NAME MASTER

SYDNEY GRAMMAR SCHOOL

2015 Half-Yearly Examination

FORM V

MATHEMATICS EXTENSION 1

Monday 18th May 2015

General Instructions

- Writing time — 1 hour 30 minutes
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 80 Marks

- All questions may be attempted.

Section I – 8 Marks

- Questions 1–8 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 72 Marks

- Questions 9–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your name, class and Master on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your name, class and Master on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Nine.

5A: DS	5B: RCF	5C: SO	5D: DNW
5E: SJE	5F: REJ	5G: DWH	5H: KWM

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 125 boys

Examiner

SO

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

If $f(x) = \dfrac{3^x + 3^{-x}}{2}$, then $f(0)$ is:

(A) 0

(B) $\dfrac{1}{2}$

(C) 1

(D) 3

QUESTION TWO

Given $f(x) = \dfrac{1}{\sqrt[3]{x}}$, the correct expression for $f'(x)$ is:

(A) $\dfrac{1}{\sqrt[3]{x^4}}$

(B) $-\dfrac{1}{\sqrt[3]{x^4}}$

(C) $\dfrac{4}{3\sqrt[3]{x^4}}$

(D) $-\dfrac{1}{3\sqrt[3]{x^4}}$

QUESTION THREE

Consider the curve $f(x) = x^3 - ax$. The x-coordinates of the points where the tangent to the curve is horizontal are:

(A) 0 and a

(B) ± 1

(C) $\pm\sqrt{\dfrac{a}{3}}$

(D) $3 - a$ and $3 + a$

Examination continues next page ...

QUESTION FOUR

Which of the following is NOT an odd function?

(A) $f(x) = x^3$

(B) $f(x) = \dfrac{1}{x}$

(C) $f(x) = \sin x$

(D) $f(x) = \cos x$

QUESTION FIVE

Which statement is TRUE?

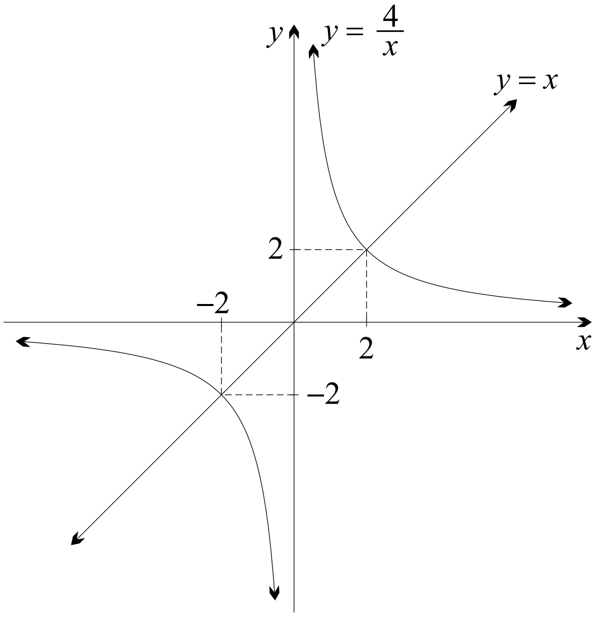
(A) $\sin(360^\circ - A) = -\sin A$

(B) $\cos(90^\circ - A) = \operatorname{cosec} A$

(C) $\cot A = \dfrac{\sin A}{\cos A}$

(D) $\operatorname{cosec}(180^\circ + A) = \dfrac{1}{\sin A}$

QUESTION SIX



Solve the inequation $\frac{4}{x} < x$ using the graph above:

(A) $x < -2$ or $0 < x < 2$

(B) $-2 < x < 0$ or $x > 2$

(C) $-2 < x < 2$

(D) $x < -2$ or $x > 2$

QUESTION SEVEN

The expression $\dfrac{\frac{1}{x^2} - \frac{1}{y^2}}{x + y}$ simplifies to:

(A) $\dfrac{x - y}{x^2 y^2}$

(B) $\dfrac{y - x}{x^2 y^2}$

(C) $\dfrac{x - y}{xy}$

(D) $\dfrac{-y - x}{xy}$

QUESTION EIGHT

The interval AB has endpoints $A(-1, 4)$ and $B(x, y)$. The point $P(14, -6)$ divides the interval AB externally in the ratio $5 : 3$. Find the value of x.

