2-(a+1)^2}{2(a+3)(a+2)} \\ &= \frac{(a+2)^2+(2a+4)(2)}{2(a+3)(a+2)} \qquad \text{(difference of squares)} \\ &= \frac{(a+2)\left((a+2)+4\right)}{2(a+3)(a+2)} \\ &= \frac{(a+6)}{2(a+3)} \end{aligned}$$

Hence the correct answer is $\boxed{\text{B}}$.

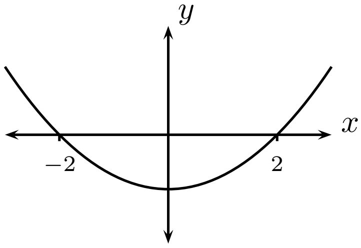
(The pupil could expand the numerator as an alternative to using difference of squares.)

SECTION II - Written Response

QUESTION NINE

(a) $$\frac{x^2-3x-10}{x+2} = \frac{(x-5)(x+2)}{(x+2)}$$
$$= x-5$$

(b) The function is defined where $x^2 - 4 \geq 0$.



Thus the domain is $x \leq -2$ or $x \geq 2$.

(c) (i) $y' = 6x + 2$

(ii) $y = 6\sqrt{x}$
$= 6x^{\frac{1}{2}}$
$y' = 6 \times (\frac{1}{2})x^{-\frac{1}{2}}$
$y' = \dfrac{3}{\sqrt{x}}$

(d) This is an arithmetic progression. The first term is $a = 3$ and the common difference is 4.
$S_n = \frac{n}{2}(2a + (n-1)d)$
$S_{12} = \frac{12}{2}\left(2(3) + (12-1)(4)\right)$
$S_{12} = 6 \times 50$
$S_{12} = 300$

(e) (i) $r = 768 \div 1536$
$= \frac{1}{2}$

(ii) $T_n = ar^{n-1}$
$T_{10} = 1536 \times (\frac{1}{2})^9$
$= 1536 \div 512$
$= 3$

(f) $f(x) = 2^x - 2^{-x}$
$f(-x) = 2^{(-x)} - 2^{-(-x)}$
$f(-x) = 2^{-x} - 2^x$
$f(-x) = -\left(2^x - 2^{-x}\right)$
$f(-x) = -f(x)$
hence the function has odd symmetry.

(g) (i) $f(x+h)-f(x) = 5(x+h)^2+4(x+h)-(5x^2+4x)$

$$= 5(x^2+2xh+h^2)+4x+4h-5x^2-4x$$

$$= 10xh+5h^2+4h$$

(ii) $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{10xh+5h^2+4h}{h}$$

$$= \lim_{h\to 0}\frac{h(10x+4+5h)}{h}$$

$$= \lim_{h\to 0}(10x+4+5h)$$

$$= 10x+4$$

QUESTION TEN

(a) $$\sum_{k=2}^{4} k^2 = 2^2+3^2+4^2$$

$$= 4+9+16$$

$$= 29$$

(b) $$x = \frac{kx_2+\ell x_1}{k+\ell}$$

$$= \frac{4(6)+1(-9)}{4+1}$$

$$= 3$$

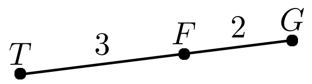
$$y = \frac{ky_2+\ell y_1}{k+\ell}$$

$$= \frac{4(3)+1(7)}{4+1}$$

$$= \frac{19}{5}$$

The required point is $P(3, \frac{19}{5})$.

(c)



$FT : TG = -3 : 5$ so $TF : FG = 3 : 2$.

(d) (i) $2\sin^2\theta - \sin\theta - 1 = 0$

$(2\sin\theta+1)(\sin\theta-1) = 0$

Hence $2\sin\theta+1=0$, $\sin\theta = -\frac{1}{2}$, $\theta = 210^\circ, 330^\circ$

Or $\sin\theta - 1 = 0$, $\sin\theta = 1$, $\theta = 90^\circ$

(ii) Let $u = 2\theta$. Then

$$\cos 2\theta = \frac{\sqrt{3}}{2}, \qquad \text{for } 0^\circ \le \theta \le 360^\circ$$

$$\cos u = \frac{\sqrt{3}}{2}, \qquad \text{for } 0^\circ \le u \le 720^\circ$$

$$u = 30^\circ, 330^\circ, 390^\circ, 690^\circ$$

$$2\theta = 30^\circ, 330^\circ, 390^\circ, 690^\circ$$

$$\theta = 15^\circ, 165^\circ, 195^\circ, 345^\circ$$

(e) This is a GP with common ratio $r = \frac{1}{3}$.

