SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2007

YEAR 11 ACCELERATED YEARLY EXAMINATION (ASSESSMENT TASK #3)

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each question is to be returned in a separate booklet.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.

Total Marks – 72

- Attempt questions 1 – 4
- All questions are of equal value.

Examiner: *AM Gainford*

Question 1. (18 Marks) **(Start a new booklet.)**

(a) Find the equation of the tangent to the curve $y = 3x + e^{-x}$ at the point where $x = 0$. Give your answer in general form ($Ax + By + C = 0$). **3**

(b) Differentiate, and simplify where possible: **8**

(i) $5(3-x)^9$

(ii) $\tan^2 2x$

(iii) $x \log_e (x+1)$

(iv) $\dfrac{1+e^x}{1-e^{-x}}$

(c) Find the second derivative of $x^2 \sin x$. **2**

(d) Write down a primitive (indefinite integral) of $\dfrac{1}{x\sqrt{x}}$. **2**

(e) Use Simpson's Rule with five function values to find an approximation to **3**

$$\int_{-2}^{2} 3^x \, dx \, .$$

Express your answer correct to one decimal place.

Question 2. (18 Marks) **(Start a new booklet.)**

(a) Write down primitive functions of **6**

(i) $\sin 3x$

(ii) $2e^{-2x}$

(iii) $\dfrac{1}{1+x} - \dfrac{1}{1-x}$ (Simplify your answer.)

(b) A particle moves along a straight line, its distance from the origin being given as a function by **6**

$$x = \cos^2 t\,.$$

(i) Sketch the displacement-time graph for $0 \le t \le 2\pi$.

(ii) Find the equations for velocity *v* and acceleration *f*, in terms of *t*.

(iii) Show that for all *t*, $f = 2 - 4x$.

(c) Evaluate **6**

(i) $\displaystyle\int_1^2 \left(\frac{x^4 + x}{x^3}\right) dx$

(ii) $\displaystyle\int_0^1 \frac{dx}{3x+1}$ (Leave your answer in exact form.)

(iii) $\displaystyle\int_0^{\ln 7} \left(1 - e^{-x}\right) dx$ (Leave your answer in exact form.)

Question 3 (18 Marks) **(Start a new booklet.)**

(a) (i) Show that the derivative of $\cot^3 x$ is $3\operatorname{cosec}^2 x - 3\operatorname{cosec}^4 x$. **3**

(ii) Hence find $\int \operatorname{cosec}^4 x \, dx$.

(b) Sketch the graph of $y = 3\cos\frac{1}{2}x + 3$ from $x = -\pi$ to $x = \pi$. **3**

(c) A function $f(x)$ is such that $f''(x) = 6x - 8$, and when $x = 0$, $f'(x) = 1$ and $f(x) = 2$. **2**

Find an expression for $f(x)$.

(d) Consider the curve $y = 7 + 4x^3 - 3x^4$ **6**

(i) Find the coordinates of the two stationary points.

(ii) Find all values of x for which $\frac{d^2y}{dx^2} = 0$.

(iii) Determine the nature of the stationary points.

(iv) Sketch the curve for the domain $-1 \le x \le 2$.

(e) The rate at which fuel burns, R kg/min, in a jet engine t minutes after it starts operation is given by the relation **4**

$$R = 10 + \left(\frac{10}{1+2t}\right).$$

(i) What is the rate of burn, R, after 7 minutes?

(ii) What value does R approach as t becomes very large?

(iii) Calculate the total amount of fuel burned in the first 7 minutes?

Question 4 (18 Marks) **(Start a new booklet)**

(a) Show that the triangle whose sides satisfy $3x + y + 1 = 0$, $x - 3y + 2 = 0$, $x + y - 1 = 0$ is right-angled. **5**

Find the length of the hypotenuse (in simplest surd form).

(b) Find the volume of the solid generated when the area bounded by the curves $y = \sec x$ and $y = x$, and the lines $x = 0$ and $x = 1$ is rotated about the x-axis. **3**

(c) A ship's engines are turned off while the ship is still moving through the water. The ship's speed, V metres per second, then decreases according to the rule **5**

$$V = Ae^{-kt}$$

where t represents time measured in seconds, and A and k are constants.