(A) 5

(B) $\dfrac{31}{5}$

(C) 23

(D) $\dfrac{109}{5}$

End of Section I

Examination continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) Solve $\dfrac{1}{m} + 6 = 1 - \dfrac{4}{m}$. **1**

(b) Evaluate $\displaystyle\sum_{k=2}^{4} 3^k$. **1**

(c) Find the limiting sum of the series $1 + \dfrac{1}{3} + \dfrac{1}{9} + \cdots$ **1**

(d) Differentiate:

(i) $2x^3 + 5$ **1**

(ii) $\dfrac{3}{x}$ **1**

(iii) $(x+1)(x-2)$ **1**

(e) A line has angle of inclination 120°. Find its gradient. **1**

(f) The line $x - y - 17 = 0$ is a tangent to a circle with centre $(2, -3)$.

(i) Find the perpendicular distance from the centre of the circle to the tangent. **2**

(ii) Hence write down the equation of the circle. **1**

(g) Find the equation of the tangent to the curve $y = x^2 - 8x$ at the point $P(-1, 9)$. **2**

Examination continues next page ...

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) An arithmetic sequence has second term 11 and eighth term -40. Find the common difference and the first term. **3**

(b) Solve the inequation $|2x+3| > 1$. **2**

(c) Solve the following equations for $0^\circ \le \theta \le 360^\circ$:

(i) $\sqrt{3}\sin\theta = \cos\theta$ **2**

(ii) $\sec^2\theta + \tan\theta = 1$ **3**

(d) In triangle XYZ, $\angle XYZ = 30^\circ$, $XZ = a\sqrt{2}$ and $YZ = 2a$. Find the two possible values of $\angle ZXY$. **2**

QUESTION ELEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate:

(i) $\dfrac{2x+x^3}{x}$ **2**

(ii) $(8+5x)^3$ **2**

(iii) $\dfrac{x^2+3}{1-x}$ **2**

(b) Consider the sequence for which $T_n = 3n + 3^n$.

(i) Find the first three terms of this sequence. **1**

(ii) Find the sum of the first eight terms. **3**

(c) Prove the identity $\cot\theta\cos\theta + \sin\theta = \operatorname{cosec}\theta$. **2**

Examination continues overleaf ...

QUESTION TWELVE (12 marks) Use a separate writing booklet. **Marks**

(a) Sketch the graph of $y = -|x|$. **1**

(b) Consider the function $f(x) = 3x^2 - 4x$.

(i) Simplify the expression $f(x+h) - f(x)$. **2**

(ii) Hence use the formula **1**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to differentiate $f(x) = 3x^2 - 4x$ from first principles.

(c) Consider the graphs of $(x-4)^2 + y^2 = 9$ and $y = x - 7$.

(i) Determine the points of intersection. **2**

(ii) Shade the region where both $(x-4)^2 + y^2 \leq 9$ and $y > x - 7$ are satisfied. **3**

(d) The lines $\ell_1 : 4x - y - 5 = 0$ and $\ell_2 : 3x + 4y + 1 = 0$ intersect at the point M.

(i) Write down the general equation of a line through M, and show that it can be written in the form **1**

$$(3k+4)x + (4k-1)y + (k-5) = 0$$

for some constant k.

(ii) Hence find the line through M and the point $P(6, 0)$. **2**

Examination continues next page ...

QUESTION THIRTEEN (12 marks) Use a separate writing booklet. **Marks**

(a) Solve the inequation $\dfrac{x-1}{x+2} \geq 2$. **3**

(b) Suppose that $\tan\alpha = \dfrac{a^2-b^2}{2ab}$, where α is acute, and both a and b are positive. Find an expression for $\cos\alpha$. **2**

(c) Consider the curve $y = \dfrac{x^2-1}{(x-3)(x+2)}$.

(i) Write down the equation(s) of any vertical asymptote(s). **1**

(ii) Find the equation of the horizontal asymptote. **1**

(iii) Find any intercepts with the x and y axes. **2**

(iv) Copy and complete the following table: **1**

x	-4	$-\frac{3}{2}$	2	4
y				

(v) Hence draw a neat sketch of the curve. **2**

QUESTION FOURTEEN (12 marks) Use a separate writing booklet. **Marks**

(a) Show that for any non-zero polynomial $Q(x)$, the curve $y = (x-a)^2\, Q(x)$ has a horizontal tangent at $x = a$. **2**

(b) Consider the function $y = 2|x+1| + |x-5|$.

