SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2006

Yearly Examination

YEAR 11

Mathematics (2 unit) Accelerated

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each Section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 79

- Attempt questions 1 – 4
- Questions are not of equal value

Examiner: *A. Fuller*

Section A - Start a new booklet

Marks

Question 1 (20 marks)

(a) Evaluate $\log_2 16$ **1**

(b) Simplify $e^{\ln(4x-2)}$ **1**

(c) What is the exact value of $\sec\frac{2\pi}{3}$? **2**

(d) Differentiate the following:

(i) $(\log_e x)^3$ **2**

(ii) $\tan(\pi x)$ **1**

(iii) $\log_e \dfrac{8+x}{8-x}$ **2**

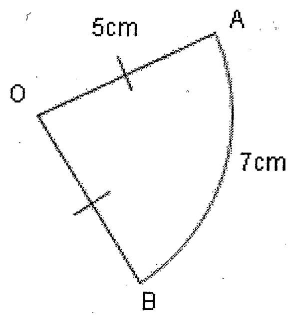
(iv) $\sin(e^x)$ **1**

(e) The third term of an arithmetic series is 18 and the eighth term is 58.

(i) Find the first term and the common difference. **3**

(ii) Show that the sum to n terms is given by $S_n = 4n^2 - 2n$. **2**

(f) In the diagram below, OAB is a sector of a circle with centre O. **2**
The radius OA is 5cm and the arc length AB is 7cm.
Find the size of angle AOB to the nearest degree.



(g) Find:

(i) $\int \frac{e^{5x} - e^{x}}{e^{2x}} dx$ **2**

(ii) $\int xe^{x^2+2} dx$ **1**

Section B – Start a new booklet

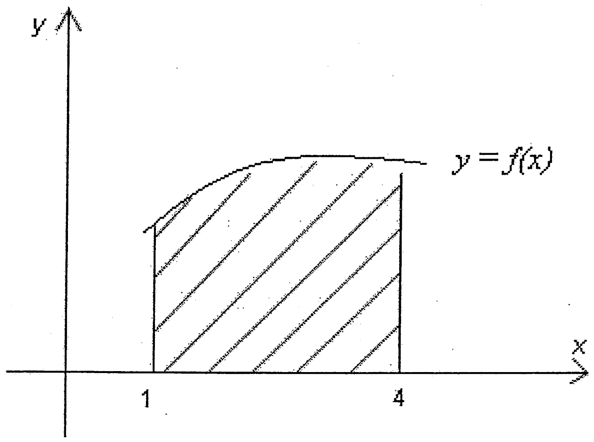
Question 2 (20 marks)

(a) The cost (C) of buying a house peaked in 2004 and then stabilised at a lower level. What does this statement imply about:

(i) $\frac{dC}{dt}$ prior to 2004? **1**

(ii) $\frac{d^2C}{dt^2}$ in 2004? **1**

(b)



When asked to calculate the volume of the solid generated when the shaded area is rotated about the x-axis a student correctly found that the volume was given by $V = \pi \int_1^4 9x dx$

(i) Calculate this volume. **2**

(ii) Find the equation of the curve $y = f(x)$. **1**

(ii) Hence, calculate the shaded area. **2**

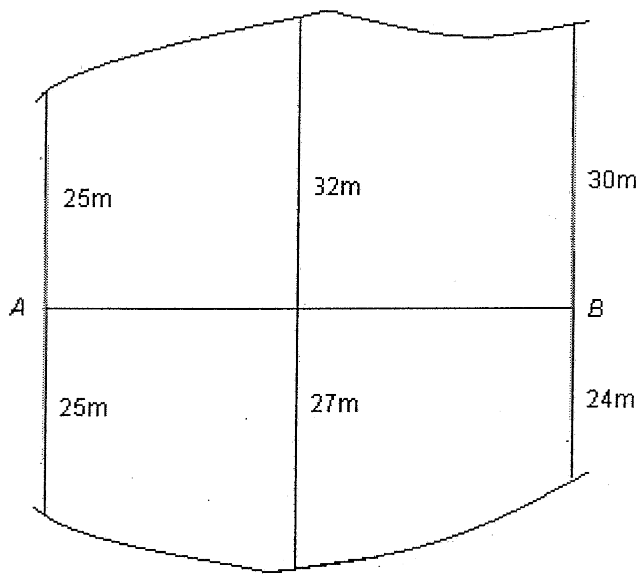
(c) The gradient function of a curve is given by $f'(x) = 3x^2 - 12$

(i) Find the x coordinates of any stationary points and determine their nature. **3**

(ii) If the curve passes through the point (1,2), find its equation. **2**

(d) Find a primitive of $\dfrac{4}{3-2x}$. **2**

(e) From a straight line AB 30 metres long perpendicular offsets are measured in both directions, at equal distances to two boundary lines as shown.



