SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2009
YEAR 11 ACCELERATED
YEARLY EXAMINATION
(ASSESSMENT TASK #3)

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each question is to be returned in a separate booklet.
- All necessary working should be shown in every question.
- Answer in simplest exact form unless otherwise stated.
- Full marks may not be awarded for careless or badly arranged work.

Total Marks – 75

- Attempt questions 1 – 5
- All questions are of equal value.

Examiner: *AM Gainford*

Question 1. (15 Marks) **(Start a new booklet.)**

(a) Find the equation of the tangent to the curve $y = x^2 + \ln x$ at the point where $x = 1$. Give your answer in general form ($Ax + By + C = 0$). **3**

(b) Differentiate, and simplify where possible: **8**

(i) $3(2x+1)^7$

(ii) $\sin^2\left(\frac{x}{2}\right)$

(iii) $(x^2 - 1)\ln x$

(iv) $\frac{x^2+1}{x+1}$

(c) Find the second derivative of $\frac{e^x + e^{-x}}{2}$. **2**

(d) Write down a primitive (indefinite integral) of $\sqrt{1-x}$. **2**

Question 2. (15 Marks) **(Start a new booklet.)**

(a) Write down primitive functions of **6**

(i) $\cos 2x$

(ii) $4e^{-2x} - \dfrac{3}{x^2}$

(iii) $\dfrac{x-1}{x^2-1}$

(b) A particle moves along a straight line, its distance from the origin being given by $x = 2\sin t - t$. **6**

(i) Find an expression for the velocity of the particle.

(ii) In what direction is the particle moving at $t = 0$?

(iii) Determine when the particle first comes to rest.

(iv) When is the acceleration positive in $0 \le t \le 2\pi$?

(v) Calculate the total distance travelled by the particle in the first π seconds.

(c) (i) Use the Trapezoidal Rule with three function values to find an approximation to **3**

$$\int_0^2 \sqrt{16 - x^2}\, dx .$$

Express your answer correct to two decimal places.

(ii) Explain whether this approximation is greater than or less than the exact value.

Question 3 (15 Marks) **(Start a new booklet.)**

(a) Evaluate (Leave your answers in exact form.) **6**

(i) $\int_0^1 \left(\frac{2}{x+1} \right) dx$

(ii) $\int_0^{\frac{\pi}{3}} \cos 2x \, dx$

(iii) $\int_0^2 \left(2 - e^{2x} \right) dx$

(b) (i) Find the derivative of $x \ln x$. **3**

(ii) Hence find $\int \ln x \, dx$.

(c) Sketch the graph of $y = 2\sin \frac{1}{2}x + 3$ from $x = 0$ to $x = 2\pi$. **3**

(d) Find the volume of the solid generated when the area bounded by the curve $y = \sqrt{x-1}$, the x-axis, and the ordinate $x = 3$ is rotated about the x-axis. **3**

Question 4 (15 Marks) **(Start a new booklet)**

(a) Consider the curve $y = 3x^4 - 16x^3 + 24x^2$ **6**

(i) Show that the curve meets the x-axis only at the origin.

(ii) Find the turning points and determine their nature.

(iii) Find the points of inflexion.

(iv) Sketch the curve showing intercepts with the axes, turning points, and points of inflexion.

(b) The cost $\$c$ of an electrical appliance is decreasing each year according to the equation **4**

$$\frac{dc}{dt} = \frac{-100}{t^2}$$

where t is the time in years that the appliance has been on the market.

(i) If the item has been on the market for 5 years, find the rate of decrease in cost.

(ii) Find an equation for c given that one year after it appeared on the market the cost was \$160.