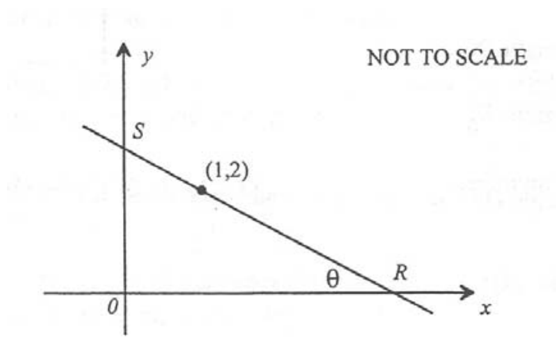
(iii) Show that $\dfrac{dV}{dt} = -kV$.

(iii) Initially the ship is moving at 12 metres per second. Find the value of A.

(iii) Six minutes after the engines are turned off, the speed has fallen to 5 metres per second. Evaluate k, correct to four significant figures.

(iv) What is the ship's speed after ten minutes? (Answer correct to four significant figures.)

(d) In the diagram the line RS meets the x and y axes at R and S respectively, and it passes through the point (1, 2). The angle ORS measures θ radians. **5**



(iii) Show that the equation of the line RS may be written as $y = -x\tan\theta + 2 + \tan\theta$.

(iii) Find the area of the triangle ORS in terms of $\tan\theta$.

(iii) Find the value of θ, correct to the nearest minute, for which this area is a minimum

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Additional Integral $\int \operatorname{cosec}^2 x dx = -\cot x + C$

2007 Yearly Accelerated Mathematics: **Solutions— Question 1**

1. (a) Find the equation of the tangent to the curve $y = 3x + e^{-x}$ at the point where $x = 0$. Give your answer in general form $[Ax + By + C = 0]$. $\boxed{3}$

Solution: When $x = 0$, $y = 1$.

$$\begin{aligned} \frac{dy}{dx} &= 3 - e^{-x}, \\ &= 2 \text{ when } x = 0. \\ \therefore \text{ The tangent is: } y - 1 &= 2(x - 0), \\ \textit{i.e. } 2x - y + 1 &= 0. \end{aligned}$$

(b) Differentiate, and simplify where possible: $\boxed{8}$

(i) $5(3 - x)^9$

Solution: $-1 \times 5 \times 9 \times (3 - x)^8 = -45(3 - x)^8$.

(ii) $\tan^2 2x$

Solution: $2 \times \sec^2 2x \times 2 \times \tan 2x = 4\sec^2 2x \tan 2x$.

(iii) $x \log_e(x + 1)$

Solution: $\ln(x + 1) + \dfrac{x}{x + 1}$.

(iv) $\dfrac{1 + e^x}{1 - e^{-x}}$

Solution:

$$\begin{aligned} \frac{(1 - e^{-x}) \times e^x - (1 + e^x) \times (-1) \times (-e^{-x})}{(1 - e^{-x})^2} &= \frac{e^x - 1 - e^{-x} - 1}{(1 - e^{-x})^2}, \\ &= \frac{e^x - e^{-x} - 2}{(1 - e^{-x})^2}. \end{aligned}$$

(c) Find the second derivative of $x^2 \sin x$. $\boxed{2}$

Solution:

$$\begin{aligned} \tfrac{d}{dx}(x^2 \sin x) &= 2x \sin x + x^2 \cos x. \\ \tfrac{d^2}{dx^2}(x^2 \sin x) &= 2 \sin x + 2x \cos x + 2x \cos x - x^2 \sin x, \\ &= (2 - x^2) \sin x + 4x \cos x. \end{aligned}$$

(d) Write down a primitive (indefinite integral) of $\dfrac{1}{x\sqrt{x}}$. $\boxed{2}$

Solution: $$\int x^{-3/2}\,dx = -2x^{-1/2} + c, \\ = -\frac{2}{\sqrt{x}} + c.$$

(e) Use Simpson's Rule with five function values to find an approximation to $\boxed{3}$

$$\int_{-2}^{2} 3^x\,dx.$$

Express your answer correct to one decimal place.

