**GIRRAWEEN HIGH SCHOOL**

**YEAR 11 – TASK4**

**2009**

# MATHEMATICS

Time Allowed: 90 minutes

**INSTRUCTIONS TO STUDENTS**

- Attempt **ALL** questions.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a new sheet of paper.

**QUESTION 1** (10 marks)

a) Evaluate the following limits:

(i) $\lim_{x\to 2}\frac{x+4}{x}$ **1**

(ii) $\lim_{x\to -5}\frac{x^2-25}{x+5}$ **2**

(iii) $\lim_{x\to 1}\frac{x^2-3x+2}{x-1}$ **2**

(iv) $\lim_{x\to \infty}\frac{4x^2+x}{x^2+2x+1}$ **2**

b) Differentiate $f(x)=x^2+2x+1$ from first principles. **3**

**QUESTION 2** (14 marks)

Differentiate:

(i) $y = 3x^4 - 5x^3 + 4x$ **1**

(ii) $y = \dfrac{2x^3 - 4x + 3}{x^2}$ **3**

(iii) $y = \sqrt{3x}$ **2**

(iv) $y = 4x^{\frac{3}{2}} - 3x^{\frac{7}{3}}$ **2**

(v) $y = \dfrac{7}{\sqrt{x}} + x^3$ **3**

(vi) $y = \dfrac{3}{x^4} + \sqrt[3]{x}$ **3**

**QUESTION 3** (17 marks)

a) Use the product rule to differentiate $y = (5x^3 + 2)(2x - x^3)$. **3**

b) Differentiate the following:

(i) $y = \dfrac{3x}{x^2 + 3}$ **3**

(ii) $y = (x^4 - 2x^3)^5$ **2**

(iii) $y = x^3(1 + x)^3$ **3**

(iv) $y = \dfrac{1}{(4x - 2)^3}$ **2**

(v) $y = \dfrac{x}{\sqrt{x^2 + 1}}$ **4**

**QUESTION 4** (13 marks)

a) At what point on the curve $y = \frac{x}{x-1}$ is the tangent parallel to the line $x + y + 5 = 0$? **4**

b) At what points on the curve $y = x^3 - x^2$ does the tangent make an angle of $45°$ with the positive $x$- axis? **4**

c) The tangent at $P(3,9)$ on the curve $y = x^2$ cuts the $x$-axis at $T$, and $PN$ is perpendicular to the $x$-axis. Find the length of $TN$. **5**

**QUESTION 5** (10 marks)

a) The line $y = mx + b$ is a tangent to the curve $y = x^3 - 3x + 1$ at the point (-2, -1). Find the values of $m$ and $b$. **4**

b) Find the equations of the *tangent* and *normal* to the curve $y = 2x^3 + 3x - 1$ at the point where $x = 1$. **6**

**QUESTION 6** (13 marks)

a) Find the quadratic equation with roots $1 + \sqrt{3}$ and $1 - \sqrt{3}$. **2**

b) For what value(s) of $m$ does the equation $(2m+1)x^2 + 2mx - 1 = 0$ have equal roots? **3**

c) Given that $\alpha$ and $\beta$ are the roots of the quadratic equation $x^2 - 5x + 4 = 0$, evaluate

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $\alpha^2 + \beta^2$ **2**

(iv) $\frac{1}{\alpha} + \frac{1}{\beta}$ **2**

(v) $(\alpha + 2)(\beta + 2)$ **2**

**QUESTION 7** (15 marks)

a) For what values of $k$ is the function $3x^2 + 8x - k$ positive definite? **3**

b) For what values of $k$ does the quadratic equation
$x^2 - 5x + (k-1) = 0$ have