(i) Sketch the graph of the function by considering separately $x \geq 5$, $x < -1$ and $-1 \leq x < 5$. **3**

(ii) Hence, or otherwise, solve the inequation $2|x+1| + |x-5| < 15$. **1**

(c) Consider the function $f(x) = \dfrac{2}{x^2+4x+8}$.

(i) Find the domain. **1**

(ii) Find the range. **2**

(d) If the sum of the first p terms of an arithmetic series is equal to zero, show that the sum of the next q terms is $\dfrac{aq(p+q)}{1-p}$, where a is the first term. **3**

End of Section II

END OF EXAMINATION

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NAME: ....

CLASS: ....MASTER: ....

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

2015 FIFTH FORM 3 UNIT HALF-YEARLY

SOLUTIONS

1. C
2. D
3. C
4. D
5. A
6. B
7. B
8. A

QUESTION 9

(a) $\frac{1}{m} + 6 = 1 - \frac{4}{m}$

$\frac{5}{m} = -5$

$m = -1$ ✓

(b) $\sum_{k=2}^{4} 3^k = 3^2 + 3^3 + 3^4$

$= 9 + 27 + 81$

$= 117$ ✓

(c) $S_\infty = \frac{1}{1 - \frac{1}{3}}$

$= \frac{3}{2}$ ✓

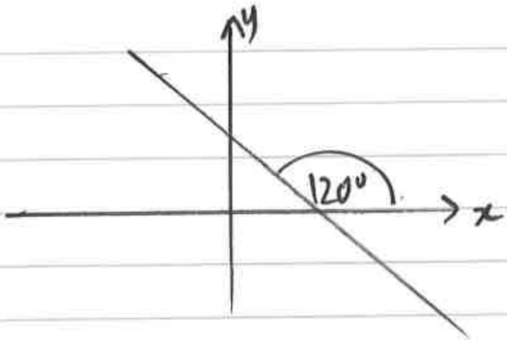
(d)(i) $\frac{d}{dx}(2x^3 + 5) = 6x^2$ ✓

(ii) $\frac{d}{dx}\left(\frac{3}{x}\right) = -\frac{3}{x^2}$ ✓

(iii) $\frac{d}{dx}\left((x+1)(x-2)\right) = \frac{d}{dx}(x^2 - x - 2)$

$= 2x - 1$ ✓

(e)



$$m = \tan 120^\circ$$
$$= -\sqrt{3}$$

$\therefore$ the gradient is $-\sqrt{3}$

(f) (i) perpendicular distance $= \dfrac{|1(2) - 1(-3) - 17|}{\sqrt{1^2 + (-1)^2}}$

$$= \frac{|2 + 3 - 17|}{\sqrt{2}}$$
$$= \frac{12}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= 6\sqrt{2} \text{ units}$$

(ii) $(x-2)^2 + (y+3)^2 = (6\sqrt{2})^2$
$(x-2)^2 + (y+3)^2 = 72$

(g) $y = x^2 - 8x$
$\dfrac{dy}{dx} = 2x - 8$

at $x = -1$, $\dfrac{dy}{dx} = 2(-1) - 8$
$= -10$

equation of tangent is $y - 9 = -10(x+1)$
$y - 9 = -10x - 10$
$10x + y + 1 = 0$

(or $y = -10x - 1$)

QUESTION 10

(a) $a + d = 11$ ①

$a + 7d = -40$ ② ✓

②−① $6d = -51$

$d = -8\frac{1}{2}$ ✓

sub back into ① : $a = 11 + 8\frac{1}{2}$

$= 10\frac{1}{2}$ ✓

$\therefore \quad a = 19\frac{1}{2}$

$d = -8\frac{1}{2}$

(b) $|2x+3| > 1$

$2x+3 > 1$

$2x > -2$

$x > -1$ ✓

or $2x+3 < -1$

$2x < -4$

$x < -2$ ✓

so $x < -2$ or $x > -1$.

