SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005
YEAR 11 ACCELERATED
YEARLY EXAMINATION
(ASSESSMENT TASK #3)

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 72

- Attempt questions 1 – 4
- All questions are of equal value.

Examiner: *AM Gainford*

Section A

(Start a new booklet.)

Question 1. (18 Marks)

(a) Find the equation of the tangent at the point (1, 7) on the parabola $y = x^2 + 3x + 3$. Give your answer in general form ($Ax + By + C = 0$). **3**

(b) Differentiate, and simplify where possible: **8**

(i) $\dfrac{1}{2x^2 + 1}$

(ii) $\sin^2 3x$

(iii) $\cos x \sin x$

(iv) $\dfrac{e^x}{1 + e^x}$

(c) Find the second derivative of xe^{-x}. **2**

(d) Write down a primitive (indefinite integral) of $\sqrt{x^5}$. **2**

(e) Use Simpson's Rule with five function values to find an approximation to **3**

$$\int_1^3 \frac{1}{1 + x^2}\,dx\,.$$

Express your answer correct to four significant figures.

Section B

(Start a new booklet.)

Question 2. (18 Marks)

(a) Write down primitive functions of **6**

(i) $\cos 2x$

(ii) $e^{2x} + e^{-2x}$

(iii) $\dfrac{x}{1+x^2}$

(b) Two particles *A* and *B* move along a straight line so that their displacements from the origin at time *t* are given respectively by: **6**

$$x_A = 6t + 5 \qquad\qquad x_B = 3t^2 - t^3$$

(i) Which is moving faster when $t = 1$?

(ii) Show that the particles never travel at the same speed.

(iii) What is the acceleration of particle *B* when $t = 2$?

(iv) What is the maximum positive displacement of particle *B*?

(c) Evaluate **6**

(i) $\displaystyle\int_1^2 \left(x^3 + \frac{1}{x^2} \right) dx$

(ii) $\displaystyle\int_0^1 \frac{dx}{2x+1}$ (Leave your answer in exact form.)

(iii) $\displaystyle\int_0^2 \frac{2x}{\sqrt{x^2+1}}\, dx$ (Leave your answer in exact form.)

Section C

(Start a new booklet.)

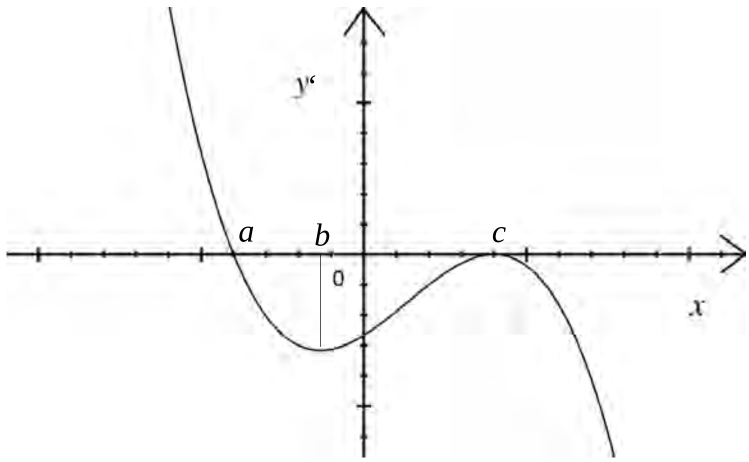
Question 3 (18 Marks)

(a) (i) Find the derivative of $\log_e(\cos x)$. **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan x \, dx$, leaving your answer in exact form.

(b) Sketch the graph of $y = 2\sin\frac{1}{2}x + 2$ from $x = -\pi$ to $x = \pi$. **3**

(c) **3**



The graph shows the derivative function y' of a function $y = f(x)$.

On a diagram in your answer booklet sketch the graph of a possible $f(x)$.

(d) Consider the curve $y = \frac{1}{12}\left(x^4 + 4x^3 + 6x^2\right)$ **5**

(i) Identify and determine the nature of any turning points.

(ii) Identify any points of inflexion.

(iii) Sketch the curve showing the above features, and intersections with the co-ordinate axes.

(e) The rate at which the solvent in nail polish will evaporate is given by **4**
$\frac{dV}{dt} = \frac{1}{400}\left(1 - \frac{t}{60}\right)$, where V ml is the volume of solvent present and t is in seconds.