Using Simpson's Rule find an approximation for the area enclosed between the two boundaries. **3**

(f) A man invests \$1500 at the beginning of each year in a superannuation scheme. If the interest is paid at a rate of 8 % p.a. on the investment compounded annually. How much will this investment be worth after 20 years? **3**

Section C – Start a new booklet

Question 3 (21 marks)

(a) The position x in centimetres of a particle at a time t seconds is given by $x = 6\sin 3t$.

(i) Find the range of values x can take. **1**

(ii) When does the particle first come to rest? **2**

(iii) What is the total distance travelled by the particle in the first 6 seconds? **2**

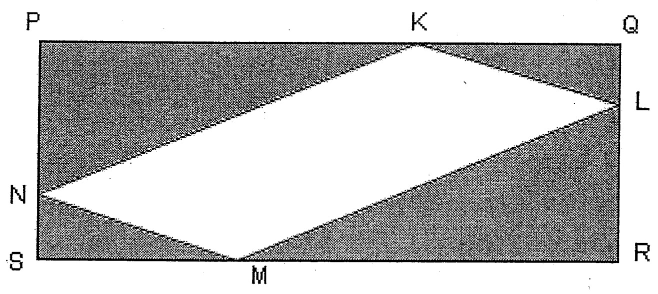
(b) The size of a colony of ants is given by the equation $P = 5000e^{kt}$ where P is the population after t days.

(i) If there are 8500 ants after 1 day, find the value of k correct to 2 decimal places. **2**

(ii) What is the size of the colony after 3 days? **1**

(iii) On what day will the colony triple in size? **2**

(c) In the diagram below, PQRS is a rectangle with PQ = 40 cm and SP = 10 cm. The shaded portions are cut away, leaving the parallelogram KLMN. QL = SN = x cm and QK = SM = $4x$ cm.



(i) Show that the area of the parallelogram KLMN is given by $A = 80x - 8x^2$ **2**

(ii) What is the domain of x ? **1**

(iii) Find the maximum area of parallelogram KLMN. **2**

(d) An infinite geometric series has a first term of -3, a common ratio of r, and a limiting sum of $4r$. Find the value(s) of r. **3**

(e) If $y = xe^x$

(i) Find $\frac{dy}{dx}$ **1**

(ii) Hence, evaluate $\int_0^1 xe^x dx$ **2**

Section D - Start a new booklet

Question 4 (18 marks)

(a) (i) Differentiate $y = \log_e(\cos x)$ **1**

(ii) Hence, find $\int \tan x\,dx$ **1**

(iii) Sketch $y = 1 + \tan x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$ clearly showing the x and y intercepts and any asymptotes. **2**

(iv) Using the integral in (ii) find the volume generated when the area under the curve $y = 1 + \tan x$ from $x = 0$ to $x = \frac{\pi}{4}$ is rotated about the x-axis. **3**

(b) A_n and B_n are two series given by

$A_n = 1^2 + 5^2 + 9^2 + 13^2 + \ldots\ldots + (4n-3)^2$

$B_n = 3^2 + 7^2 + 11^2 + 15^2 + \ldots\ldots$ for $n = 1, 2, 3, 4, 5, \ldots$

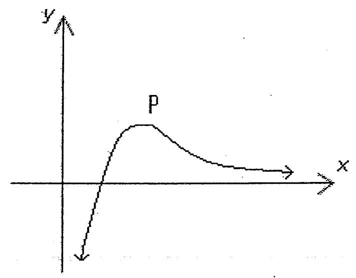
(i) Find the nth term of B_n. **1**

(ii) If $S_{2n} = A_n - B_n$ prove that $S_{2n} = -8n^2$ **2**

(iii) Hence, or otherwise, evaluate **2**

$$101^2 - 103^2 + 105^2 - 107^2 + \ldots\ldots + 2001^2 - 2003^2$$

(c) The graph of $y = \frac{\log_e x}{x}$ is sketched below. The point P is a maximum turning point.