Solution:

x	-2	-1	0	1	2
3^x	0.111	0.333	1	3	9

$$\int_{-2}^{2} 3^x\,dx \approx \tfrac{1}{3}\{1\times 0.111 + 4\times 0.333 + 2\times 1 + 4\times 3 + 1\times 9\}, \\ \approx 8.1.$$

Question 2

a) i) $\int \sin 3x \, dx = -\frac{1}{3}\cos 3x + c$ (2)

ii) $\int 2e^{-2x} dx = \frac{2}{-2} e^{-2x} + c$

$= -e^{-2x} + c$ (2)

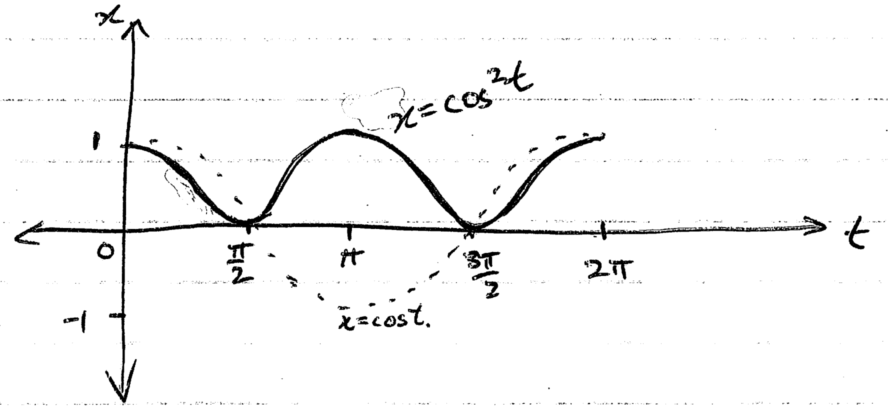
iii) $\int \left(\frac{1}{1+x} - \frac{1}{1-x}\right) dx$

$= \int \left(\frac{1}{1+x} + \frac{-1}{1-x}\right) dx$

$= \ln(1+x) + \ln(1-x) + c, \; -1 < x < 1$

$= \ln(1-x^2) + c$ (2)

b) i) $x = \cos^2 t$



(1)

ii) $x = (\cos t)^2$

$\dot{x} = 2(\cos t) . -\sin t$

$V = \underline{\dot{x} = -2\sin t \cos t}$

$u = -2\sin t \quad v = \cos t$

$u' = -2\cos t \quad v' = -\sin t$

$f = \underline{\ddot{x} = 2\sin^2 t - 2\cos^2 t}$ (3)

iii) $f = 2\sin^2 t - 2\cos^2 t$ $\qquad \sin^2 x + \cos^2 x = 1$

$f = 2(1 - \cos^2 t) - 2\cos^2 t$

$f = 2 - 4\cos^2 t$

$f = 2 - 4x$ (2)

c) i) $\int_1^2 \left(\frac{x^4+x}{x^3}\right) dx$

$= \int_1^2 (x + x^{-2})\,dx$

$= \left[\frac{x^2}{2} + \frac{x^{-1}}{-1}\right]_1^2$

$= \left[\frac{x^2}{2} - \frac{1}{x}\right]_1^2$

$= \frac{(2)^2}{2} - \frac{1}{2} - \left(\frac{(1)^2}{2} - \frac{1}{1}\right)$

$= 2$ (2)

ii) $\int_0^1 \frac{dx}{3x+1}$

$= \frac{1}{3}\int_0^1 \frac{3\,dx}{3x+1}$

$= \left[\frac{1}{3}\ln(3x+1)\right]_0^1$

$= \frac{1}{3}\ln(3(1)+1) - \frac{1}{3}\ln(3(0)+1)$

$= \frac{1}{3}\ln 4 - \cancel{\frac{1}{3}\ln 1}$

$= \frac{1}{3}\ln 4$ (2)

iii) $\int_0^{\ln 7} (1 - e^{-x})\,dx$

$= \left[x - \frac{e^{-x}}{-1}\right]_0^{\ln 7}$

$= \left[x + e^{-x}\right]_0^{\ln 7}$

$$= \ln 7 + e^{-\ln 7} - (0 + e^{0})$$

$$= \ln 7 + e^{\ln 7^{-1}} - 1$$

$$= \ln 7 + \frac{1}{7} - 1$$

$$= \ln 7 - \frac{6}{7}$$ 2

QUESTION 3

(a) (i) Given $\int \operatorname{cosec}^2 x\,dx = -\cot x$

$$\Rightarrow \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\therefore \frac{d}{dx}(\cot^3 x) = 3(\cot^2 x).\frac{d}{dx}\cot x$$

$$= 3(\cot^2 x).-\operatorname{cosec}^2 x$$

$$= -3\cot^2 x.\operatorname{cosec}^2 x$$

$$= -3(\operatorname{cosec}^2 x - 1).\operatorname{cosec}^2 x$$

$$= 3\operatorname{cosec}^4 x - 3\operatorname{cosec}^2 x$$

(ii) $3\operatorname{cosec}^4 x = 3\operatorname{cosec}^2 x - \frac{d}{dx}(\cot x)^3$