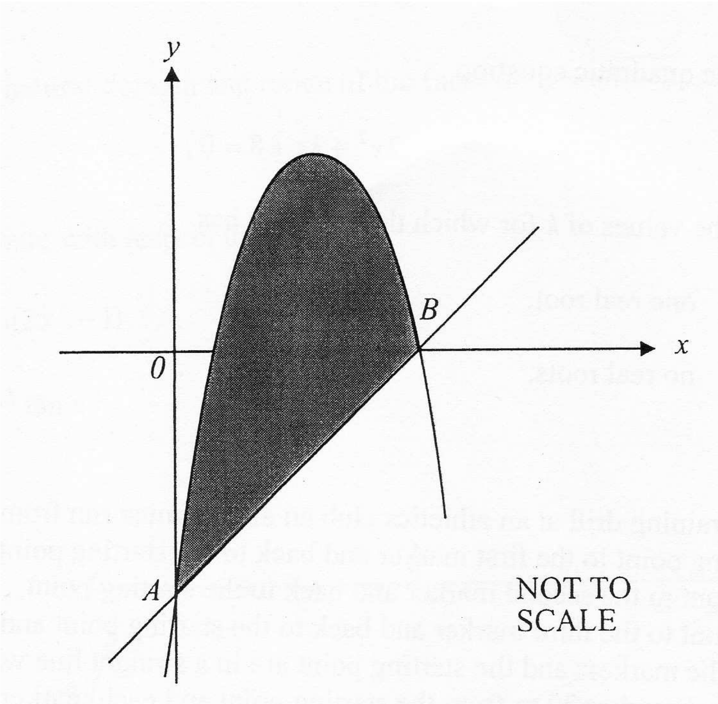
(iii) Find the long-term cost as t increases without bound.

(c) A function $f(x)$ is such that $f''(x) = 2 - \frac{2}{x^2}$, and when $x = 1$, $f'(x) = 0$, and $f(x) = -1$. Find an expression for $f(x)$. **2**

(d) Find the volume of the solid generated when the area bounded by the curves $y = \sec x$ and $y = x$, and the lines $x = 0$ and $x = 1$ is rotated about the x-axis. **3**

Question 5 (15 Marks) **(Start a new booklet)**

(a) **5**



The graphs of $y = x - 5$ and $y = -x^2 + 6x - 5$ intersect at the points A and B as shown on the diagram.

(i) Find the x-coordinates of point A and point B.

(ii) Find the area of the shaded region bounded by the two curves.

(b) In a laboratory, a food product is heated in an oven set at 210°C in an effort to kill harmful bacteria. The number of bacteria recorded when the food product is first placed in the oven is 1800. After ten minutes the number of bacteria recorded is 550. **5**
It is found that the number N, of bacteria remaining t minutes after being in the oven is given by

$$N = N_0 e^{-kt}$$

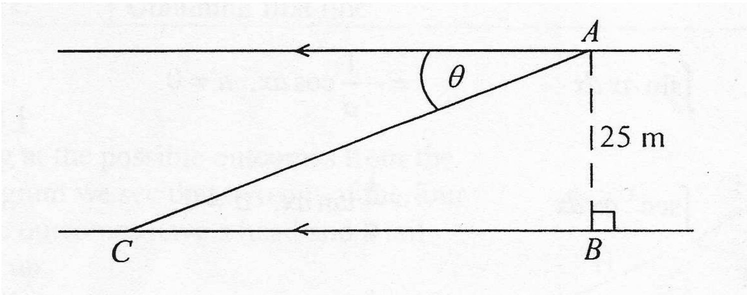
where N_0 and k are constants.

(i) Find the values of N_0 and k, correct to four significant figures.

(ii) If an acceptable number of bacteria present is 100 or less, find how long the food product should be in the oven, correct to the nearest second.

Question 5 continues overleaf.

(c) George is walking along a section of a still river, which is 25m wide, and has parallel banks. When he is at a point A, he spots his heart's desire, Mildred, directly opposite at point B. **5**
George dives into the river and swims in a straight line at an angle of θ to the riverbank, at a speed of 1 m/s.
Mildred, not having seen George, continues walking along her side of the river towards a point C at a speed of 2 m/s. George reaches Mildred's side of the river at the point C.



