Question 1 *(20marks)*

(a) Factor fully (i) $8x^3 - y^3$

(ii) $x^3 - x^2 - x + 1$

(b) Simplify $\dfrac{c^2+5c+6}{c^2-16} \div \dfrac{c+3}{c-4}$

(c) Solve $\dfrac{3}{x-2} \le 5$

(d) i) Simplify $\dfrac{10\sqrt{21}\times\sqrt{22}}{\sqrt{55}\times 3\sqrt{35}}$

ii) Given that $a = 4 - \sqrt{15}$ simplify

α) $a + \dfrac{1}{a}$ β) $a^2 + \dfrac{1}{a^2}$

(e) If $x + \sqrt{y-x} = 4 + \sqrt{5}$ find x and y

(f) Given $f(x) = \dfrac{x^3 - 3x^2 + 1}{x(1-x)}$ for $x \ne 0,1$ find $f\left(\dfrac{1}{x}\right)$ in simplest form

Question 2 *(20 marks)*

(a) What is the natural domain of $y = \dfrac{1}{\sqrt{1-x^2}}$

(b) If $f(x) = \dfrac{1-x}{x}$ find $f^{-1}(x)$

(c) On separate axes sketch (i) $y = (x+1)^2 + 2$

(ii) $y = x(3-x)^2$

(iii) $y = \dfrac{1}{x(3-x)^2}$

(d) Simplify (i) $\dfrac{|x|}{x}$ $x \ne 0$ (ii) $|2x-5| - |x-9|$ for $2\frac{1}{2} < x < 9$

(e) By finding $f(-x)$ decide if the following functions are odd, even or neither

(i) $f(x) = \dfrac{a^2 - x^2}{a^2 + x^2}$ where a is a constant

(ii) $f(x) = |x^3 - x|$

(f) By drawing appropriate graphs, or otherwise, solve (i) $|2x-6| \le 4$ (ii) $x + 6 > 2|x|$

END OF PAPER

Year 11 Extension 1 2001

Question 1

(a) (i) $8x^3 - y^3 = (2x)^3 - (y)^3$
$= (2x-y)(4x^2 + 2xy + y^2)$

(ii) $x^3 - x^2 - x + 1 = x^2(x-1) - 1(x-1)$
$= (x^2-1)(x-1)$
$= (x-1)(x+1)(x-1)$
$= (x+1)(x-1)^2$

(b) $\dfrac{c^2+5c+6}{c^2-16} \div \dfrac{c+3}{c-4}$

$= \dfrac{(c+3)(c+2)}{(c-4)(c+4)} \times \dfrac{(c-4)}{(c+3)}$

$= \dfrac{(c+2)}{(c+4)}$

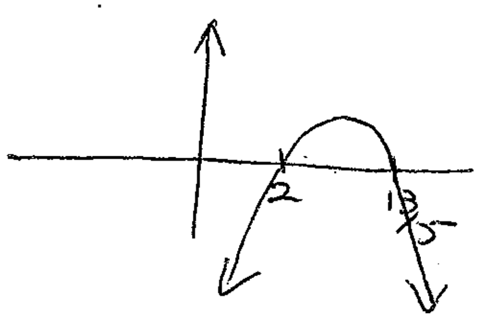
(c) $\dfrac{3}{x-2} \leq 5$

$(x-2)^2 \times \dfrac{3}{(x-2)} \leq 5(x-2)^2$

$3(x-2) \leq 5(x-2)^2$
$3(x-2) - 5(x-2)^2 \leq 0$
$(x-2)(3 - 5(x-2)) \leq 0$
$(x-2)(3 - 5x + 10) \leq 0$
$(x-2)(13 - 5x) \leq 0$

$x = 2$ or $\frac{13}{5}$

$\therefore x \leq 2$ or $x \geq \frac{13}{5}$



(d) (i) $\frac{10\sqrt{21} \times \sqrt{22}}{\sqrt{55} \times 3\sqrt{35}} = \frac{10\cancel{\sqrt{7}} \times \sqrt{3} \times \sqrt{2} \times \cancel{\sqrt{11}}}{\sqrt{5} \times \cancel{\sqrt{11}} \times 3 \times \cancel{\sqrt{7}} \times \sqrt{5}}$