(i) How much solvent will evaporate in the first 30 seconds?

(ii) When does the evaporation cease, and how much solvent will have evaporated then?

Section D

(Start a new booklet)

Question 4 (18 Marks)

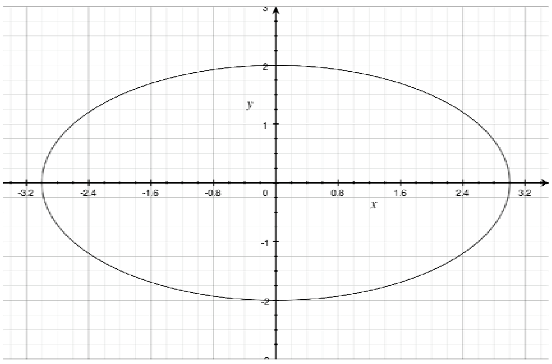
(a) Consider the curves $y = \ln x$ and $y = \ln 3x$. **7**

(i) Show that the tangents to the curves at $x = 1$ are parallel.

(ii) The line $x = 1$ cuts the curves at P and Q respectively. Find the distance PQ.

(iii) Show that the vertical distance between any two points on the curves with the same x-value is constant.

(b) A toy football has a long cross-section described by the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. (Measurements are in centimeters.) **3**



The part of the curve above the x-axis is rotated about the x-axis to sweep out the solid shape.

Find the volume of the football in exact form.

(c) The mass in grams of a certain radioactive substance may be expressed as $M = M_0 e^{-kt}$, where t is the time in years, and k and M_0 are positive constants. **4**

(i) Show that $\dfrac{dM}{dt} = -kM$.

(ii) At time $t = 0$, $M = 20$. Find M_0.

(iii) After 5 years the mass is 18 grams. Find the value of k (correct to 4 significant figures).

(iv) Find when the mass will be 10 grams.

(d) (i) Indicate by shading on a diagram the region of the plane bounded by the x-axis, the line $x = e$, and the curve $y = \ln x$. **4**

(ii) Calculate the area of this region, giving your answer in exact form.

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right)\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

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Mathematics

Sample Solutions

Section	Marker
A	PB
B	DH
C	CK
D	PSP

Section A

QUESTION 1.

(a) $y = x^2 + 3x + 3$

$\frac{dy}{dx} = 2x + 3$

at $(1,7)$ $m = 2+3 = 5$

T: $\frac{y-7}{x-1} = 5$

$y - 7 = 5x - 5$

$\boxed{5x - y + 2 = 0}$

(b) (i) $y = \frac{1}{2x^2+1} = (2x^2+1)^{-1}$

$y' = -(2x^2+1)^{-2} \times 4x.$

$\boxed{y' = \frac{-4x}{(2x^2+1)^2}}$

(ii) $y = \sin^2 3x = (\sin 3x)^2$

$y' = 2(\sin 3x)^1 . 3.\cos 3x$

$\boxed{y' = 6\sin 3x . \cos 3x}$

(iii) $y = \cos x . \sin x.$

$y' = \cos x . \cos x + -\sin x . \sin x.$

$\boxed{y' = \cos^2 x - \sin^2 x.}$

(iv) $y = \frac{e^x}{1+e^x}$

$y' = \frac{(1+e^x).e^x - e^x.e^x}{(1+e^x)^2}$ $\boxed{= \frac{e^x}{(1+e^x)^2}}$

(c) $y = xe^{-x}$

$y' = x \cdot -e^{-x} + e^{-x}$

$= -xe^{-x} + e^{-x}$

$y'' = -(-xe^{-x} + e^{-x}) + -e^{-x}$

$= \boxed{xe^{-x} - 2e^{-x}}$

(d) $\int \sqrt{x^5}\,dx = \int x^{5/2}\,dx$

$= \boxed{\frac{2}{7}x^{7/2} + c}$

(e)

x	1	1.5	2	2.5	3
y	$\frac{1}{2}$	$\frac{4}{13}$	$\frac{1}{5}$	$\frac{4}{29}$	$\frac{1}{10}$

$$\int_1^3 \frac{1}{1+x^2}\,dx \doteq \frac{0.5}{3}\left[\frac{1}{2} + \frac{1}{10} + 4\left(\frac{4}{13} + \frac{4}{29}\right) + 2 \times \frac{1}{5}\right]$$