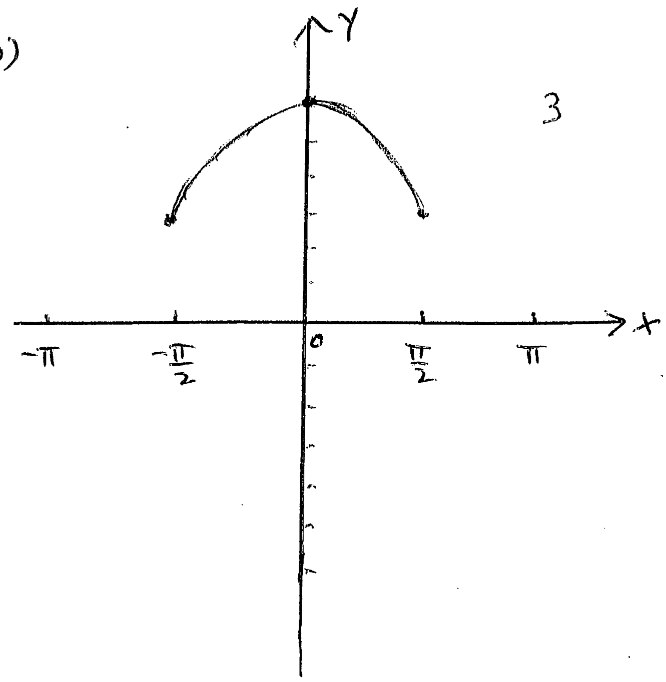
$$\operatorname{cosec}^4 x = \operatorname{cosec}^2 x - \frac{d}{dx}$$

$$\operatorname{cosec}^4 x = \frac{d}{dx}\left[-\cot x - \frac{1}{3}(\cot x)^3\right]$$

$$\therefore \int \operatorname{cosec}^4 x\,dx = -\cot x - \frac{1}{3}\cot^3 x + c$$

$1\frac{1}{2}$

(b) 3



(c) $f''(x) = 6x - 8$ 2

$$f'(x) = 3x^2 - 8x + c \quad \tfrac{1}{2}$$

$$f'(0) = 3(0) - 8(0) + c = 1$$

$$\Rightarrow \boxed{c = 1} \quad \tfrac{1}{2}$$

$$\therefore f'(x) = 3x^2 - 8x + 1$$

and $f(x) = x^3 - 4x^2 + x + c_1$ $\tfrac{1}{2}$

$$f(0) = 0 - 0 + 0 + c_1 = 2$$

$$\Rightarrow \boxed{c_1 = 2} \quad \tfrac{1}{2}$$

$$\therefore f(x) = x^3 - 4x^2 + x + 2$$

(d) $y = 7 + 4x^3 - 3x^4$

(i) St. pts $y' = 12x^2 - 12x^3 = 0$ (2)

ie $-12x^2(x-1) = 0$

$\therefore \begin{pmatrix} x=0 \\ y=7 \end{pmatrix}$ or $\begin{pmatrix} x=1 \\ y=8 \end{pmatrix}$

(ii) $y'' = 24x - 36x^2 = 0$

when $12x(2-3x) = 0$ (1)

ie when $\boxed{x=0}$ or $\boxed{x=\frac{2}{3}}$

(iii) * When $x=0$, $y''=0$ $\therefore$ possible P.O.I. (1)

x	$-\frac{1}{3}$	0	$\frac{1}{3}$
y''	-12	0	4

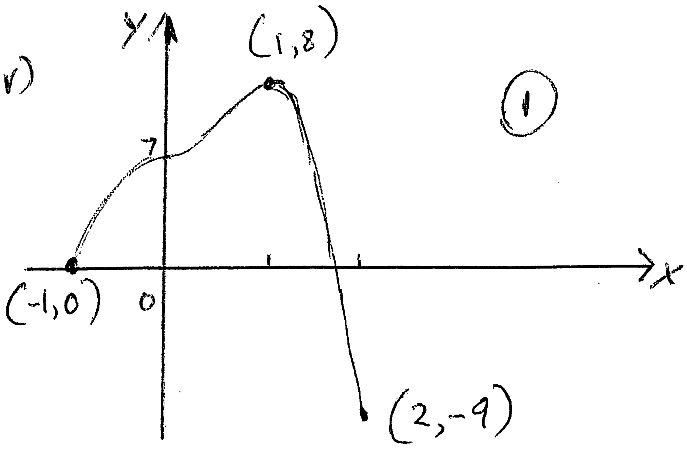
Change of concavity $\Rightarrow$ H.P.O.I at $(0,7)$

* When $x=1$, $y''=-12<0$

$\Rightarrow$ Max T.P. at $(1,8)$ (1)

Note: Not required to test $x=\frac{2}{3}$ since it is not a stationary point.

