SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2010

Year 11 Yearly

Mathematics Accelerated

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Start each **NEW** question in a separate answer booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All answers must be given in exact simplified form unless otherwise stated.
- All necessary working should be shown in every question.

Total Marks – 72

- Attempt questions 1-5.

Examiner: *D.McQuillan*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

START A NEW ANSWER BOOKLET

Question One [16 marks]

(a) Find [4 marks]

(i) $\displaystyle\int \frac{dx}{2x-1}$

(ii) $\displaystyle\int 2^x\,dx$

(b) Evaluate [7 marks]

(i) $\displaystyle\int_{-5}^{-2} \frac{x^4-1}{x^2+1}\,dx$

(ii) $\displaystyle\int_{3\pi/8}^{3\pi/4} \sec^2\left(2\theta-\frac{\pi}{2}\right)d\theta$

(iii) $\displaystyle\int_{1}^{8} \frac{x-1}{\sqrt[3]{x^2}}\,dx$

(c) Solve the equation $2\ln 2x = \ln 4 + \ln(2x+3)$. [2 marks]

(d) Use the Trapezoidal Rule with five equal subintervals to estimate $\displaystyle\int_{1}^{6} \log_e\left(\frac{1}{x}\right)dx$ correct to two decimal places. [3 marks]

End of Question One

START A NEW ANSWER BOOKLET

Question Two [15 marks]

(a) Consider the function $y = x^3 - x^2 - x$. [5 marks]

(i) Find the value(s) of x for which the function:

(α) increases;

(β) changes from increasing to decreasing.

(ii) Sketch the graph of $y = x^3 - x^2 - x$, locating its turning points.

(b) Find the maximum value of $f(x) = x^3 + x^2 - 16x + 7$ on the interval $-5 \le x \le 5$. [2 marks]

(c) The volume, V, cubic metres, of a rectangular block of ice is given by $V = 2000 - 50t$, where t is time in hours. [4 marks]

(i) What is the initial volume of the block of ice?

(ii) At what rate is the ice melting at the end of 10 hours?

(iii) How long does it take to completely melt the ice?

(d) A particle moves in a straight line such that its displacement $x(t)$ metres from a fixed point O, at time t seconds is given by $x(t) = 3 + \ln(t+1)$. [4 marks]

(i) Show that the particle starts from 3 metres to the right of O.

(ii) Determine whether the particle ever is at rest. Why?

(iii) What happens to the acceleration as t approaches infinity?

End of Question Two

START A NEW ANSWER BOOKLET

Question Three [15 marks]

(a) Find the area bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. [4 marks]

(b) A botanist growing trees under experimental conditions discovered that for a particular species the diameter *D* (cm) of the tree increased according to the formula $D = D_0 e^{kt}$, where D_0 and *k* are constants and where *t* is the time in years. [5 marks]

(i) Given that the diameter *D* of the tree doubles every 5 years, calculate the value of *k* correct to 3 significant figures.

(ii) If at the beginning of 2005 the diameter of the tree was 50 cm, what will the diameter be at the beginning of 2018 (2 decimal places)?

(iii) In how many years will the diameter of the tree be three times its initial diameter?

(c) Sketch the graph of the function that satisfies the given conditions:
$\lim_{x \to 3} f(x) = -\infty$, $f''(x) < 0$ if $x \neq 3$, $f'(0) = 0$,
$f'(x) > 0$ if $x < 0$ or $x > 3$, $f'(x) < 0$ if $0 < x < 3$. [3 marks]

(d) Sketch $y = \dfrac{1}{1 - x^2}$ showing all major features. [3 marks]

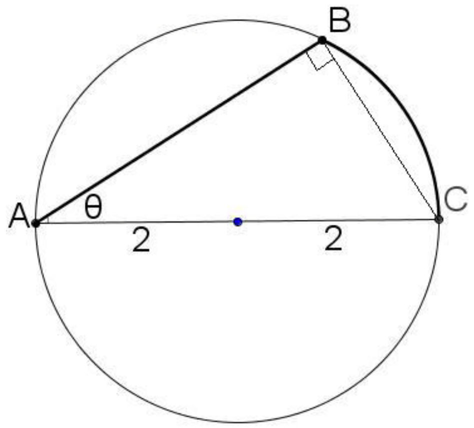
End of Question Three

START A NEW ANSWER BOOKLET

Question Four [13 marks]

(a) If f has a minimum value at c, show that the function $g(x) = -f(x)$ has a maximum value at c. [2 marks]

(b) Mark is at a point A on the shore of a circular lake with a radius of 2 km and he wants to be at the point C diametrically opposite A on the other side of the lake in the shortest possible time. [6 marks]



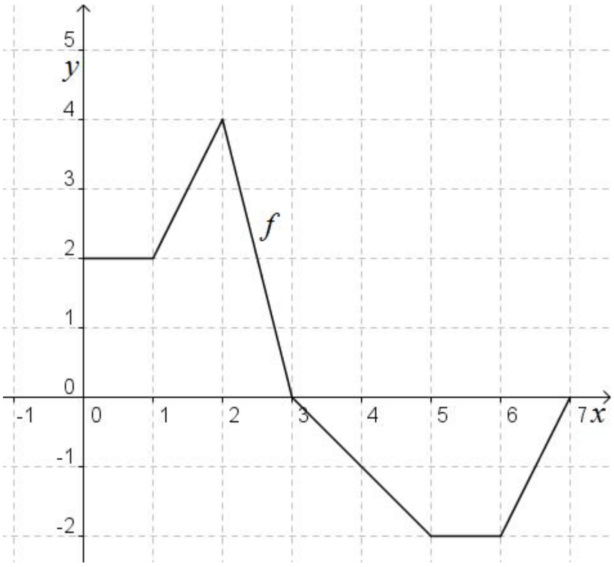
(i) Find the length of the arc BC in terms of θ.