(i) two real roots? **3**
(ii) one real root? **1**
(iii) no real roots? **1**

c) Find the values of $A$, $B$ and $C$ if

$$4x^2 - 12x + 9 \equiv A(x-1)^2 + B(x-1) + C$$ **4**

d) Solve: $3^{2x} - 10 \times 3^x + 9 = 0$. **3**

Year 11 Mathematics
Task 4 2009 SOLUTIONS

## Question 1 (10 marks)

a) i) $\lim_{x \to 2} \frac{x+4}{x} = 3$ (1)

ii) $\lim_{x \to -5} \frac{x^2-25}{x+5} = \lim_{x \to -5} \frac{(x+5)(x-5)}{(x+5)} = \lim_{x \to -5} (x-5) = -10$ (2)

iii) $\lim_{x \to 1} \frac{x^2-3x+2}{x-1} = \lim_{x \to 1} \frac{(x-1)(x-2)}{x-1} = \lim_{x \to 1} (x-2) = -1$ (2)

iv) $\lim_{x \to \infty} \frac{4x^2+x}{x^2+2x+1} = \lim_{x \to \infty} \frac{4+\frac{1}{x}}{1+\frac{2}{x}+\frac{1}{x^2}} = 4$ (2)

b) $f(x) = x^2+2x+1$

$f(x+h) = (x+h)^2+2(x+h)+1$
$= x^2+2xh+h^2+2x+2h+1$

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
$= \lim_{h \to 0} \frac{x^2+2xh+h^2+2x+2h+1-x^2-2x-1}{h}$
$= \lim_{h \to 0} \frac{2xh+h^2+2h}{h}$
$= \lim_{h \to 0} 2x+h+2$
$= 2x+2$ (3)

## Question 2 (14 marks)

i) $y = 3x^4-5x^3+4x$

$\frac{dy}{dx} = 12x^3-15x^2+4$ (1)

ii) $y = \frac{2x^3-4x+3}{x^2}$

$= 2x - \frac{4}{x} + \frac{3}{x^2}$

$y = 2x - 4x^{-1} + 3x^{-2}$

$\frac{dy}{dx} = 2 + 4x^{-2} - 6x^{-3}$

$= 2 + \frac{4}{x^2} - \frac{6}{x^3}$ (3)

iii) $y = \sqrt{3x} = (3x)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(3x)^{-1/2} \cdot 3$

$= \frac{3}{2\sqrt{3x}}$ (2)

iv) $y = 4x^{3/2} - 3x^{7/3}$

$\frac{dy}{dx} = 6x^{1/2} - 7x^{4/3}$

$= 6\sqrt{x} - 7\sqrt[3]{x^4}$ (2)

v) $y = \frac{7}{\sqrt{x}} + x^3$

$= 7x^{-1/2} + x^3$

$\frac{dy}{dx} = -\frac{7}{2}x^{-3/2} + 3x^2$

$= \frac{-7}{2\sqrt{x^3}} + 3x^2$ (3)

vi) $y = \frac{3}{x^4} + \sqrt[3]{x}$

$= 3x^{-4} + x^{1/3}$

$\frac{dy}{dx} = -12x^{-5} + \frac{1}{3}x^{-2/3}$

$= \frac{-12}{x^5} + \frac{1}{3x^{2/3}}$

$= \frac{-12}{x^5} + \frac{1}{3\sqrt[3]{x^2}}$ (3)

## Question 3 (17 marks)

a) $y = (5x^3+2)(2x-x^3)$

$\frac{dy}{dx} = vu' + uv'$

$u = 5x^3+2$, $u' = 15x^2$, $v = 2x-x^3$, $v' = 2-3x^2$

$= 15x^2(2x-x^3) + (5x^3+2)(2-3x^2)$

$= 30x^3 - 15x^5 + 10x^3 - 15x^5 + 4 - 6x^2$

$= -30x^5 + 40x^3 - 6x^2 + 4$ (3)

b) i) $y = \frac{3x}{x^2+3}$

$u = 3x$, $u' = 3$, $v = x^2+3$, $v' = 2x$

$\frac{dy}{dx} = \frac{vu'-uv'}{v^2}$

$= \frac{3(x^2+3)-3x(2x)}{(x^2+3)^2}$

$= \frac{3x^2+9-6x^2}{(x^2+3)^2}$

$= \frac{9-3x^2}{(x^2+3)^2}$

$= \frac{3(3-x^2)}{(x^2+3)^2}$ (3)