$\boxed{= 0.4637}$ (4 SIG. FIGS)

Section B

(a) Write down primitive functions of

(i) $\cos 2x$

Solution: $\dfrac{\sin 2x}{2}$

(ii) $e^{2x} + e^{-2x}$

Solution: $\dfrac{e^{2x} - e^{-2x}}{2}$

(iii) $\dfrac{x}{1+x^2}$

Solution: $\frac{1}{2} \displaystyle\int \frac{2x}{1+x^2} = \frac{1}{2} \ln(1+x^2)$

(b) Two particles A and B move along a straight line so that their displacements from the origin at time t are given respectively by:

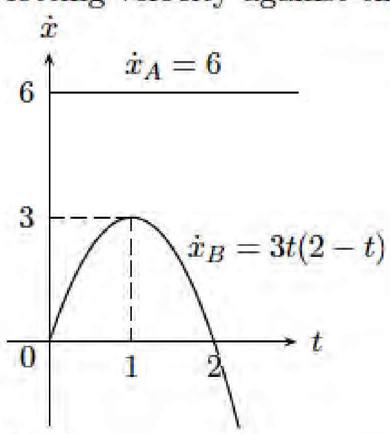
$$x_A = 6t + 5 \qquad\qquad x_B = 3t^2 - t^3$$

(i) Which is moving faster when $t = 1$?

Solution:
$\dot{x}_A = 6, \quad \dot{x}_B = 6t - 3t^2.$
When $t = 1, \quad \dot{x}_B = 3.$
$\therefore$ A is moving faster.

(ii) Show that the particles never travel at the same speed.

Solution: Method 1, Plotting velocity against time:



Clearly $\dot{x}_A \neq \dot{x}_B$.

Solution: **Method 2**, Assume travel at the same speed:

$$\begin{aligned} i.e.,\ 6 &= 6t - 3t^2, \\ t^2 - 2t + 2 &= 0. \\ \Delta &= 2^2 - 4 \times 2, \\ &= -4. \end{aligned}$$

$\therefore$ There is no real solution and travel at the same speed is not possible.

(iii) What is the acceleration of particle B when $t = 2$?

Solution:

$$\begin{aligned} \ddot{x}_B &= 6 - 6t \\ \text{When } t = 2, \quad \ddot{x}_B &= -6. \end{aligned}$$

(iv) What is the maximum positive displacement of particle B?

Solution: For maximum or minimum, $\dot{x}_B = 0$

$$\begin{aligned} i.e,\ 6t - 3t^2 &= 0, \\ t(2-t) &= 0, \\ t &= 0,\ 2. \end{aligned}$$

When $t = 0$, $x_B = 0$, $\ddot{x}_B = 6$.
When $t = 2$, $x_B = 4$, $\ddot{x}_B = -6$.
$\therefore$ Maximum positive displacement is 4.

(c) Evaluate

(i) $\displaystyle\int_1^2 \left(x^3 + \frac{1}{x^2}\right) dx$

Solution:

$$\begin{aligned} \int_1^2 \left(x^3 + \frac{1}{x^2}\right) dx &= \left[\frac{x^4}{4} - \frac{1}{x}\right], \\ &= \left(4 - \tfrac{1}{2}\right) - \left(\tfrac{1}{4} - 1\right), \\ &= 4\tfrac{1}{4}. \end{aligned}$$

(ii) $\displaystyle\int_0^1 \frac{dx}{2x+1}$ (Leave your answer in exact form.)

Solution:

$$\begin{aligned} \tfrac{1}{2}\int_1^2 \frac{2\,dx}{2x+1} &= \tfrac{1}{2}\left[\ln(2x+1)\right]_1^2, \\ &= \tfrac{1}{2}\left\{\ln 3 - \ln 1\right\}, \\ &= \tfrac{1}{2}\ln 3. \end{aligned}$$

(iii) Not counted towards exam.