$= \frac{10\sqrt{6}}{15}$

$= \frac{2\sqrt{6}}{3}$

(ii) $a = 4 - \sqrt{15}$

α) $a + \frac{1}{a}$

$= \frac{4 - \sqrt{15}}{1} + \frac{1}{4 - \sqrt{15}}$

$= \frac{(4 - \sqrt{15})^2 + 1}{(4 - \sqrt{15})}$

$= \frac{16 - 8\sqrt{15} + 15 + 1}{(4 - \sqrt{15})}$

$= \frac{32 - 8\sqrt{15}}{4 - \sqrt{15}}$

$= \frac{8(4 - \sqrt{15})}{4 - \sqrt{15}}$

$= 8$

β) $a^2 + \frac{1}{a^2}$

$(a + \frac{1}{a})^2 = a^2 + \frac{1}{a^2} +$

$a^2 + \frac{1}{a^2} = (a + \frac{1}{a})^2 -$

$= (a + \frac{1}{a})^2 - 2$

$= (8)^2 - 2$

$= 64 - 2$

$= 62$

Question 1 continued.

e) If $x + \sqrt{y-x} = 4 + \sqrt{5}$

$\therefore \quad x = 4$ and

$y - x = 5$

$\therefore y - 4 = 5$

$y = 9$

$\therefore x = 4$ and $y = 9$.

(f) $f(x) = \dfrac{x^3 - 3x^2 + 1}{x(1-x)} \qquad x \neq 0, 1$

$$f\left(\tfrac{1}{x}\right) = \frac{\left(\frac{1}{x}\right)^3 - 3\left(\frac{1}{x}\right)^2 + 1}{\frac{1}{x}\left(1 - \frac{1}{x}\right)}$$

$$= \frac{\frac{1}{x^3} - \frac{3}{x^2} + 1}{\frac{1}{x} - \frac{1}{x^2}} \times \frac{x^3}{x^3}$$

$$= \frac{x^3\left(\frac{1}{x^3} - \frac{3}{x^2} + 1\right)}{x^3\left(\frac{1}{x} - \frac{1}{x^2}\right)}$$

$$= \frac{1 - 3x + x^3}{x^2 - x}$$

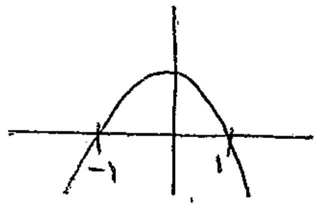
Question 2

(a) $y = \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2} > 0$

$1-x^2 > 0$

$(1-x)(1+x) > 0$



$\therefore$ Domain: $-1 < x < 1$

(b) if $y = \frac{1-x}{x} = f(x)$

find. $f^{-1}(x) \rightarrow \quad x = \frac{1-y}{y}$

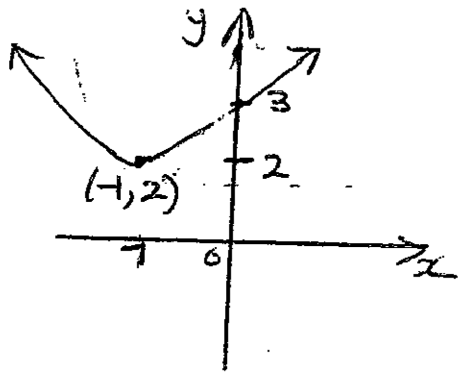
$xy = 1-y$

$xy + y = 1$

$y(x+1) = 1$

$y = \frac{1}{x+1}$

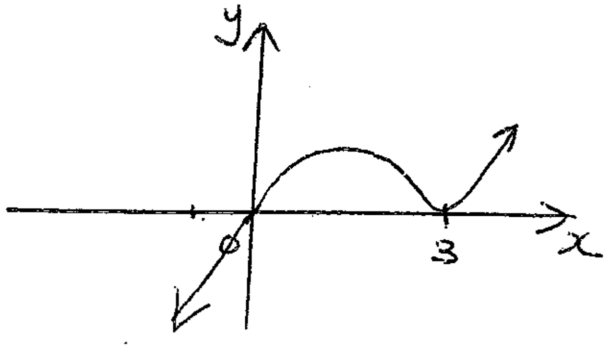
(c)(i) $y = (x+1)^2 + 2$



(ii) $y = x(3-x)^2$

$x = 0, 3$

x	-1	0	1	3	4
y	-16	0	4	0	4



Question 2 continued (5)(a)

$$y = \frac{1}{x(3-x)^2}$$

Domain: all real values, $x \neq 0$, $x \neq 3$

Odd / Even

$$f(x) = \frac{1}{x(3-x)^2}$$

$$f(-x) = \frac{1}{(-x)(3--x)^2}$$

$$= \frac{1}{-x(3+x)^2}$$

$$= \frac{-1}{x(3+x)^2}$$

$$-f(x) = \frac{-1}{x(3-x)^2}$$

$\therefore$ not odd nor even.