(i) Since $-1 < r < 1$, the limiting sum exists.

(ii) $$S_\infty = \frac{a}{1-r}$$

$$= \frac{45}{1-\frac{1}{3}} \begin{matrix}\times 3\\ \times 3\end{matrix}$$

$$= \frac{135}{3-1}$$

$$= 67{\cdot}5$$

QUESTION ELEVEN

(a)
$$y' = 2x - 3$$
$$= 2(4) - 3 \qquad \text{at } x = 4$$
$$= 5$$

and
$$y = x^2 - 3x + 2$$
$$= (4)^2 - 3(4) + 2$$
$$= 6$$

Hence
$$(y - y_1) = m(x - x_1)$$
$$(y - 6) = 5(x - 4)$$
$$y = 5x - 14$$

(b) (i) $$y = (2x-1)(x+3)$$
$$= 2x^2 + 5x - 3$$
$$y' = 4x + 5$$

(ii) Let $y = 7 - 4x$.

Then
$$y = (7-4x)^9$$
$$= u^9$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 9u^8 \times -4$$
$$= -36(7-4x)^8$$

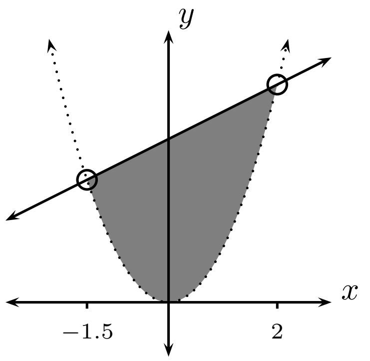
$$\frac{dy}{du} = 9u^8$$
$$\frac{du}{dx} = -4$$

(c) The graphs intersect when:

$$x^2 = \tfrac{1}{2}x + 3$$
$$2x^2 - x - 6 = 0$$
$$(2x+3)(x-2) = 0$$

Hence they intersect at $(-\frac{3}{2}, \frac{9}{4})$ and $(2, 4)$.

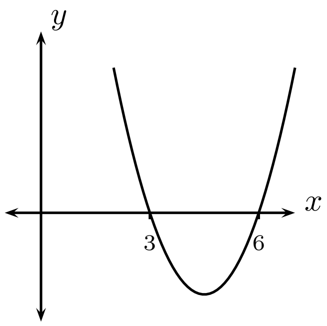
(i)



QUESTION TWELVE

(a)
$$\frac{2x}{x-3} > 4, \qquad x \neq 3$$

$\boxed{\times(x-3)^2}$
$$2x(x-3) > 4(x-3)^2$$
$$0 > 4(x-3)^2 - 2x(x-3)$$
$$0 > (x-3)(4x - 12 - 2x)$$
$$0 > (x-3)(2x-12)$$



Hence $3 < x < 6$.

(b) We have $T_5 = ar^4$ and $T_9 = ar^8$. Hence

$ar^4 = 18$ $\boxed{1}$

$ar^8 = 1458$ $\boxed{2}$

Thus $\boxed{2} \div \boxed{1}$ gives:

$r^4 = 81$

$r = \pm 3$

Hence from $\boxed{1}$,

$$a = \frac{18}{81} = \frac{2}{9}$$

(c) Note first that $\angle ABC = 75^\circ$ (angle sum of triangle). Now by the Sine Rule;

$$\frac{BC}{\sin 50^\circ} = \frac{2}{\sin 75^\circ}$$
$$BC = \frac{2 \times \sin 50^\circ}{\sin 75^\circ}$$
$$\doteqdot 1{\cdot}59\,\text{km}$$

Construct $BX \perp AC$ with X on AC. Thus BX is the required height of the balloon above the road. Then by right-angled triangle trigonometry,

(i) $$\frac{BX}{1{\cdot}59} = \sin 55^\circ$$
$$BX \doteqdot 1{\cdot}59 \times \sin 55^\circ$$
$$\doteqdot 1{\cdot}30\,\text{km}$$