(i) Find an expression in terms of θ for the time taken by Mildred to walk from B to C.

(ii) Find an expression in terms of θ for the time taken by George to swim from A to C.

(iii) Show that George cannot arrive at point C at the same time as Mildred.

(iv) Find the least time by which George can miss Mildred at point C, to the nearest second.

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Question 1.

(a) $y = x^2 + \ln x.$

$\frac{dy}{dx} = 2x + \frac{1}{x}.$

Grad at $x = 1$

$m = 3.$

Point of contact at $x = 1.$

$y = 1$

$(1, 1)$

$y - 1 = 3(x - 1)$

$y = 3x - 2$

$3x - y - 2 = 0.$

(b)(i) $21(2x+1)^6 \times 2$

$= 42(2x+1)^6$

(ii) $2\sin(\frac{x}{2})\cos(\frac{x}{2}) \times \frac{1}{2}$

$= \sin(\frac{x}{2})\cos(\frac{x}{2}).$

(iii) $(x^2 - 1)\frac{1}{x} + 2x\ln x.$

$= x - \frac{1}{x} + 2x\ln x.$

(iv) $\dfrac{(x+1)2x - x^2 - 1}{(x+1)^2}$

$= \dfrac{x^2+2x-1}{(x+1)^2}$ 2

(c) $f(x) = \dfrac{e^x}{2} + \dfrac{e^{-x}}{2}$

$f'(x) = \dfrac{e^x}{2} - \dfrac{e^{-x}}{2}$

$f''(x) = \dfrac{e^x}{2} + \dfrac{e^{-x}}{2}$ 2.

$= \dfrac{e^x + e^{-x}}{2}$

(d) $\int \sqrt{1-x}\,dx = \int (1-x)^{\frac{1}{2}}\,dx$

$= \frac{2}{3}(1-x)^{\frac{3}{2}} \times -1 \quad + C$ 2

$= -\frac{2}{3}(1-x)^{\frac{3}{2}} + C$

YR 11 2U Accelerated Term 3 Yearly exam 2009.

2 (a) (i) $\frac{1}{2}\int 2\cos 2x \, dx = \frac{1}{2}\sin 2x + C$ 1

/15

(ii) $\int 4e^{-2x} - \frac{3}{x^2} \, dx$

$= \int 4e^{-2x} dx - \int 3x^{-2} dx$

$= \frac{-4}{2}\int -2e^{-2x} dx - 3\int x^{-2} dx$

$= -2e^{-2x} - \frac{3x^{-1}}{-1} + C$

$= -2e^{-2x} + \frac{3}{x} + C$ 3

(iii) $\int \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} dx = \int \frac{1}{(x+1)} dx = \ln(x+1) + C$ 2