(i) Find the coordinates of P. **2**

(ii) For what values of h are there two distinct solutions of the equation $\frac{\log_e x}{x} = h$? **1**

(iii) For what values of k are there two distinct solutions of the equation $\frac{\log_e x}{x} = kx$? **3**

END OF EXAM

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

QUESTION 1

(a) 4

(b) $4x-2$

(c) $\sec\frac{2\pi}{3} = \frac{1}{\cos 2\pi/3}$

$= \frac{1}{-1/2}$

$= -2$

(d) $3(\log_e x)^2 \times \frac{1}{x} = \frac{3}{x}(\log_e x)^2$

(ii) $\pi \sec^2(\pi x)$

(iii) $\log_e \frac{8+x}{8-x} = \log_e(8+x) - \log_e(8-x)$

$= \frac{1}{8+x} - \frac{1}{8-x} \times -1$

$= \frac{1}{8+x} + \frac{1}{8-x}$

$= \frac{16}{64-x^2}$

(iv) $e^x \cos e^x$

(e) (i) $a+2d = 18$ ①

$a+7d = 58$ ②

$5d = 40$ ②-①

$d = 8$

$a + 2\times 8 = 18 \quad a = 2$

$T_1 = 2 \quad d = 8$

(e) (ii) $S_n = \frac{n}{2}[2a+(n-1)d]$

$= \frac{n}{2}(4+8n-8)$

$= 4n^2 - 2n$

(f) $l = r\theta$

$7 = 5\theta$

$\theta = 7/5$ rads

$= \frac{7}{5} \times \frac{180}{\pi}$ degrees

$= 80°$ (nearest degree)

(g) (i) $\int \frac{e^{5x} - e^{x}}{e^{2x}}dx = \int (e^{3x} - e^{-x})dx$

$= \frac{1}{3}e^{3x} + e^{-x} + c$

(ii) $\frac{d}{dx} e^{x^2+2} = 2x e^{x^2+2}$

$\frac{1}{2}\int 2x e^{x^2+2}dx = \frac{1}{2}e^{x^2+2} + C$

SECTION B Question 2

(a)

(i) $\frac{dC}{dt} > 0$ 1

(ii) $\frac{d^2C}{dt^2} < 0$ 1

(b)

(i) $V = \pi \int_1^4 9x\,dx$

$= 9\pi \left[\frac{x^2}{2}\right]_1^4$ 1

$= \frac{135\pi}{2}$ units^3 1

(ii) $y^2 = 9x$ or $y = +3\sqrt{x}$ 1 (Since $y > 0$) $1 \le x \le 4$

(iii) Area $= \int_1^4 y\,dx$

$= \int_1^4 3\sqrt{x}\,dx$ 1

$= 2\left[x^{\frac{3}{2}}\right]_1^4$

$= 2\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= 2[8 - 1]$ 1

$= 14$ units^2

(c) $f'(x) = 3x^2 - 12$ & $f''(x) = 6x$

(i) $f'(x) = 3(x^2 - 4) = 3(x-2)(x+2)$ ①

$= 0$ when $x = \pm 2$

$f''(2) = 12 > 0$ ① MIN $(2, \ )$ & $f''(-2) = -12 < 0$ ① Max $(-2, \ )$

(ii) $y = \int (3x^2 - 12)\,dx$

i.e. $y = x^3 - 12x + c$ ①

$x = 1, y = 2$: $2 = 1 - 12 + c \Rightarrow \boxed{c = 13}$ ①

Eq^n is $\boxed{y = x^3 - 12x + 13}$

(d) $-\log_e(3 - 2x) + C$ ②

(e) Area $\doteqdot \frac{30-0}{6}\left[25 + 4(32) + 30\right]$ ① (915)

$+ \frac{30-0}{6}\left[25 + 4(27) + 24\right]$ ① 785

$\boxed{\text{Area} \doteqdot 1700\text{m}^2}$ ①

(f) 1st \$1500 → $1500(1.08)^{20}$

2nd \$1500 → $1500(1.08)^{19}$

⋮

last \$1500 → $1500(1.08)^1$

$\therefore S_{20} = 1500(1.08)^1 + 1500(1.08)^2 + \ldots 1500(1.08)^{20}$ ①

Geom. Series

$\therefore S_{20} = 1500(1.08)\frac{[1.08^{20} - 1]}{1.08 - 1}$ ①

$S_{20} = \$74134.38$ ①

2U ACCEL YEARLY 06

QUESTION 3 (a) $x = 6\sin 3t$

(i) $R: \{x: -6 \le x \le 6\}$ [1]