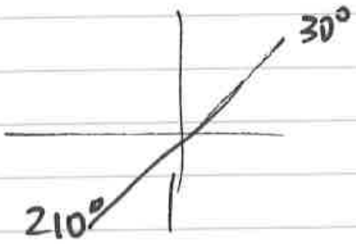
(c) (i) $\sqrt{3}\sin\theta = \cos\theta$

$\frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$ ✓

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= 30°, 210°$ ✓



(ii) $\sec^2\theta + \tan\theta = 1$ ✓

$1 + \tan^2\theta + \tan\theta = 1$

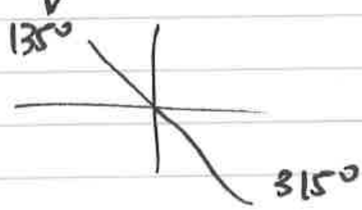
$\tan^2\theta + \tan\theta = 0$

$\tan\theta(\tan\theta + 1) = 0$ ✓

$\tan\theta = 0, \quad \tan\theta = -1$

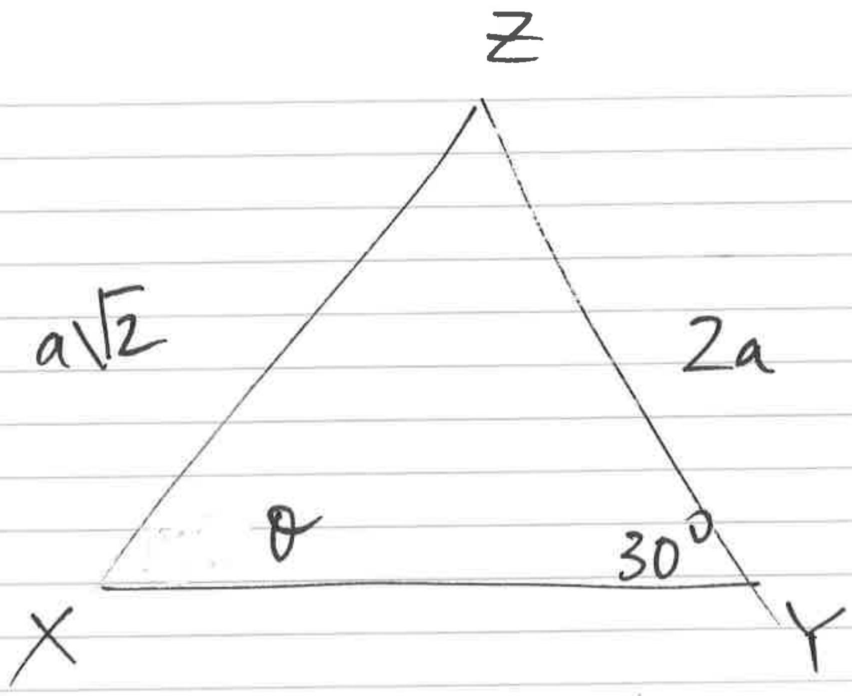
$\theta = 0°, 180°, 360°$

acute $\theta = 45°$



$\therefore \theta = 0°, 135°, 180°, 315°, 360°$ ✓

(d)



$$\frac{\sin\theta}{2a} = \frac{\sin 30^\circ}{a\sqrt{2}}$$ ✓

$$\sin\theta = \frac{2a}{2a\sqrt{2}}$$

$$\sin\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 45^\circ$$

θ could also $= 135^\circ$ $(135 + 30 = 165 < 180)$ ✓

$\therefore\ \theta = 45^\circ, 135^\circ$

$\boxed{\text{QUESTION 11}}$

(a) (i) $\frac{d}{dx}\left(\frac{2x+x^3}{x}\right) = 2x$

(ii) $\frac{d}{dx}\left((8+5x)^3\right) = 3(8+5x)^2 \times 5$

$= 15(8+5x)^2$

(iii) $\frac{d}{dx}\left(\frac{x^2+3}{1-x}\right) = \frac{2x(1-x)+x^2+3}{(1-x)^2}$

$= \frac{2x-2x^2+x^2+3}{(1-x)2}$

$= \frac{-x^2+2x+3}{(1-x)^2}$

(b) (i) $T_n = 3n+3^n$

$$\left.\begin{aligned} T_1 &= 3+3 \\ &= 6 \\ T_2 &= 6+3^2 \\ &= 15 \\ T_3 &= 9+3^3 \\ &= 36 \end{aligned}\right\}$$

(ii)

$3n$ is an AP with $a=3$ and $l=24, n=8$
so let its sum be S_a

$S_a = \frac{8}{2}(3+24)$
$= 108$

3^n is a GP with $a=3$, $r=3$, $n=8$; let its sum be S_G.

