SYDNEY TECHNICAL HIGH SCHOOL

YEAR 11 MATHEMATICS EXTENSION 1

PRELIMINARY ASSESSMENT TASK 2

JULY 2006

Time allowed: 70 minutes

Instructions:

- Show full working (this is important, especially in "show that" questions)
- Start each question on a new page
- Full marks may not be awarded for careless or badly arranged work
- Non-programmable calculators may be used
- This paper must be handed in on top of your answer sheets
- Answers must be written in blue or black pen
- Answers must be arranged in order and stapled securely. No claims for missing pages will be allowed.

Name:____________________________

Class:_______________________

Question 1	Question 2	Question 3	Question 4	Question 5	Total
/11	/11	/11	/11	/11	/55

Question 1

(a) Evaluate i) $\lim_{x\to\infty}\frac{x^2+3x+1}{2x^2-4}$ **1**

ii) $\lim_{x\to 3}\frac{x^2+3x-18}{x^2-5x+6}$ **1**

(b) Consider the interval joining the points A(1, -4) and B(5, 12).

i) Find the point which divides the interval AB externally in the ratio 3:1 **2**

(ii) Find the ratio in which the x axis divides the interval AB. **1**

(c) Show on the Cartesian plane the union of the regions represented by the following inequalities: $y<x-2$, $y\geq 0$ & $x\leq 0$. **3**

(d) The acute angle between the lines $y=3x+5$ and $y=mx+4$ is 45°. **3**
Find the possible values of m.

Question 2 (Start a new page)

(a) Using the notation $\frac{d}{dx}f(x)=f'(x)$ and $\frac{d}{dx}g(x)=g'(x)$, write in simplest form: **4**

i) $\frac{d}{dx}[f(x)+g(x)]$

ii) $\frac{d}{dx}[f(x).g(x)]$

iii) $\frac{d}{dx}\frac{1}{f(x)}$

iv) $\frac{d}{dx}f(g(x))$

Q2 continues

Question 2 (cont.)

(b) For the curve. $y = \dfrac{x+1}{x-1}$ **7**

i) The curve cuts the x axis at P. Find the x value at P.

ii) Find $\dfrac{dy}{dx}$.

iii) Find the gradient of the tangent to the curve at P.

iv) Using the point gradient form of a straight line, show how the equation of the <u>normal</u> at P, $y = 2x + 2$, may be derived.

v) State clearly the x value of the point where the normal again cuts the curve.

Question 3 (Start a new page)

(a) Simplify $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}$ **2**

(b) ABCD is a parallelogram in which A is the point (4, 1). The equation of side BC is $3x - y + 5 = 0$ and the equation of side CD is $x + 2y + 2 = 0$. **5**

i) Find the equation of the side AD.

ii) Find the equation of the diagonal AC.

(c) (Angles in this part may be left in decimal form, correct to 2 dec. places)

i) Write $8\cos x + 6\sin x$ in the form $A\cos(x - \alpha)$ where α is acute. **2**

ii) Hence solve $8\cos x + 6\sin x = 5$ for $0° \le x \le 360°$. **2**

Question 4 (Start a new page)

(a) If $f(x)=\dfrac{1}{x}$

i) Write an expression for $f(a+h)-f(a)$ as a single fraction. **1**

ii) Hence evaluate $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$. **2**

iii) What does the <u>answer</u> to ii) represent? **1**

iv) Describe briefly the graphical process which is represented **1**

by the expression $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

(b) i) By expanding $\sin(x+2x)$ and using the double angle formulas, **3**

prove the identity $\sin 3x = 3\sin x - 4\sin^3 x$.

ii) Hence solve the equation $\sin 3x = 2\sin x$ for $0° \le x \le 360°$. **3**

Question 5 (Start a new page)

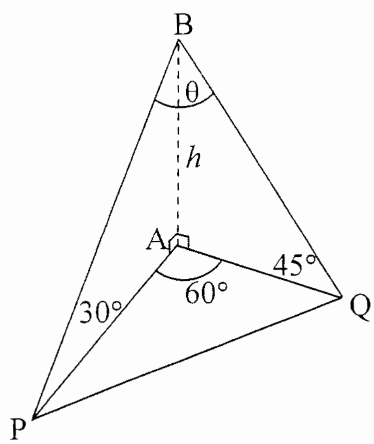
(a) i) Differentiate $y = \sqrt{x^2 + 1}$ **1**

ii) Hence find the derivative of $y = x^2\sqrt{x^2 + 1}$ as a fraction in simplest form. **2**