Intercepts $x \neq 0$ $\therefore$ no intercepts.
$y \neq 0$

Vertical asymptotes

$x = 0$, $x = 3$

Horizontal asymptotes $x \to \infty$ $y \to ?$

$$\lim_{x\to\infty} \frac{1}{x(9-6x+x^2)}$$

$$\lim_{x\to\infty} \frac{1}{9x-6x^2+x^3}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x^3}}{\frac{9x}{x^3} - \frac{6x^2}{x^3} + \frac{x^3}{x^3}}$$

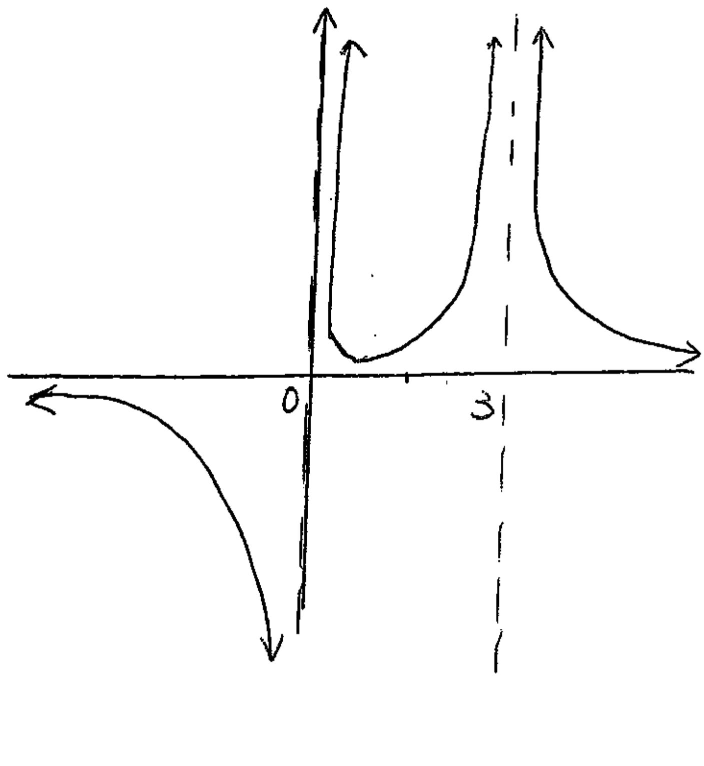
$= 0$

As $x \to +\infty$ $y \to 0^+$

As $x \to -\infty$ $y \to 0^-$

$0 < x < 3$

x	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	$\frac{1}{2}$
y	$+\frac{1}{4}$	0.3	$\frac{1}{2}$	1.6	0.32



Question 2 continued.

(c) (iii) $y = \frac{1}{x(3-x)^2}$

Using $y = x(3-x)^2$

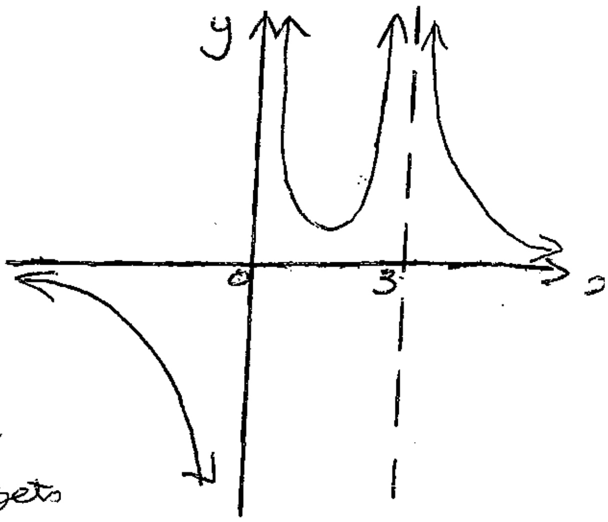
$y = 0$ at $x = 0$ and 3

$\therefore \frac{1}{y} \neq x = 0$ and 3 (asymptotes)

for $x > 3$ $x \Rightarrow \infty$ $\therefore$ for $\frac{1}{y}$, $x > 0$ $\frac{1}{y} \Rightarrow {}^{+}0$

for $x < 0$ $x \Rightarrow -\infty$ $\therefore$ for $\frac{1}{y}$ $x < 0$ $\frac{1}{y} \Rightarrow -0$

for $0 < x < 3$, the largest point will become the minimum point. And as the function gets close to zero $\frac{1}{y}$ will get closer to ∞.