(ii) $x' = 18\cos 3t$

$x' = 0$ for $18\cos 3t = 0$

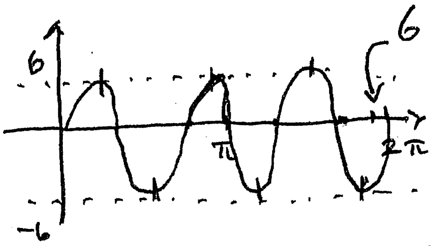
$\cos 3t = 0$

$3t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$t = \frac{\pi}{6}, \frac{3\pi}{6}, \ldots$

First comes to rest when $t = \frac{\pi}{6}$ [2]

(iii)



Distances (unsigned)

0 to $\frac{\pi}{6}$: 6

$\frac{\pi}{6}$ to $\frac{11\pi}{6}$: 12 (×5)

$\frac{11\pi}{6}$ to 6 : 1.494

Total Distance 67.494 [2]

(b) $P = 5000e^{kt}$

(i) When $t = 1$, $P = 8500$

$\therefore 8500 = 5000e^{k}$

$e^{k} = \frac{8.5}{5}$

$k = \ln\left(\frac{8.5}{5}\right)$

$\doteqdot 0.53$ [2]

(ii) When $t = 3$

$P = 5000e^{3k}$

$= 24565$ (24519) [1]

(iii) When $t = 0$, $P = 5000$

We seek t for $P = 15000$

$15000 = 5000e^{kt}$

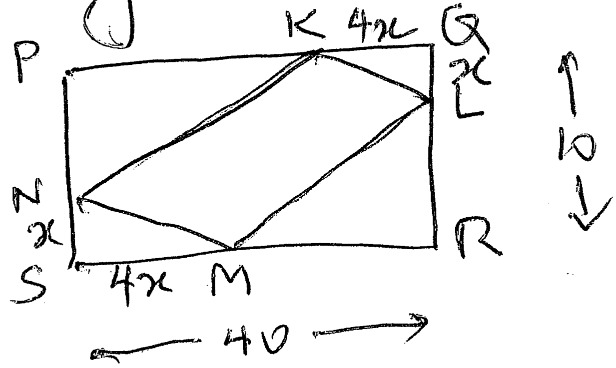
$3 = e^{kt}$

$t = \frac{\ln 3}{k}$

$= 2.07$ (2.07)

Triples on the third day. [2]

(c)



(i) Area Par'm $= 40 \times 10 - 4x^2 - (40-4x)(10-x)$

$= 400 - 4x^2 - 400 + 40x + 40x - 4x^2$

$= 80x - 8x^2$

[2]

Q3 (Contd) (2)

(c)(ii) $0 \le x \le 10$ [1]

(iii) $\frac{dA}{dx} = 80 - 16x$

$\frac{d^2A}{dx^2} = -16$

$\frac{dA}{dx} = 0$ when $80 - 16x = 0$

$x = 5$

This is a max as $\frac{d^2A}{dx^2} < 0$ when $x = 5$

$\therefore A_{max} = 80(5) - 8(5^2)$

$= 400 - 200$

$= 200\ cm^2$ [2]

(d) $a = -3 \quad S_\infty = 4r$

$4r = \frac{-3}{1-r}$

$4r - 4r^2 = -3$

$0 = 4r^2 - 4r - 3$

$0 = (2r-3)(2r+1)$

$r = \frac{3}{2}$ or $-\frac{1}{2}$

For limiting sum $|r| < 1$

$\therefore r = -\frac{1}{2}$ [3]

(e) $y = xe^x$

(i) $\frac{dy}{dx} = x.e^x + e^x.1$

$= e^x(x+1)$ [1]

(ii) $\int_0^1 xe^x\,dx$

$= \int_0^1 (xe^x + e^x - e^x)\,dx$

$= \int_0^1 ((xe^x + e^x) - e^x)\,dx$

$= [xe^x - e^x]_0^1$

$= (e - e) - (0 - 1)$

$= 1.$ [2]