(b) i) Express $\operatorname{cosec}\theta$ in terms of t, where $t = \tan\frac{\theta}{2}$. **1**

ii) Hence show that $\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}$. **2**

(c)



AB is a vertical tower of height h which stands on level ground. Angles of elevation of B from P and Q are 30° and 45° respectively. Angle PAQ is 60° and angle PBQ is θ.

i) Write PA and QA in terms of h **1**

ii) By considering triangle PAQ show that $PQ^2 = (4 - \sqrt{3})h^2$ **1**

iii) By finding a second expression for PQ^2, show that **3**

$$\cos\theta = \frac{2 + \sqrt{3}}{4\sqrt{2}}$$

End of Paper

QUESTION 1

a) i) $\lim_{x \to \infty} \dfrac{x^2 + 3x + 1}{2x^2 - 4}$

$= \lim_{x \to \infty} \dfrac{1 + \frac{3}{x} + \frac{1}{x^2}}{2 - \frac{4}{x^2}}$

$= \dfrac{1}{2}$ ✓

ii) $\lim_{x \to 3} \dfrac{x^2 + 3x - 18}{x^2 - 5x + 6}$

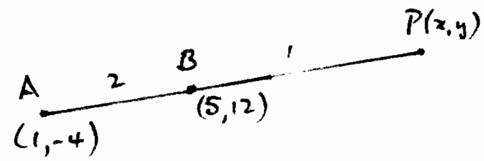
$= \lim_{x \to 3} \dfrac{(x+6)(x-3)}{(x-3)(x-2)}$

$= \lim_{x \to 3} \dfrac{(x+6)}{x-2}$

$= 9$ ✓

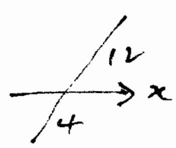
(b)

(i)



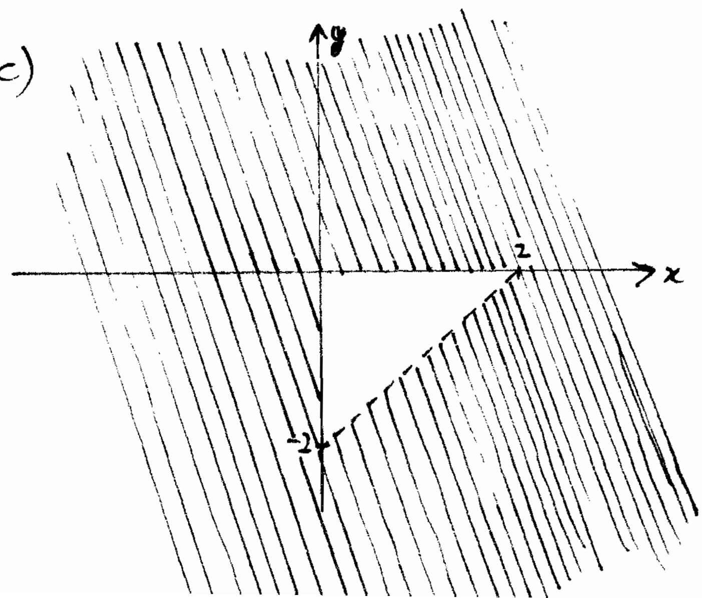
P is (7 ✓, 20 ✓)

(ii)



Ratio is 1 : 3.

(c)



a) i) Correct answer – 1 mark.

ii) Correct answer – 1 mark

b) (i) • Correct point – 2 marks

• x OR y value correct – 1 mark

• A correct substitution into the formula – 1 mark.

ii) Correct answer – 1 mark

(c)

Correct diagram – 3 marks.
(1 off each error)

(d) $y = 3x + 5$, $y = mx + 4$

$m_1 = 3$, $m_2 = m$

$$\tan 45^\circ = \left| \frac{3 - m}{1 + 3m} \right| \checkmark$$

Correct values - 3 marks.
1 Correct value - 2 marks
Correct use of formula - 1 mark.

ie $\frac{3-m}{1+3m} = 1$ or $\frac{3-m}{1+3m} = -1$

$3 - m = 1 + 3m$ $\quad$ $3 - m = -1 - 3m$

$\therefore m = 1/2 \checkmark$ $\quad$ $2m = -4$

$\therefore m = -2 \checkmark$

QUESTION 2

a) i) $\frac{d}{dx}\left(f(x) + g(x)\right] = f'(x) + g'(x)$ ①

ii) $\frac{d}{dx}\left[f(x).g(x)\right] = f'(x)g(x) + f(x).g'(x)$ ①

iii) $\frac{d}{dx}\frac{1}{f(x)} = \frac{-f'(x)}{[f(x)]^2}$ ①

iv) $\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$ ①

b) $y = \frac{x+1}{x-1}$

i) $x = -1$ ①

ii) $\frac{dy}{dx} = \frac{(x-1) - (x+1)}{(x-1)^2}$ ✓①

$= \frac{-2}{(x-1)^2}$

iii) at $P(x=-1)$, $m = -\frac{2}{4} = -\frac{1}{2}$ ①

Q2 (cont)

(iv) Slope of Normal is 2 ✓
since $2x - \frac{1}{2} = -1$.