(d)(i) $\frac{|x|}{x}$ $x \neq 0$

If $x < 0$

$|x| = -x$

$\therefore \frac{|x|}{x} = \frac{-x}{x}$

$= -1$

If $x > 0$

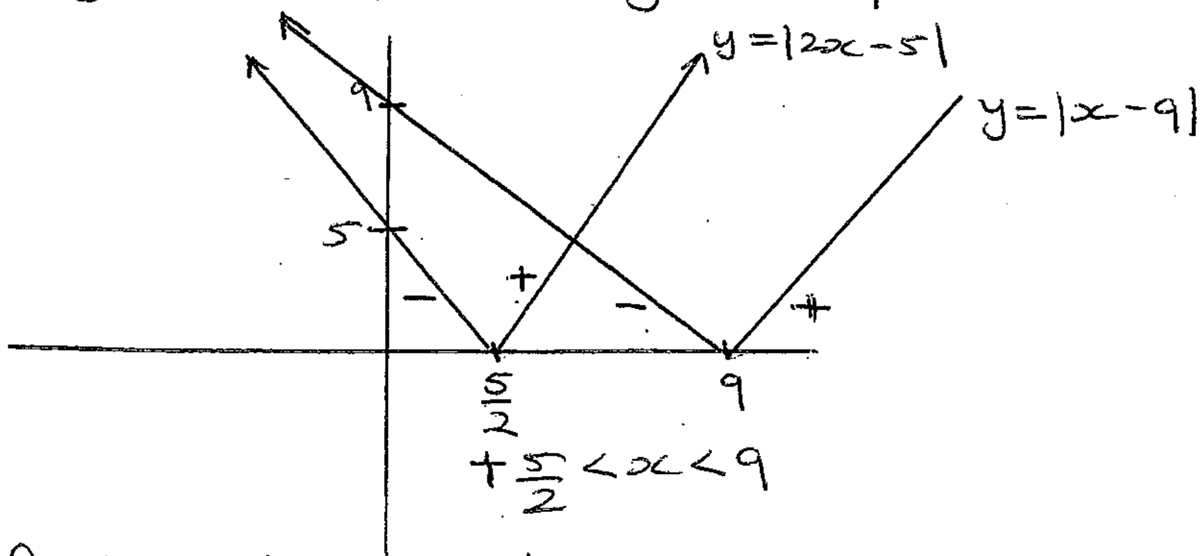
$|x| = +x$

$\frac{|x|}{x} = \frac{x}{x}$

$= 1$

(ii) $|2x - 5| - |x - 9|$

~~sketch~~ $y = |2x - 5|$ and $y = |x - 9|$



Simplify $|2x - 5| - |x - 9|$

for $+\frac{5}{2} < x < 9$ $= +(2x - 5) - {}^{-}(x - 9)$

$= 2x - 5 + x - 9$

$= 3x - 14$

Question 2 continued

(e) (i) $f(x) = \dfrac{a^2 - x^2}{a^2 + x^2}$

$$f(-x) = \frac{a^2 - (-x)^2}{a^2 + (-x)^2}$$

$$= \frac{a^2 - x^2}{a^2 + x^2}$$

$\therefore \quad f(-x) = f(x)$

$\therefore \quad f(x)$ is an even function.

(ii) $f(x) = |x^3 - x|$

$$f(-x) = |(-x)^3 - (-x)|$$

$$= |-x^3 + x|$$

$$= |-1\,(x^3 - x)|$$

$$= |-1| \cdot |x^3 - x|$$

$$= 1 \cdot |x^3 - x|$$

$$= f(x)$$

$\therefore f(x)$ is an even function.

(f) (i) $|2x - 6| \le 4$

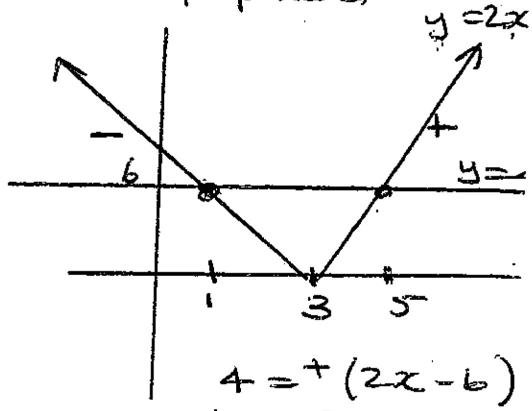
$2x - 6 \le 4$ or $2x - 6 \ge \bar{4}$

$2x \le 10$ or $2x \ge$

$x \le 5$ $\quad x \ge 1$

$\therefore \quad 1 \le x \le 5$

Graphical



$4 = {}^{+}(2x - 6)$

$10 = 2x$

$5 = x$

$4 = -(2x - 6)$

$4 = -2x + 6$

$-2 = -2x$

$1 = x$

$1 \le x \le 5$