(b) $x = 2\sin t - t$

(i) $\dot{x} = 2\cos t - 1$ 1

(ii) $t = 0$, $\dot{x} = 2\cos 0 - 1 = 2 - 1 = 1$ 1

moving to the right

(iii) when $\dot{x} = 0$, $2\cos t - 1 = 0$

$\cos t = \frac{1}{2}$

1st when $t = 60^\circ = \frac{\pi}{3}$ 1

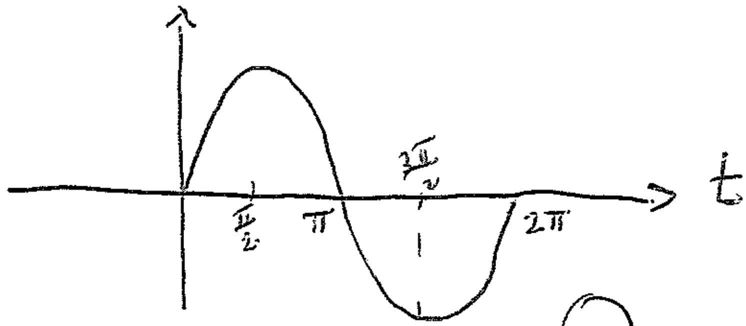
(iv) $x = 2\sin t - t$

$\dot{x} = 2\cos t - 1$

$\ddot{x} = -2\sin t$

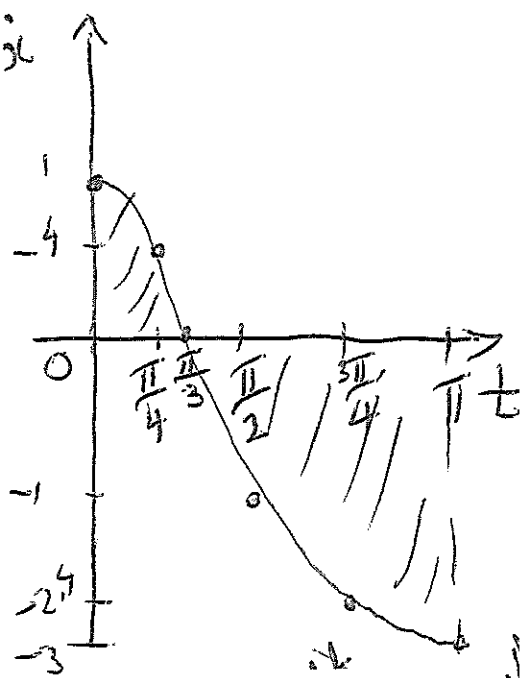
when is $-2\sin t > 0$

$\sin t < 0$



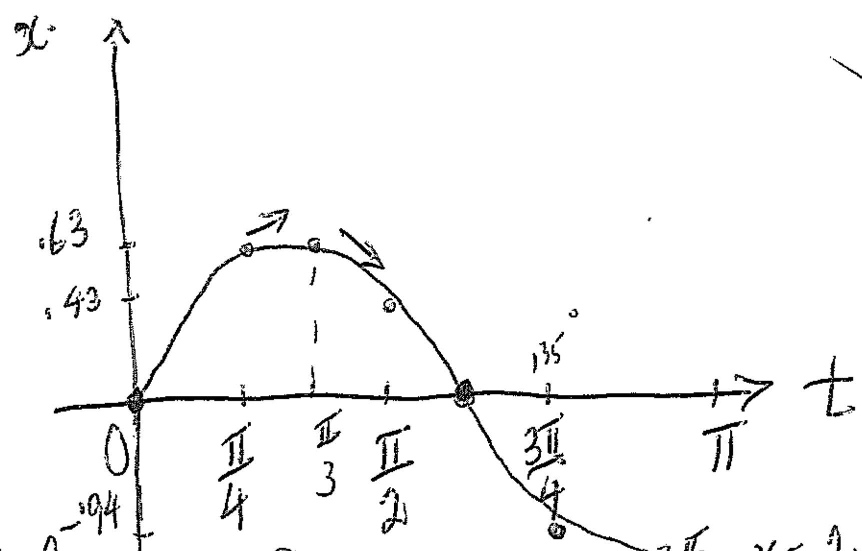
$\pi < t < 2\pi$ 1

$\dot{x} = 2\cos t - 1$



(v) total distance travelled in first π seconds

$x = 2\sin t - t$.



$t = 0 \quad x = 2\sin 0 - 0 = 0$

$t = \frac{\pi}{4} \quad x = 2\sin\frac{\pi}{4} - \frac{\pi}{4} = .63$

$t = \frac{\pi}{2} \quad x = 2\sin\frac{\pi}{2} - \frac{\pi}{2} =$

$t = \frac{3\pi}{4} \quad x = 2\sin\frac{3\pi}{4} - \frac{3\pi}{4}$

$t = \pi \quad x = 2\sin\pi - \pi = -\pi$

(v) $$\int_0^{\frac{\pi}{3}} (2\cos t - 1)\, dt + \left| \int_{\frac{\pi}{3}}^{\pi} (2\cos t - 1)\, dt \right|$$