$\therefore$ at $P(-1, 0)$

$y - 0 = 2(x+1)$ ✓

ie. $y = 2x + 2$.

Correct working – 2 marks.
Use of point/gradient form MUST be shown.
Otherwise 1 mark for correctly stating gradient of normal.

(v) when $2x + 2 = \dfrac{x+1}{x-1}$ ✓

$2(x+1)(x-1) = x+1$

$\therefore x = -1, \ 1\frac{1}{2}$

$\therefore$ Normal meets curve again when $x = 1\frac{1}{2}$. ✓

$x = 1\frac{1}{2}$ must explicitly be shown as the answer to this part. ("State clearly ...")
1 for initial equation
1 for answer.

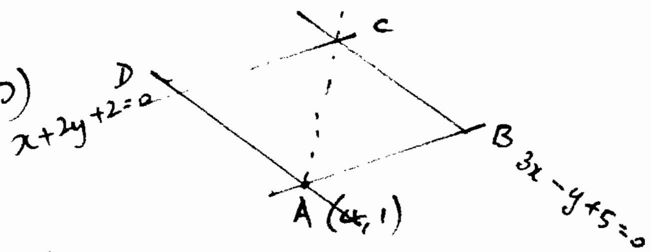
QUESTION 3

a) $\dfrac{1+\tan^2\theta}{1+\cot^2\theta} = \dfrac{\sec^2\theta}{\csc^2\theta}$

$= \tan^2\theta$

Correct answer 2 marks
Knowing 1 Pythagorean identity 1 mark.

(b)



i) Eqn AD:

$3(x-4) - 1(y-1) = 0$

ie $3x - y - 11 = 0$

Correct answer 2 marks.
Either correct gradient or ratio of coefficients 1 mark.

ii) Eqn AC is

$x + 2y + 2 + k(3x - y + 5) = 0$

and contains $(4, 1)$

$\therefore 4 + 2 + 2 + k(12 - 1 + 5) = 0$

$8 + 16k = 0$

✓ for set up

$\therefore$ Eqn AC is

$2x + 4y + 4 - 3x + y - 5 = 0$

ie. $-x + 5y - 1 = 0$

or $x - 5y + 1 = 0$

Correct equation – (3) marks.

(c) i) $8\cos x + 6\sin x = A\cos(x - \alpha)$

$\therefore A\sin\alpha = 6$

$A\cos\alpha = 8$

$\therefore A = \sqrt{8^2 + 6^2} = 10$

(1) for value of A

$\alpha = \tan^{-1}\frac{3}{4} \doteqdot 36.87^\circ$

(1) for α

$\therefore 8\cos x + 6\sin x = 10\cos(x - 36.87^\circ)$

ii) $10\cos(x - 36.87^\circ) = 5$

$\therefore \cos(x - 36.87^\circ) = \frac{1}{2}$

$\therefore$

$x - 36.87^\circ = 60^\circ, 300^\circ$

$\therefore x \doteqdot 96.87^\circ, 336.87^\circ$

Correct answers (2) marks.

-1 for each error

$\boxed{\text{QUESTION 4}}$

(a) i) $f(a+h) - f(a)$

$= \frac{1}{a+h} - \frac{1}{a}$

$= \frac{a - a - h}{a(a+h)}$

$= \frac{-h}{a(a+h)}$

(1) for correct answer.

ii) $\lim_{h \to 0} \frac{\frac{-h}{(a)(a+h)}}{h} = \lim_{h \to 0} \frac{-1}{(a)(a+h)}$

(1) for dividing by h

OR

$= -1$

(2) for correct answer

(iii) Gradient of the curve at point where $x = a$.
OR Slope of (tangent to) the curve at the point $(a, \frac{1}{a})$.

Answer must refer to $x = a$ or the point where $x = a$. (1) mark.
Eg $(a, \frac{1}{a})$.
Simply stating "the derivative" or "the slope" is in sufficient.