$S_G = \frac{3(3^8-1)}{3-1}$

$= 9840$. ✓

So $S_8 = S_a + S_G$

$= 108 + 9840$ ✓

$= 9948$

(c) Prove $\cot\theta\cos\theta + \sin\theta = \operatorname{cosec}\theta$

$LHS = \cot\theta\cos\theta + \sin\theta$

$= \frac{\cos\theta}{\sin\theta} \cdot \cos\theta + \sin\theta$

$= \frac{\cos^2\theta}{\sin\theta} + \sin\theta$ ✓

$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta}$

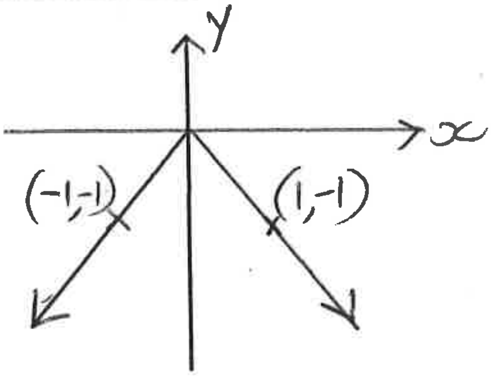
$= \frac{1}{\sin\theta}$

$= \operatorname{cosec}\theta$ ✓

$= RHS$

QUESTION 12

a)