$$= \left. 2\sin t - t \right]_0^{\frac{\pi}{3}} + \left| \left. 2\sin t - t \right]_{\frac{\pi}{3}}^{\pi} \right|$$

$$= \left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(2\sin 0 - 0\right) + \left| \left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(2\sin\pi - \pi\right) \right|$$

$$= \cancel{2} \times \frac{\sqrt{3}}{\cancel{2}} - \frac{\pi}{3} - 0 + \left| \cancel{2} \times \frac{\sqrt{3}}{\cancel{2}} - \frac{\pi}{3} + \pi \right|$$

$$= \sqrt{3} - \frac{\pi}{3} + \left| \sqrt{3} + \frac{2\pi}{3} \right|$$

$$= 2\sqrt{3} + \frac{\pi}{3} \ u$$

2

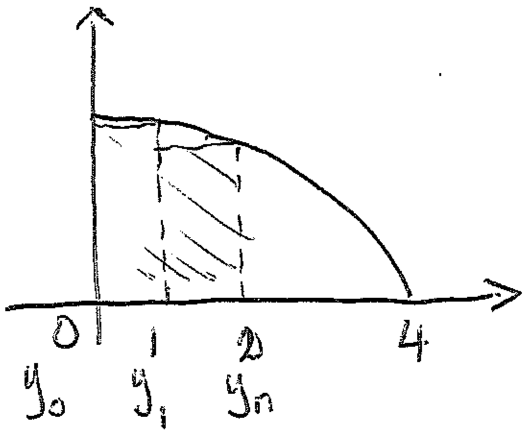
OR 4.51 u

C (i) Trapezoidal Rule:
3 function values

$$\int_0^2 \sqrt{16-x^2}\,dx \doteq \frac{1}{2}\left[(4+\sqrt{12})+2(\sqrt{15})\right]$$

$$\doteq 7.61 \ (2DP)$$

2



(ii) less than.
Rectangles of the trapezium formula is under the curve.

1

3) a) i) $\int_0^1 \left(\frac{2}{x+1}\right) dx$

$= 2 \int_0^1 \frac{dx}{x+1}$

$= 2\left[\ln(x+1)\right]_0^1$

$= 2\left[\ln(1+1) - \cancel{\ln(0+1)}\right]$

$= 2 \ln 2$

ii) $\int_0^{\frac{\pi}{3}} \cos 2x \, dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{3}}$

$= \left[\frac{1}{2}\sin 2\left(\frac{\pi}{3}\right) - \cancel{\frac{1}{2}\sin 2(0)}\right]$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{4}$