(d) (i) The perpendicular distance d from ℓ to $P(x_0, y_0)$ is given by

$$d = \frac{|12x_0 - 5y_0 + c|}{\sqrt{12^2 + 5^2}}$$
$$= \frac{|12x_0 - 5y_0 + c|}{13}$$

(ii) If ℓ is a tangent, then its distance d from the centre of the circle is exactly the radius 1. Thus

$$\frac{|12(2) - 5(3) + c|}{13} = 1$$
$$|12(2) - 5(3) + c| = 13$$
$$|c + 9| = 13$$

Hence $c + 9 = 13$ or $c + 9 = -13$.
Thus $c = 4$ or $c = -22$.

QUESTION THIRTEEN

(a) We want the x values for which the cubic $y = x(x-3)(x-4)$ is ABOVE the line $y = 2x$. Thus the solution set is $0 < x < 2$ or $x > 5$.

(b) $$\text{LHS} = \frac{1}{1 - \cos\theta} + \frac{1}{1 + \cos\theta}$$
$$= \frac{(1 + \cos\theta) + (1 - \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)}$$
$$= \frac{2}{1 - \cos^2\theta}$$
$$= \frac{2}{\sin^2\theta}$$
$$= 2\,\text{cosec}^2\,\theta$$

(c)

A. First we shall check the case $n = 1$;

$LHS = \frac{1}{1\times 3}$ $RHS = \frac{1}{2\times 1+1}$

$= \frac{1}{3}$ $= \frac{1}{3}$

Thus the result is true for $n = 1$.

B. Now we assume the result holds for $n = k$, that is that

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}\,.$$

We want to show that the result holds for $n = k+1$, namely that

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2(k+1)-1)(2(k+1)+1)}=\frac{k+1}{2(k+1)+1}\,.$$

Consider the LHS of this expression;

$$\begin{aligned}
LHS &= \frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(2(k+1)-1)(2(k+1)+1)}\\
&= \frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}\\
&= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)}\\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)}\\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}\\
&= \frac{(k+1)}{(2k+3)}\\
&= RHS
\end{aligned}$$

C. The result now follows for all positive integers $n = 1, 2, \ldots$, by the principle of mathematical induction.

(d) (i) There is no y-intercept, since $x = 0$ is not on the domain.
The x-intercept is $(4, 0)$.

(ii) The vertical asymptotes are $x = 0$ and $x = 2$.

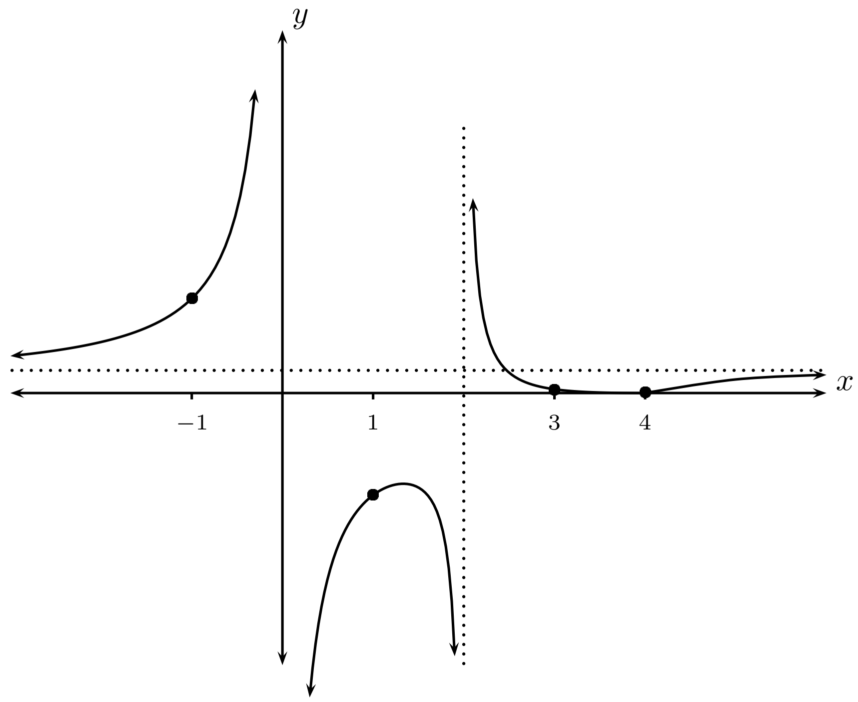
(iii)
$$\begin{aligned}
y &= \frac{(x-4)^2}{4x(x-2)}\\
&= \frac{x^2-8x+16}{4x^2-8x}\quad \begin{matrix}\div x^2\\ \div x^2\end{matrix}\\
&= \frac{1-\frac{8}{x}+\frac{16}{x^2}}{4-\frac{8}{x}}\\
&\to \frac{1}{4}\qquad \text{as } \tfrac{1}{x}\to 0
\end{aligned}$$