ii) $y = (x^4-2x^3)^5$

$\frac{dy}{dx} = 5(x^4-2x^3)^4(4x^3-6x^2)$

$= 5(x^4-2x^3)^4 \cdot 2x^2(2x-3)$

$= 10x^2(x^4-2x^3)^4(2x-3)$ (2)

iii) $y = x^3(1+x)^3$

$\frac{dy}{dx} = vu' + uv'$

$u = x^3$, $u' = 3x^2$, $v = (1+x)^3$, $v' = 3(1+x)^2$

$= 3x^2(1+x)^3 + 3x^3(1+x)^2$

$= 3x^2(1+x)^2(1+x+x)$

$= 3x^2(1+x)^2(1+2x)$ (3)

iv) $y = \frac{1}{(4x-2)^3} = (4x-2)^{-3}$

$\frac{dy}{dx} = -3(4x-2)^{-4} \cdot 4$

$= \frac{-12}{(4x-2)^4}$ (2)

v) $y = \frac{x}{\sqrt{x^2+1}}$

$u = x$, $u' = 1$, $v = (x^2+1)^{1/2}$, $v' = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$

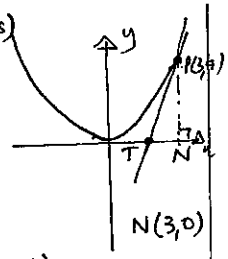
$\frac{dy}{dx} = \frac{vu'-uv'}{v^2}$

$= \frac{\sqrt{x^2+1} - x \cdot \frac{x}{\sqrt{x^2+1}}}{x^2+1}$

$= \frac{x^2+1-x^2}{(x^2+1)\sqrt{x^2+1}}$

$= \frac{1}{(x^2+1)^{3/2}}$ (4)

## Question 4 (13 marks)




a) $y = x^2$

$\frac{dy}{dx} = 2x$

at $x = 3$,

$m = 2(3) = 6$ ; pt $(3,9)$

Eqn : $y - y_1 = m(x - x_1)$

$y - 9 = 6(x-3)$

$y - 9 = 6x - 18$

$y = 6x - 9$

$x$ int $\Rightarrow y = 0$

$6x - 9 = 0$

$x = \frac{9}{6} = \frac{3}{2}$

$\therefore$ coordinates of T $(1\frac{1}{2}, 0)$

Coordinates of N $= (3,0)$

$\therefore TN = 1\frac{1}{2}$ units (5)

a) $y = \frac{x}{x-1}$ ; $x + y + 5 = 0$

$y = -x - 5$

$m = -1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{x - 1 - x}{(x-1)^2}$

$= \frac{-1}{(x-1)^2}$

$m_{tangent} = m_{x+y+5=0}$

$\frac{-1}{(x-1)^2} = -1$

$(x-1)^2 = 1$ (4)

$x - 1 = \pm 1$

$x = 0, 2$

when $x = 0$, $y = 0$ $\therefore$ At $(0,0)$ and $(2,2)$

when $x = 2$, $y = 2$

b) $y = x^3 - x^2$

$\frac{dy}{dx} = 3x^2 - 2x$ ; $m = \tan 45^\circ$

$m = 1$

$\therefore 3x^2 - 2x = 1$

$3x^2 - 2x - 1 = 0$

$(3x+1)(x-1) = 0$

$x = -\frac{1}{3}, 1$

when $x = -\frac{1}{3}$, $y = \frac{-4}{27}$ (4)

$x = 1$, $y = 0$

$\therefore$ at $(1,0)$ and $(-\frac{1}{3}, \frac{-4}{27})$

## Question 5 (10 marks)

a) $y = x^3 - 3x + 1$ $E_{tangent}$ : $y = mx + b$

$\frac{dy}{dx} = 3x^2 - 3$

at $x = -2$

$m = 3(-2)^2 - 3$

$= 9$

$E_{tangent}$ : $y + 1 = 9(x+2)$

$y + 1 = 9x + 18$

$y = 9x + 17$ (4)