iii) $\int_0^2 \left(2 - e^{2x}\right) dx$

$= \left[2x - \frac{1}{2}e^{2x}\right]_0^2$

$= 2(2) - \frac{1}{2}e^{2(2)} - \left(2(0) - \frac{1}{2}e^{2(0)}\right)$

$= 4 - \frac{1}{2}e^4 + \frac{1}{2}$

$= \frac{9}{2} - \frac{1}{2}e^4$

b) i) $\dfrac{d(x\ln x)}{dx} = x \cdot \dfrac{1}{x} + 1 \cdot \ln x$

$= 1 + \ln x.$

ii) $\int \ln x \, dx = \int ((1 + \ln x) - 1) \, dx$

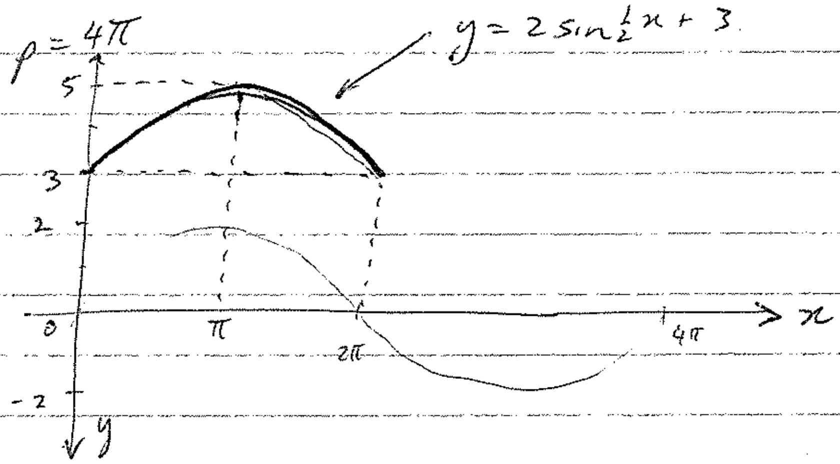
$= x\ln x - x + C$

c) $y = 2\sin\frac{1}{2}x + 3$

$p = \dfrac{2\pi}{n}$ $\quad a = 2$

$p = \dfrac{2\pi}{(\frac{1}{2})}$

$p = 4\pi$



d) $y = \sqrt{x-1}$

when $y = 0$

$x = 1.$

$V = \pi \int_a^b y^2 \, dx$

$V = \pi \int_1^3 (x-1) \, dx$

$V = \pi \left[\dfrac{x^2}{2} - x \right]_1^3$

$V = \pi \left[\dfrac{(3)^2}{2} - (3) - \left(\dfrac{(1)^2}{2} - (1) \right) \right]$

$V = \pi \left[\dfrac{9}{2} - 3 - \left(\dfrac{1}{2} - 1 \right) \right]$

$V = 2\pi$ units^3.

Question 4

(a) $y = 3x^4 - 16x^3 + 24x^2$

(i) $0 = x^2(3x^2 - 16x + 24)$

when $x = 0$ or $3x^2 - 16x + 24 = 0$

$\Rightarrow x = \dfrac{16 \pm \sqrt{256-288}}{6}$

(1) No real roots

$\therefore$ meets x axis at origin only

(ii) st. pts $\dfrac{dy}{dx} = 12x^3 - 48x^2 + 48x = 0$

ie $12x(x^2 - 4x + 4) = 0$

$\Rightarrow x = 0$ or $x = 2$ (1)

$\dfrac{d^2y}{dx^2} = 36x^2 - 96x + 48$

When $x = 0$, $\dfrac{d^2y}{dx^2} = 48 > 0$ (½)

$\Rightarrow$ MIN T.P. at $(0,0)$

At $x = 2$, $\dfrac{d^2y}{dx^2} = 0$

$\therefore$ Possible P.O.I.

(iii) P.O.I.

$\dfrac{d^2y}{dx^2} = 36x^2 - 96x + 48 = 0$

when $12(3x^2 - 8x + 4) = 0$

ie when $x = \dfrac{8 \pm \sqrt{16}}{6}$

$\therefore x = 2$ or $\dfrac{2}{3}$

Possible P.O.I.

Test $x = 2$

x	1	2	3
y''	−12	0	+84

(½)

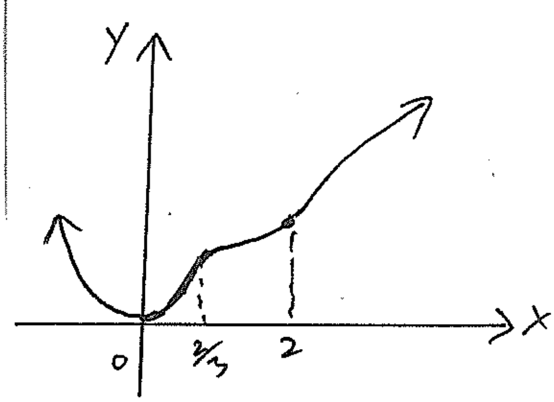
Change of concavity $\Rightarrow$ P.O.I. (Horizontal) at $(2, 16)$ (½)