(iv) (1)



(e) $R = 10 + \left(\frac{10}{1+2t}\right)$

(i) $t=7$, $R = 10\frac{2}{3}$ kg/min (1)

(ii) $\lim_{t\to\infty}\left(10 + \frac{10}{1+2t}\right) = 10$ (1)

Since $\frac{10}{1+2t} \to 0$ as $t \to \infty$

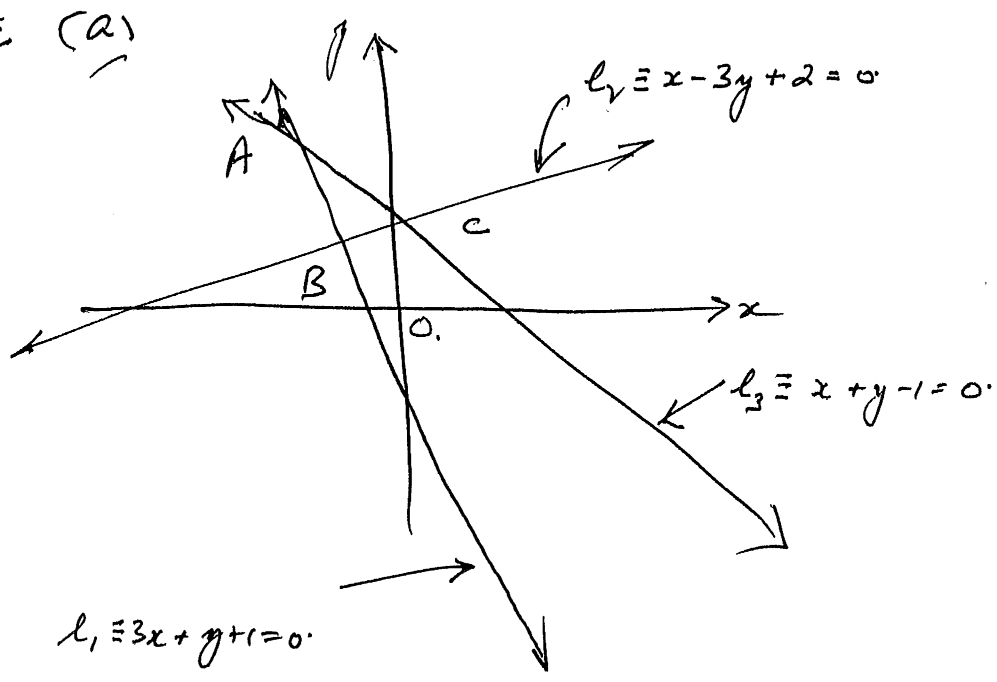
(iii) $A = \int_0^7 \left(10 + \frac{10}{1+2t}\right) dt$

$= \left[10t + \frac{10}{2}\log_e(1+2t)\right]_0^7$

$= \left(70 + \frac{10}{2}\ln 15\right)$ kg

$\doteqdot 83.5$ kg (2)

Q4. (a)



now $l_1 \equiv y = -3x-1$. $\therefore$ slope $m_1 = -3$.

$l_2 \equiv y = \frac{1}{3}x + \frac{2}{3}$ $\therefore$ slope $m_2 = \frac{1}{3}$

($\checkmark\checkmark$)

$\therefore$ Right-angle at B because $-3 \times \frac{1}{3} = -1$.

Solving l_1 and l_3 we get the co-ords of A ie $(-1, 2)$

Similarly solving l_2 & l_3 we get $C\left(\frac{1}{4}, \frac{3}{4}\right)$
(NB working to be shown)

$$\therefore AC = \sqrt{\left(-1-\tfrac{1}{4}\right)^2 + \left(2-\tfrac{3}{4}\right)^2}$$