✓

b) $f(x) = 3x^2 - 4x$

(i) $f(x+h) - f(x) = 3(x+h)^2 - 4(x+h) - [3x^2 - 4x]$ ✓

$= \cancel{3x^2} + 6xh + 3h^2 - \cancel{4x} - 4h - \cancel{3x^2} + \cancel{4x}$

$= h(6x - 4 + 3h)$ ✓

(ii) $f'(x) = \lim_{h \to 0} (6x - 4 + 3h)$

$= 6x - 4$ ✓

c) $(x-4)^2 + y^2 = 9 \quad y = x - 7$

(i) $\therefore (x-4)^2 + (x-7)^2 = 9$ ✓

$x^2 - 8x + 16 + x^2 - 14x + 49 = 9$

$2x^2 - 22x + 56 = 0$

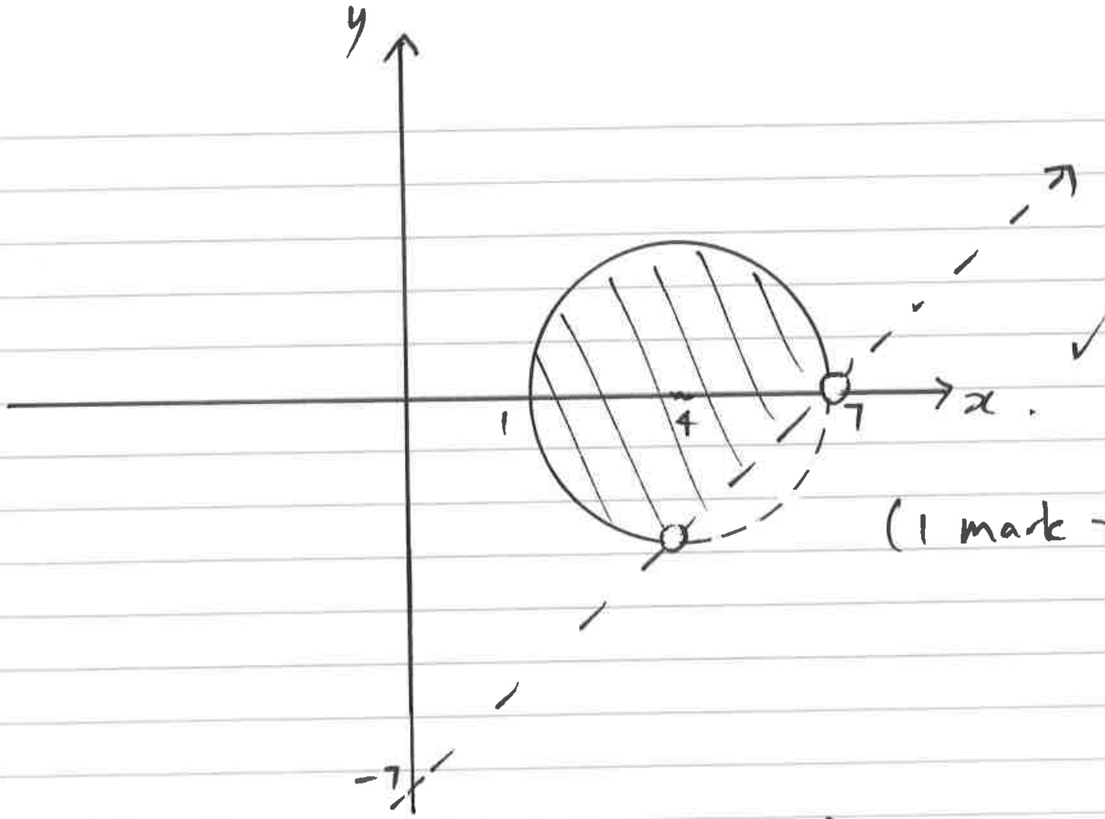
$x^2 - 11x + 28 = 0$

$(x-7)(x-4) = 0$

$\therefore x = 7$ OR 4

$y = 0$ OR -3

so $(7,0)$ and $(4,-3)$ ✓



(1 mark for shading)

d(i) $4x - y - 5 + k(3x + 4y + 1) = 0$

$4x - y - 5 + 3kx + 4ky + k = 0$

$(3k + 4)x + (4k - 1)y + (k - 5) = 0$ ✓

(ii) P is a line to

$(3k+4)\cdot 6 + (4k-1)\cdot 0 + k - 5 = 0$

$18k + 24 + k - 5 = 0$

$19k + 19 = 0$

$k = -1$ ✓

So the equation of the line is

$(3(-1)+4)x + (4(-1)-1)y + (-1-5) = 0$

$x - 5y - 6 = 0$ ✓

QUESTION 13

(a) $\dfrac{x-1}{x+2} \geq 2 \quad (x \neq -2)$

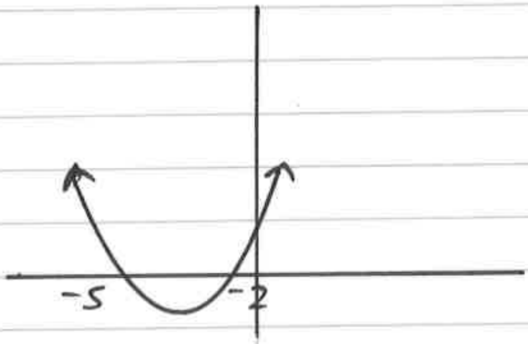
$(x-1)(x+2) \geq 2(x+2)^2$ ✓

$2(x+2)^2 - (x+2)(x-1) \leq 0$

$(x+2)[2x+4 - x+1] \leq 0$

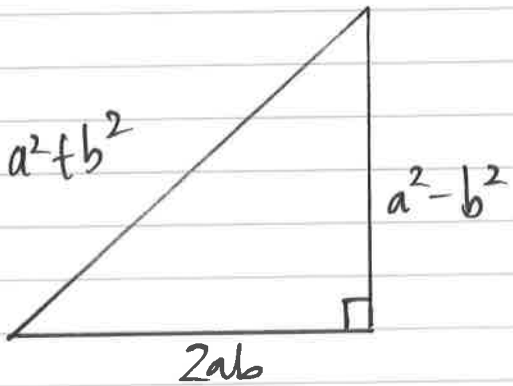
$(x+2)(x+5) \leq 0$ ✓

$-5 \leq x < -2$ ✓



(b) $a, b > 0$

$\tan\alpha = \dfrac{a^2 - b^2}{2ab}$



By Pythagoras' theorem:

$(a^2-b^2)^2 + (2ab)^2$

$= a^4 - 2a^2b^2 + b^4 + 4a^2b^2$

$= a^4 + 2a^2b^2 + b^4$

$= (a^2+b^2)^2$

hypotenuse $= a^2+b^2$ ✓

$\cos\alpha = \dfrac{2ab}{a^2+b^2}$ ✓

(c) $y = \dfrac{x^2-1}{(x-3)(x+2)}$

(i) $x=3$ and $x=-2$ ✓

(ii) $y = \dfrac{\frac{x^2}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{6}{x^2}}$