Test $x = \frac{2}{3}$

x	1/3	2/3	1
y''	+20	0	−12

(½)

Change of concavity $\Rightarrow$ P.O.I. at $\left(\frac{2}{3}, 6\frac{14}{27}\right)$ (½)



(1)

$$\frac{dc}{dt} = -\frac{100}{t^2}$$

(i) When $t = 5$ (1)

$$\frac{dc}{dt} = \frac{-100}{5^2} = -4$$

$\therefore$ decreases by \$4. ~~every~~

(ii) $c = -100\int t^{-2}\,dt$

$c = +\frac{100}{t} + k$ (1)

When $t=1$, $c = 160$

$\Rightarrow 160 = \frac{100}{1} + k$ (1/2)

$\therefore \boxed{k=60}$

$\therefore \boxed{c = 60 + \frac{100}{t}}$ (1/2)

(iii) At $\to +\infty$, $c = 60$

since $\frac{100}{t} \to 0$

$\checkmark$ (1)

(c) $f''(x) = 2 - 2x^{-2}$

$f'(x) = 2x + 2x^{-1} + c$ (1/2)

$x=1$, $f'(1) = 2 + 2 + c = 0$

$\Rightarrow \boxed{c=-4}$ (1/2)

$\therefore f'(x) = 2x + 2x^{-1} - 4$

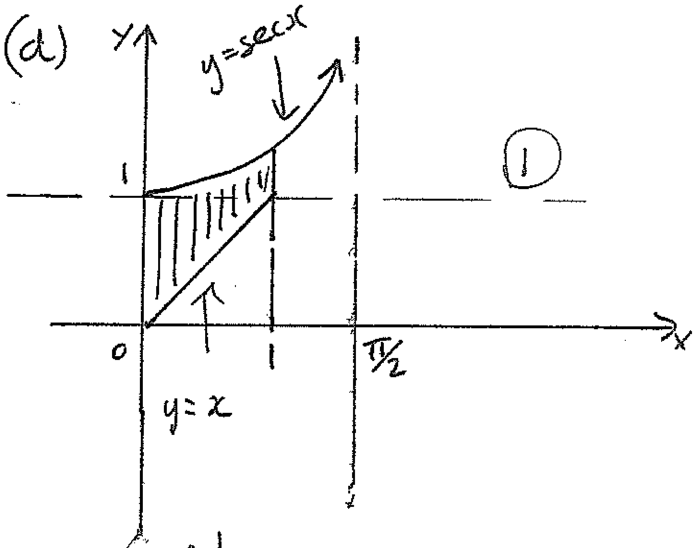
$f(x) = x^2 + 2\log_e x - 4x + c_1$ (1/2)

$f(1) = 1 + 0 - 4 + c_1 = -1$

$\Rightarrow \boxed{c_1 = 2}$ (1/2)

$\therefore \boxed{f(x) = x^2 + 2\ln x - 4x + 2}$

(d)



(1)

$V = \pi\int_0^1 (\sec^2 x - x^2)\,dx$ (1/2)

$= \pi\left[\tan x - \frac{x^3}{3}\right]_0^1$ (1)

$= \pi\left[\tan 1 - \frac{1}{3}\right]$ (1/2)

$=$

Q5 (a) (i) Solving $y = x - 5$ — ①

$\quad$ & $y = -x^2 + 6x - 5$ — ②

Let $x - 5 = -x^2 + 6x - 5$

$x^2 - 5x = 0$

$x(x-5) = 0$

$\boxed{x = 0, 5}$ ✓

$\therefore y = -5, 0$

$\boxed{A \text{ is } (0,-5) \quad B \text{ is } (5,0)}$ NOT REQUIRED

(ii) Shaded Region $= \displaystyle\int_0^5 \left[(-x^2+6x-5) - (x-5)\right] dx$

$= \displaystyle\int_0^5 (-x^2 + 5x)\, dx$

$= \left[\dfrac{-x^3}{3} + 5\dfrac{x^2}{2}\right]_0^5$

$= -\dfrac{125}{3} + \dfrac{125}{2}$

$= \boxed{\dfrac{125}{6}\ u^2}$ $\quad$ ($20\tfrac{5}{6}\ u^2$) ✓✓

(b) (i) $N = N_0 e^{-kt}$.

when $t=0$ $N = 1800$

$\therefore$ $1800 = N_0 e^0$

$\boxed{N_0 = 1800}$

when $t=10$, $N = 550$

$\therefore$ $550 = 1800 e^{-10k}$.