$\therefore m = 9$, $b = 17$

b) $y = 2x^3 + 3x - 1$ at $x = 1$

$y(-1) = 2(1)^3 + 3(1) - 1$ pt $(1, 4)$

$= 4$

$\frac{dy}{dx} = 6x^2 + 3$

at $(1,4)$ ; $m_{tangent} = 6(1)^2 + 3$

$m_{tangent} = 9$

$E_{tangent}$ : $y - 4 = 9(x-1)$

$y - 4 = 9x - 9$

$9x - y - 5 = 0$

$m_{normal} = -\frac{1}{9}$ pt $(1,4)$

$E_{normal}$ : $y - 4 = -\frac{1}{9}(x-1)$

$9(y-4) = -x + 1$

$9y - 36 = -x + 1$

$x + 9y - 37 = 0$ (5)

## Question 6 (13 marks)

a) roots : $1 + \sqrt{3}$, $1 - \sqrt{3}$

$\alpha + \beta = 2$

$\alpha\beta = -2$ (2)

Equation : $x^2 - 2x - 2 = 0$

b) $(2m+1)x^2 + 2mx - 1 = 0$

Equal roots $\Rightarrow b^2 - 4ac = 0$

$(2m)^2 - 4(2m+1)(-1) = 0$

$4m^2 + 8m + 4 = 0$

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$ (3)

$m = -1$

$\therefore$ equal roots when $m = -1$

c) $x^2 - 5x + 4 = 0$

i) $\alpha + \beta = -\frac{b}{a} = 5$ (1)

ii) $\alpha\beta = \frac{c}{a} = 4$ (1)

iii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 25 - 8$

$= 17$ (2)

iv) $\frac{1}{\alpha} + \frac{1}{\beta}$

$= \frac{\alpha + \beta}{\alpha\beta} = \frac{5}{4}$ (2)

v) $(\alpha + 2)(\beta + 2)$

$= \alpha\beta + 2\alpha + 2\beta + 4$

$= \alpha\beta + 2(\alpha + \beta) + 4$

$= 4 + 2(5) + 4$ (2)

$= 18$

## Question 7 (15 marks)

a) $3x^2 + 8x - k$

positive definite $\Rightarrow a > 0$, $\Delta < 0$

$a = 3 > 0$

$b^2 - 4ac < 0$

$64 + 4(3)k < 0$

$12k < -64$

$k < -5\frac{1}{3}$ (3)

b) $x^2 - 5x + (k-1) = 0$

i) two real roots $\Rightarrow \Delta > 0$

$b^2 - 4ac > 0$

$25 - 4(k-1) > 0$

$25 - 4k + 4 > 0$

$-4k > -29$

$k < 7\frac{1}{4}$ (3)

ii) one real root

$\Delta = 0$

ie $k = 7\frac{1}{4}$ (1)

iii) no real roots

$\Delta < 0$

ie. $k > 7\frac{1}{4}$ (1)

c) $4x^2 - 12x + 9 \equiv A(x-1)^2 + B(x-1) + C$

$= A(x^2 - 2x + 1) + Bx - B + C$

$= Ax^2 - 2Ax + A + Bx - B + C$

$= Ax^2 - (2A - B)x + (A - B + C)$

$\therefore A = 4$

$2A - B = 12$

$8 - B = 12$

$-B = 4$

$B = -4$

$A - B + C = 9$

$4 + 4 + C = 9$

$C = 1$ (4)

$\therefore A = 4, B = -4, C = 1$

d) $3^{2x} - 10.3^x + 9 = 0$

let $u = 3^x$

then

$u^2 - 10u + 9 = 0$

$(u - 9)(u - 1) = 0$

$u = 1, 9$

ie. $3^x = 1$ or $3^x = 9$

$3^x = 3^0$ $3^x = 3^2$

$\therefore x = 0, 2$ (3)