$= \dfrac{1 - \frac{1}{x^2}}{1 - \frac{1}{x} - \frac{6}{x^2}}$

Horizontal asymptote $y=1$ ✓

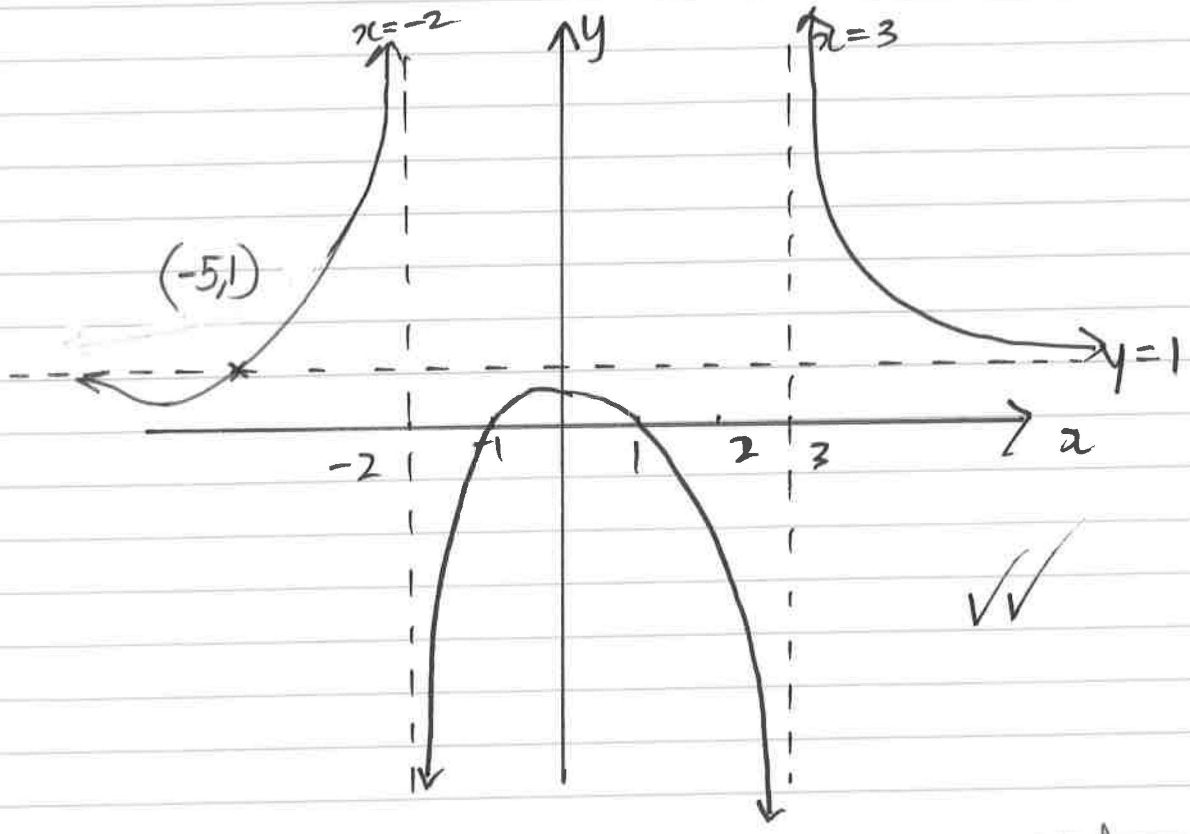
(iii) y-intercept: When $x=0$, $y = \dfrac{-1}{-3\times 2}$ $= -\frac{1}{6}$ $(0, \frac{1}{6})$ ✓

x-intercept: when $y=0$, $x^2-1=0$
$x = 1$ or -1 $(1,0)$ and $(-1,0)$ ✓

(iv)

x	-4	$-\frac{3}{2}$	2	4
y	$\frac{15}{4}$	$-\frac{5}{9}$	$-\frac{3}{4}$	$\frac{15}{6}$

✓



✓✓

(NB: Lefthand branch actually crosses horizontal asymptote at (-5,1) and approaches from below - No penalty for students not recognising this)

QUESTION 14

(a) $f(x) = (x-a)^2 \cdot Q(x)$

$f'(x) = 2(x-a)\cdot Q(x) + Q'(x)(x-a)^2$ ✓

at $x=a$, $f'(a) = 2(a-a)Q(x) + Q'(x)(a-a)^2$
$= 2\times 0 + 0$
$= 0$

So the polynomial has a horizontal tangent at $x=a$. ✓

(b)(i) when $x \geq 5$, $y = 2x+2+x-5$
$= 3x-3$

when $-1 \leq x < 5$, $y = 2x+2-x+5$
$= x+7$

when $x<-1$, $y = -2x-2-x+5$
$= -3x+3$

✓(branches)