$e^{10k} = \frac{180}{55}$

$10k = \ln \frac{180}{55}$

$k = \frac{1}{10} \ln \frac{180}{55}$.

$\boxed{k = 0.1186}$ (correct to 4 sig. fig.)

(ii). now $N = 1800 e^{-0.1186t}$.

if $1800 e^{-0.1186t} \leq 100$

$18 e^{-0.1186t} \leq 1$

$e^{0.1186t} \geq 18$.

$0.1186t \geq \ln 18$

$t \geq \frac{\ln 18}{0.1186}$.

≥ 24.3707 mins

$\therefore$ $\boxed{24^{\circ}22' \text{ or } 24^{\circ}23'}$

(c) (i) Let $CB = d$

and clearly $A\hat{C}B = \theta$ (alternate angles)

$$\frac{25}{d} = \tan\theta$$

$$d = \frac{25}{\tan\theta}$$ now $v = \frac{d}{t} \Rightarrow t = \frac{d}{v}$.

$$\therefore t_1 = \frac{25}{\tan\theta} \div 2$$

$$\boxed{t_1 = \frac{25}{2\tan\theta}}$$

(ii) Let $AC = f \Rightarrow f = \frac{25}{\sin\theta}$.

$$\therefore t_2 = \frac{f}{1} = \boxed{\frac{25}{\sin\theta}}.$$

(iii) If $t_1 = t_2$

$$\frac{25}{\sin\theta} = \frac{25}{2\tan\theta}.$$

$$2\tan\theta = \sin\theta.$$

$$\frac{2\sin\theta}{\cos\theta} - \sin\theta = 0$$

$$\sin\theta\left(\frac{2}{\cos\theta} - 1\right) = 0$$

$\sin\theta = 0$ OR $\cos\theta = 2$

IMPOSSIBLE IMPOSSIBLE

$(|\cos\theta| \leq 1)$

$\therefore$ cannot arrive at C

at the same time.

(iv). Let the difference in times be T.

where

$$T = t_2 - t_1$$

$$= \frac{25}{\sin\theta} - \frac{25}{2\tan\theta}.$$

$$\frac{dT}{d\theta} = -25(\sin\theta)^{-2} \times \cos\theta + \frac{25}{2}(\tan\theta)^{-2}\sec^2\theta.$$

$$= \frac{25}{2}\left[\frac{\sec^2\theta}{\tan^2\theta} - \frac{2\cos\theta}{\sin^2\theta}\right]$$

$$= \frac{25}{2}\left[\frac{1}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta}\right]$$

$$= \frac{25}{2\sin^2\theta}(1 - 2\cos\theta)$$

now $\frac{dT}{d\theta} = 0$ when $1 - 2\cos\theta = 0$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

Test.

θ	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{5}$
T'	$2(1-\sqrt{2})$	0	5.3

<0   >0

∴ MIN.

∴ MIN TIME.

$$= \frac{25}{\sin\frac{\pi}{3}} - \frac{25}{2\tan\frac{\pi}{3}}$$

$$= 25\left(\frac{2}{\sqrt{3}} - \frac{1}{2\sqrt{3}}\right)$$

$$= 25 \times \frac{3\sqrt{3}}{6}$$

$$= \frac{25\sqrt{3}}{2}.$$

$$\doteqdot \boxed{22 \text{ secs}}$$