$$= \sqrt{\left(\tfrac{5}{4}\right)^2 + \left(-\tfrac{5}{4}\right)^2}$$

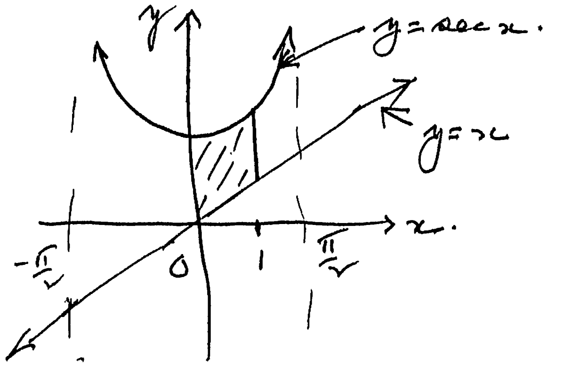
$$= \sqrt{\tfrac{50}{16}}$$

$$= \frac{\sqrt{50}}{4}$$

$$= \frac{5\sqrt{2}}{4}$$

($\checkmark\checkmark\checkmark$).

(b)



$$V = \pi \int_0^1 (\sec^2 x - x^2)\,dx.$$

$$= \pi \left[\tan x - \frac{x^3}{3} \right]_0^1$$

$$= \pi \left[\tan 1 - \frac{1}{3} \right] u^3 \quad (\checkmark\checkmark\checkmark)$$

(OR $\sim 1.224\pi\, u^3$ or $\sim 3.8456\, u^3$)

(c). (i) $V = A e^{-kt}.$

$$\frac{dV}{dt} = A.-k e^{-kt}.$$

$$= -k \left(A e^{-kt}\right)$$

$$\therefore \boxed{\frac{dV}{dt} = -kV} \quad (\checkmark)$$

(ii) when $t = 0$.

$V = 12.$

$\therefore 12 = A e^0$

$\therefore \boxed{A = 12.} \quad (\checkmark).$

(iii). Now when $t = 6$ mins $= 360$ seconds., $V = 5$.

$$\therefore 5 = 12 e^{-360k}.$$

$$\frac{5}{12} = e^{-360k}.$$

$$\ln \frac{5}{12} = -360k.$$

$$k = -\frac{1}{360} \ln \frac{5}{12}.$$

$$\boxed{k \doteqdot 2.432 \times 10^{-3}} \quad (\checkmark).$$

(iv) $V = 12 e^{-2.432 \times 10^{-3} \times 600}$

$\boxed{V \doteqdot 2.789}$ (4.s.f.s) $\quad (\checkmark\checkmark)$

(d) (i) $m = \tan(180 - \theta) = -\tan\theta$. ①

$\therefore$ using $y = mx + b$ where $(1, 2)$ lies on the line.

$2 = -\tan\theta \times 1 + b$.

$b = 2 + \tan\theta$. ②

$\therefore$ from ① & ②

$y = (-\tan\theta)x + 2 + \tan\theta$.

ie $\boxed{y = -x\tan\theta + 2 + \tan\theta}$ (✓).

(ii) OS is the y-intercept of the line in (i).

ie. $OS = 2 + \tan\theta$.

OR is the x-intercept of the line in (i)

ie. $OR = \dfrac{2 + \tan\theta}{\tan\theta}$.

$\therefore$ Area of $\triangle ORS = \frac{1}{2} \cdot (2 + \tan\theta)\left(\dfrac{2 + \tan\theta}{\tan\theta}\right)$

$= \boxed{\dfrac{(2 + \tan\theta)^2}{2\tan\theta}}$ (✓).

(iii) $$A = \frac{(2+\tan\theta)^2}{2\tan\theta}$$

$$= \frac{4 + 4\tan\theta + \tan^2\theta}{2\tan\theta}$$

$$= \frac{2}{\tan\theta} + 2 + \frac{\tan\theta}{2}$$

$$\frac{dA}{d\theta} = -2\operatorname{cosec}^2\theta + \tfrac{1}{2}\sec^2\theta$$

If $\frac{dA}{d\theta} = 0$

$$2\operatorname{cosec}^2\theta = \tfrac{1}{2}\sec^2\theta$$

$$\frac{2}{\sin^2\theta} = \frac{1}{2\cos^2\theta}$$

$$4 = \frac{\sin^2\theta}{\cos^2\theta}$$

$$\tan^2\theta = 4$$

$$\tan\theta = 2$$ (clearly θ is acute)

$$\boxed{\theta = 63^\circ 26'}$$

Test.

θ	60°	$63^\circ 26'$	70°
A'	$-\frac{2}{3}$	0	~ 2.5
	$\searrow$	$-$	$\nearrow$

(✓✓✓).

$\therefore$ $\boxed{\text{min. when } \theta = 63^\circ 26'}$