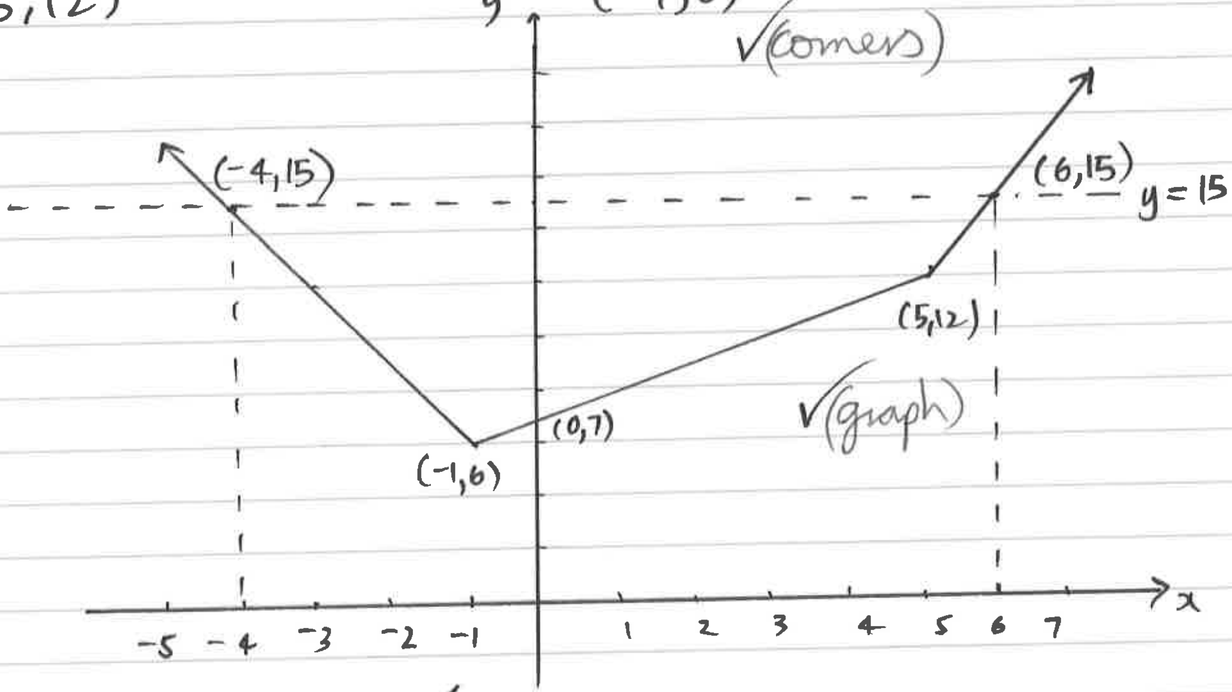
Points of Intersection

$3x-3 = x+7$
$2x = 10$
$x = 5$
when $x=5$, $y=12$
$(5,12)$

$-3x+3 = x+7$
$-4 = 4x$
$x = -1$
when $x=-1$, $y=6$
$(-1,6)$

y-intercept
$(0,7)$

✓(corners)



(ii) $-4 < x < 6$ ✓

(c) $f(x) = \dfrac{2}{(x^2+4x+4)+4}$

$= \dfrac{2}{(x+2)^2+4}$

(i) Domain : all real x ✓

(ii) when $x=0$, $y=\frac{1}{2}$

Range : $0 < y \leq \frac{1}{2}$ ✓✓

(d) $\underbrace{a + a+d + a+2d + \dots a+(p-1)d}_{p \text{ terms}}$

$S_p = \frac{p}{2}(2a+(p-1)d) = 0$ (*)

$p \neq 0$, so $2a+(p-1)d = 0$ ✓

$d = -\dfrac{2a}{p-1}$

$= \dfrac{2a}{1-p}$

$S_{p+q} = \dfrac{p+q}{2}[2a+(p+q-1)d]$

$= \dfrac{p+q}{2}[2a+(p-1)d+qd]$

$= \dfrac{p+q}{2} \cdot qd$ (using (*)) ✓

Also from (*) $d = \dfrac{2a}{1-p}$.

So $S_{p+q} = \frac{p+q}{2} \cdot q \cdot \frac{2a}{1-p}$

$$= \frac{\cancel{2}aq(p+q)}{\cancel{2}(1-p)}$$

$$= \frac{aq(p+q)}{1-p}.$$

$\checkmark$

Since $S_p = 0$, S_{p+q} is the sum of the next q terms.