Hence the horizontal asymptote is $y = \frac{1}{4}$.

(iv)

x	-1	1	3
y	$\frac{25}{12}$	$\frac{9}{-4}$	$\frac{1}{12}$

(v)



QUESTION FOURTEEN

(a) Let $u = 8x^{-1} - x^2$, and then

Then $y = u^5$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 5u^4 \times (-8x^{-2} - 2x)$$
$$= 5(\tfrac{8}{x} - x^2)^4(-\tfrac{8}{x^2} - 2x)$$

$$\frac{dy}{du} = 5u^4$$
$$\frac{du}{dx} = -8x^{-2} - 2x$$

The tangent is horizontal when

$\frac{8}{x} = x^2$ OR $-\frac{8}{x^2} = 2x$

$8 = x^3$ $\qquad -4 = x^3$

$x = 2$ $\qquad x = -\sqrt[3]{4}$

(b) The graphs intersect where:

$$3x + 3 - x^2 = -(2x - 1)$$
$$0 = x^2 - 5x - 2$$

The solutions of this quadratic are:

$$x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times (-2)}}{2 \times 1}$$
$$= \frac{5 \pm \sqrt{33}}{2}$$

We need the negative root, i.e. $x = \dfrac{5 - \sqrt{33}}{2}$.

(c) (i) $y' = 3ax^2 + 2bx + c$

$= c \qquad \text{at } x = 0$

The tangent to the cubic at $(0, 0)$ is $y = cx$.

(ii) This tangent intersects the cubic when

$$ax^3 + bx^2 + cx = cx$$
$$ax^3 + bx^2 = 0$$
$$x^2(ax + b) = 0$$

Hence it intersects at $x = 0$ and again at $x = -\frac{b}{a}$, $y = -\frac{cb}{a}$.

(iii) We need to find the equation of the tangent at Q. Thus

$$y' = 3ax^2 + 2bx + c$$
$$= 3a(-\tfrac{b}{a})^2 + 2b(-\tfrac{b}{a}) + c \qquad \text{at } x = -\tfrac{b}{a}$$
$$= \tfrac{3b^2}{a} - \tfrac{2b^2}{a} + c$$
$$= \tfrac{b^2}{a} + c$$

and the tangent is

$$y - y_1 = m(x - x_1)$$
$$y + \tfrac{cb}{a} = (\tfrac{b^2}{a} + c)(x + \tfrac{b}{a})$$
$$y + \tfrac{cb}{a} = (\tfrac{b^2}{a} + c)x + \tfrac{b^3}{a^2} + \tfrac{cb}{a}$$
$$y = (\tfrac{b^2}{a} + c)x + \tfrac{b^3}{a^2}$$

The y-intercept is $(0, \frac{b^3}{a^2})$, which is independent of c.

(iv) Now intersect THIS tangent and the original cubic:

$$ax^3 + bx^2 + cx = (\tfrac{b^2}{a} + c)x + \tfrac{b^3}{a^2}$$
$$ax^3 + bx^2 - (\tfrac{b^2}{a})x - \tfrac{b^3}{a^2} = 0$$
$$ax^2(x + \tfrac{b}{a}) - \tfrac{b^2}{a}(x + \tfrac{b}{a}) = 0$$
$$(ax^2 - \tfrac{b^2}{a})(x + \tfrac{b}{a}) = 0$$
$$a(x - \tfrac{b}{a})(x + \tfrac{b}{a})^2 = 0$$

The tangent and cubic intersect again at $x = \frac{b}{a}$.

The coordinates $\frac{b}{a}$, 0 and $-\frac{b}{a}$ form an AP with common difference $-\frac{b}{a}$.

BDD