Section C

QUESTION 3

(a) (i) $\log_e[\cos x] = -\frac{\sin x}{\cos x}$

$= -\tan x$

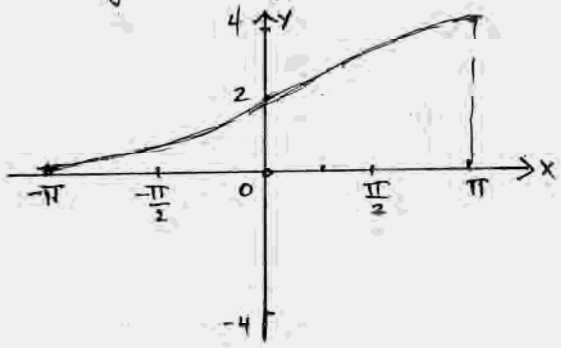
(ii) $\int_0^{\pi/4} \tan x \, dx = -\left[\log_e \cos x\right]_0^{\pi/4}$

$= -\left[\log_e \frac{1}{\sqrt{2}} - \log_e 1\right]$

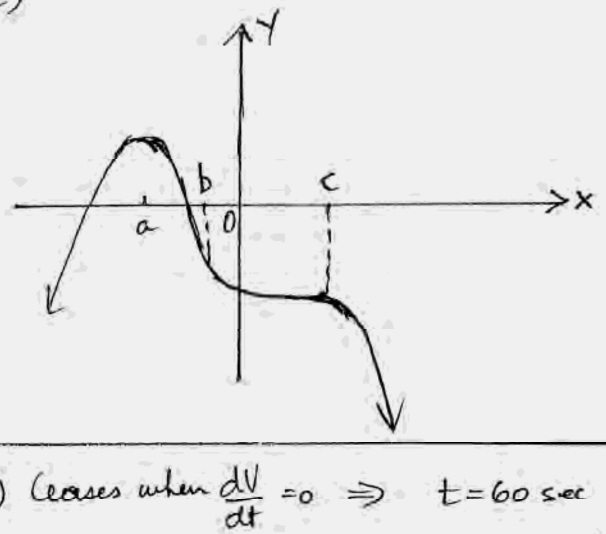
$= -\log_e \frac{1}{\sqrt{2}}$

or $= \frac{1}{2}\log_e 2$

b) $y = 2\sin\frac{x}{2} + 2$



(c)



(d) $y = \frac{1}{12}\left[x^4 + 4x^3 + 6x^2\right]$

(i) $y' = \frac{1}{3}x^3 + x^2 + x = 0$

when $x(x^2+3x+3) = 0$

ie when $x = 0$ $\qquad$ $x^2+3x+3=0$ has no real solutions

$y'' = x^2 + 2x + 1$

When $x=0$, $y''=1 \Rightarrow$

MIN T.P at $(0,0)$

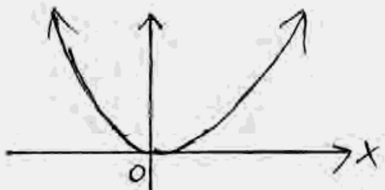
(ii) For P.O.I. $y''=0$ when $x=-1$

x	<	-1	>
y''	+	0	+

No change of concavity at $x=-1$

$\Rightarrow$ No P.O.I.

(iii)



(e) (i) $\frac{dV}{dt} = \frac{1}{400}\left[1 - \frac{t}{60}\right]$

$V = \int \frac{1}{400}\left[1 - \frac{t}{60}\right] dt$

ie $V = \frac{t}{400} - \frac{t^2}{48000} + C$

$t=0$, $V=C$

$t=30$, $V = \frac{9}{160} + C$

$\therefore V(30) - V(0) = \frac{9}{160}$ mL

$= 0.05625$ mL

(e) (ii) Ceases when $\frac{dV}{dt} = 0 \Rightarrow t = 60$ sec

$V(60) - V(0) = \left(\frac{60}{400} - \frac{3600}{48000} + C\right) - C$

$= \frac{3}{40}$

$= 0.075$ mL

Section D

(4)

(a) (i) $y=\ln x \Rightarrow y'=\dfrac{1}{x}$

$\therefore m_{x=1}=\dfrac{1}{1}=1$

$y=\ln 3x \Rightarrow y'=\dfrac{3}{3x}=\dfrac{1}{x}$

$\therefore m_{x=1}=\dfrac{1}{1}=1$

So the two tangents are parallel as they have the same gradient.

(ii) $x=1\Rightarrow$

$P(1,\ln 1)=P(1,0)$

$Q(1,\ln 3)$

$\therefore PQ=\ln 3$ units

(iii) $\ln 3x=\ln 3+\ln x$

So the vertical separation is always $\ln 3$ units.