Q 4 (a)

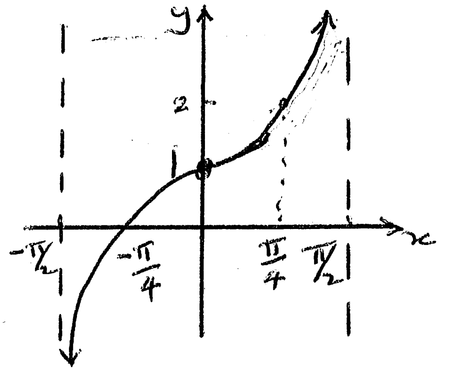
(i) $y = \log_e(\cos x)$

$y' = \frac{1}{\cos x} \times -\sin x$

$= -\tan x$ 1

(ii) $\int \tan x \, dx = -\log_e(\cos x) + c.$ 1

(iii)



2

(iv) $V = \pi \int_0^{\pi/4} (1 + \tan x)^2 dx$

$= \pi \int_0^{\pi/4} (1 + 2\tan x + \tan^2 x) dx$

$= \pi \int_0^{\pi/4} (\sec^2 x + 2\tan x) dx$

$= \pi [\tan x - 2\log_e(\cos x)]_0^{\pi/4}$

$= \pi\{[1 - 2\log_e \frac{1}{\sqrt{2}}] - [0 - 2\log_e 1]\}$

$= \pi[1 + \log_e 2]$ 3

(b) $A_n = 1^2 + 5^2 + \dots + (4n-3)^2$

$B_n = 3^2 + 7^2 + \dots$

(i) nth term of $B_n = (4n-1)^2$ 1

(ii) $S_{2n} = A_n - B_n$

$= 1^2 + 5^2 + \dots + (4n-3)^2$
$- (3^2 + 7^2 + \dots + (4n-1)^2)$

$= (1^2 - 3^2) + (5^2 - 7^2) + \dots + ((4n-3)^2 - (4n-1)^2)$

$= (1-3)(1+3) + (5-7)(5+7) + \dots + (4n-3-4n+1)(4n-3+4n-1)$

$= -2.4 + -2.12 + \dots + -2.(8n-4)$

$= -2(4 + 12 + \dots + (8n-4))$

$= -2.\frac{n}{2}(8 + (n-1).8)$

$= -n(8 + 8n - 8)$

$= -8n^2$ 2

(iii) $101^2 - 103^2 + 105^2 - 107^2 + \dots + 2001^2 - 2003^2$

$= 1^2 - 3^2 + 5^2 - 7^2 + \dots + 2001^2 - 2003^2$

$- 1^2 - 3^2 + 5^2 - \dots - 99^2$

$= (A_{501} - B_{501}) - (A_{25} - B_{25})$

$= -8 \times 501^2 - -8 \times 25^2$

$= -2003\,008$ 2

(c) (i) $y' = \frac{x \cdot \frac{1}{x} - \log_e x \cdot 1}{x^2}$

$= \frac{1 - \log_e x}{x^2}$

At P $y' = 0$

$\therefore 1 - \log_e x = 0$

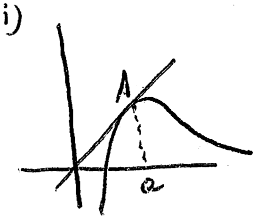
$\therefore x = e$

$\therefore$ P is $(e, \frac{1}{e})$ 2

(ii) $\frac{\log_e x}{x} = h$ has 2 solutions

for $0 < h < \frac{1}{e}$ 1

(iii)



Eqn of tangent at A is

$y - \frac{\log_e a}{a} = \frac{1 - \log_e a}{a^2}(x - a)$

If tangent passes through origin:

$\frac{-\log_e a}{a} = \frac{1 - \log_e a}{-a}$

$\therefore -\log_e a = -1 + \log_e a$

$\therefore 2\log_e a = 1$

$\log_e a = \frac{1}{2}$

$\therefore a = e^{\frac{1}{2}}$

$\therefore K = \frac{1 - \log_e e^{\frac{1}{2}}}{e}$

$= \frac{1 - \frac{1}{2}}{e}$

$= \frac{1}{2e}$ 3

$\therefore$ 2 solutions for $0 < K < \frac{1}{2e}$.

OR Similar to (i) and (ii) using $\frac{\log_e x}{x^2} = K$

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