(iv) Gives the gradient of the secant in its final position as tangent to the curve.

ie Secant becomes tangent as h becomes zero.

(1) mark - idea of tangent as limiting position of secant is required.

b) i) $\sin(x + 2x)$

$= \sin x \cos 2x + \cos x \sin 2x$

$= \sin x(1 - 2\sin^2 x) + \cos x(2\cos x \sin x)$

$= \sin x - 2\sin^3 x + 2\cos^2 x \sin x$

$= \sin x - 2\sin^3 x + 2(1 - \sin^2 x)\sin x$

$= \sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

(b)
(1) for initial expansion.
(2) for showing another 2 correct substitutions - either $\sin 2x$ or $\cos 2x$ or
TOTAL (3) MARKS.

ii) $3\sin x - 4\sin^3 x = 2\sin x$

$\therefore 4\sin^3 x - \sin x = 0$

ie. $\sin x(4\sin^2 x - 1) = 0$

$\sin x(2\sin x - 1)(2\sin x + 1) = 0$

$\therefore \sin x = 0, \frac{1}{2}, -\frac{1}{2}$

$\therefore x = 0°, 30°, 150°, 210°, 330°, 180°, 360°$

6 correct angles - (3) MARKS
IF $0°, 360°$ missing - (2) MARKS.
OR
Either finding $\sin x = \frac{1}{2}$ or $x = 30°$ or equivalent (1) MARK.

QUESTION 5

a) i) $y = (x^2+1)^{\frac{1}{2}}$

$y' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x$

$= \dfrac{x}{\sqrt{x^2+1}}$ ✓

1 MARK.

ii) $y' = 2x\sqrt{x^2+1} + x^2 \cdot \dfrac{x}{\sqrt{x^2+1}}$

PRODUCT RULE CORRECT (1)

$= \dfrac{2x(x^2+1) + x^3}{\sqrt{x^2+1}}$

$= \dfrac{3x^3+2x}{\sqrt{x^2+1}}$

Correct answer (2) marks.

(b) i) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{1+t^2}{2t}$

(1) MARK.

ii) LHS $= \operatorname{cosec}\theta + \cot\theta$

$= \dfrac{1+t^2}{2t} + \dfrac{1-t^2}{2t}$

$= \dfrac{2}{2t}$

$= \dfrac{1}{t}$

$= \cot\dfrac{\theta}{2}$ = RHS.

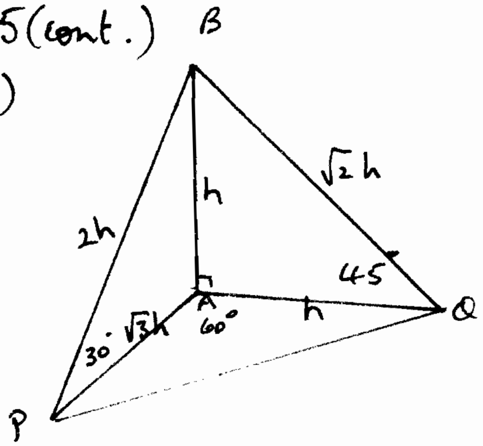
Correct substitution for $\cot\theta$ or equivalent (1) MARK.

OR

Correct down to $\frac{1}{t}$ – (2) MARKS.

Q5 (cont.)

(c)



i) $PA = \sqrt{3}h$, $QA = h$

① MARK - both correct

ii) Using cosine rule

$PQ^2 = (\sqrt{3}h)^2 + h^2 - 2(\sqrt{3}h)(h) \cdot \cos 60^\circ$

$= 3h^2 + h^2 - \sqrt{3}h^2$

$= 4h^2 - \sqrt{3}h^2$ as req^d.

Cosine rule correct - ① mark.

iii) In ΔPBQ

$PQ^2 = (2h)^2 + (\sqrt{2}h)^2 - 2 \cdot 2h \cdot \sqrt{2}h \cdot \cos\theta$

$= 4h^2 + 2h^2 - 4\sqrt{2}h^2 \cdot \cos\theta$

$= 6h^2 - 4\sqrt{2}h^2 \cdot \cos\theta$

PB and QB correct ① MARK.

OR

Correct expression for PQ^2 - ② MARKS

OR

$\therefore\therefore \quad (4-\sqrt{3})h^2 = 6h^2 - 4\sqrt{2}h^2 \cdot \cos\theta$

Correct to here - ③ MARKS

$\therefore \cos\theta = \dfrac{4-\sqrt{3}-6}{-4\sqrt{2}}$

$= \dfrac{2+\sqrt{3}}{4\sqrt{2}}$ as req^d.