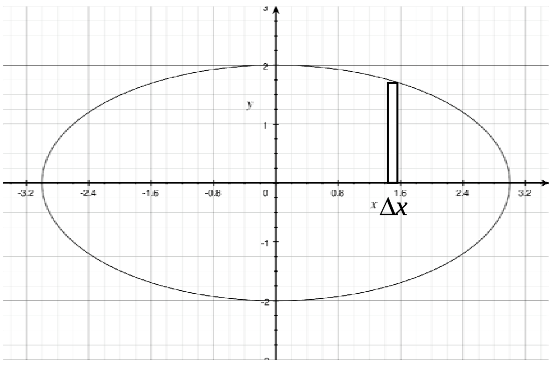
Alternatively

$P'(x,\ln x),\ x>0$

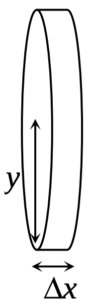
$Q'(x,\ln 3x).,x>0$

$\therefore P'Q'=\ln 3x-\ln x=\ln\left(\dfrac{3x}{x}\right)=\ln 3$

(b)



When the slice of thickness Δx is rotated about the x axis, it will approximately a cylinder of radius y units and height of Δx units. Let V be the volume of the toy and let ΔV be the volume of this approximate cylinder.



$$\Delta V\approx \pi y^2\Delta x$$
$$=\pi\left(\dfrac{36-4x^2}{9}\right)\Delta x$$

$$V=\frac{\pi}{9}\int_{-3}^{3}\left(36-4x^2\right)dx=2\times\frac{\pi}{9}\int_{0}^{3}\left(36-4x^2\right)dx \qquad \left[\text{the integrand is an even function}\right]$$

$$=\frac{2\pi}{9}\left[36x-\frac{4x^3}{3}\right]_0^3$$

$$=\frac{2\pi}{9}(108-36)$$

$$=16\pi\,\text{units}^3$$

(c) (i) $M=M_0e^{-kt}$

$$\text{LHS}=\frac{dM}{dt}$$

$$=-k\underbrace{M_0e^{-kt}}_{M}$$

$$=-kM$$

$$=\text{RHS}$$

(ii) $t=0, M=20=M_0e^0\Rightarrow M_0=20$

(iii) $M=20e^{-kt}$

$$\therefore 18=20e^{-5k}$$

$$\therefore e^{-5k}=\frac{9}{10}$$

$$\therefore -5k=\ln\frac{9}{10}$$

$$\therefore k=-\frac{1}{5}\ln\frac{9}{10}=\frac{1}{5}\ln\frac{10}{9}\approx 0\cdot 02107$$

(iv) $10=20e^{-kt}$

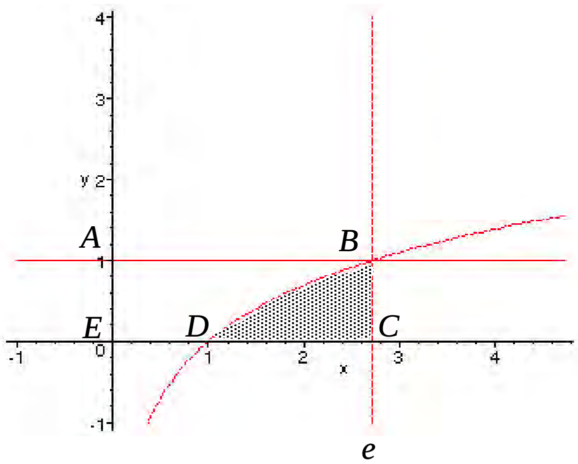
$$\therefore e^{-kt}=\frac{1}{2}$$

$$\therefore -tk=\ln\frac{1}{2}=-\ln 2$$

$$\therefore t=\frac{\ln 2}{k}\approx 32\cdot 9$$

So in approximately 33 years.

(d) (i)



(ii) The shaded area is given by $\int_1^e \ln x \, dx$.

This area is equal to the difference in the areas of *ABCD* and *ABDE*.

$y = \ln x \Rightarrow x = e^y$.

The area of *ABDE* is given by $\int_0^1 e^y dy$.

The area of *ABCD* is simply the rectangle with area e square units.

$$\begin{aligned}\int_1^e \ln x \, dx &= e - \int_0^1 e^y dy \\ &= e - \left[e^y\right]_0^1 \\ &= e - (e-1) \\ &= 1\end{aligned}$$

So the shaded area is 1 square unit.