(ii) Find the length of chord AB in terms of θ.

(iii) He can walk at the rate of 4 km/h and row a boat at 2 km/h. Find the time taken to get from A to B and from B to C in terms of θ.

(iv) At what angle θ to the diameter should he row?

(c) Let $g(x) = \int_0^x f(t)dt$ where *f* is the function whose graph is shown. [5 marks]



(i) Evaluate $g(0)$, $g(3)$ and $g(6)$.

(ii) On what intervals is g decreasing?

(iii) Where does g have a maximum value?

End of Question Four

START A NEW ANSWER BOOKLET

Question Five [13 marks]

(a) [8 marks]

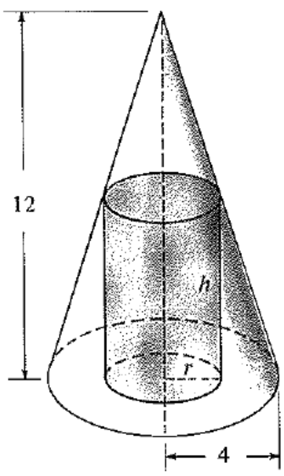
(i) Show that the points of intersection of $y = \sec x$ and $y = \frac{3}{\pi}x + 1$ for the interval $0 \le x \le \pi$ are (0, 1) and $\left(\frac{\pi}{3}, 2\right)$.

(ii) Use graphical means to show that there are the only two points of intersection for the interval $0 \le x \le \pi$.

(iii) Find the volume of the solid generated when $y = \sec x$ is rotated about the *x*-axis from $x = 0$ to $x = \frac{\pi}{3}$.

(iv) Hence find the volume generated when the region bounded by $y = \sec x$ and $y = \frac{3}{\pi}x + 1$ is rotated about the *x*-axis from $x = 0$ to $x = \frac{\pi}{3}$.

(b) Find the maximum volume of a right circular cylinder that can be inscribed in a cone of altitude 12 cm and base radius 4 cm, if the axes of the cylinder and cone coincide. [5 marks]



End of Exam

QUESTION 1

(a)(i) $\frac{1}{2}\log(2x-1) + C$

(ii) $y = 2^x$

$\log_e y = \log_e 2^x$

$= x\log_e 2$

$y = e^{x\log_e 2}$

$\int 2^x = \frac{1}{\ln 2} e^{x\ln 2}$

(iii) $\int_1^8 (x-1) \div x^{\frac{2}{3}}$

$= \int_1^8 (x^{\frac{1}{3}} - x^{-\frac{2}{3}})\,dx$

$= \left[\frac{3}{4}x^{\frac{4}{3}} - 3x^{\frac{1}{3}}\right]_1^8$

$= 12 - 6 - \left(\frac{3}{4} - 3\right)$

$= 8\frac{1}{4}$

(b) (i) $\int_{-5}^{-2} \frac{x^4-1}{x^2+1}\,dx = \int_{-5}^{-2}(x^2-1)\,dx$

$= \left[\frac{x^3}{3} - x\right]_{-5}^{-2}$

$= \frac{-8}{3} + 2 - \left[\frac{-125}{3} + 5\right]$

$= 36$

(ii) $\int_{3\pi/8}^{3\pi/4} \sec^2\left(2\theta - \frac{\pi}{2}\right)dx$

$= \frac{1}{2}\left[\tan\left(2\theta - \frac{\pi}{2}\right)\right]_{3\pi/8}^{3\pi/4}$

$\frac{1}{2}\left[\tan\pi - \tan\frac{\pi}{4}\right]$

$= -\frac{1}{2}$

(c) $2\log_e 2x = \log_e 4 + \log(2x+3)$

$\log_e(2x)^2 = \log_e 4(2x+3)$

$4x^2 = 8x + 12$

$x^2 - 2x - 3 = 0$

$x = 3, -1$

$x = 3 \quad (x > 0)$

(d) $\int_1^6 \log\frac{1}{x}\,dx = \frac{1}{2}\left[\log 1 + \log\frac{1}{6} + 2\left(\log\frac{1}{2} + \log\frac{1}{3} + \log\frac{1}{4} + \ldots\right.\right.$

$= \frac{1}{2}\left(0 - 1.79 + 2(-1.1239)\right)$

$= -5.68 \quad$ 2 d.p.

2) a) $y = x^3 - x^2 - x$

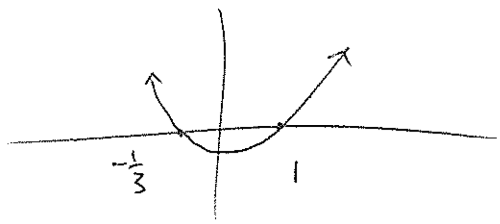
(i) $y' = 3x^2 - 2x - 1$

(α) $3x^2 - 2x - 1 > 0$

$$\frac{(3x-3)(3x+1)}{3} > 0$$

$(x-1)(3x+1) > 0$

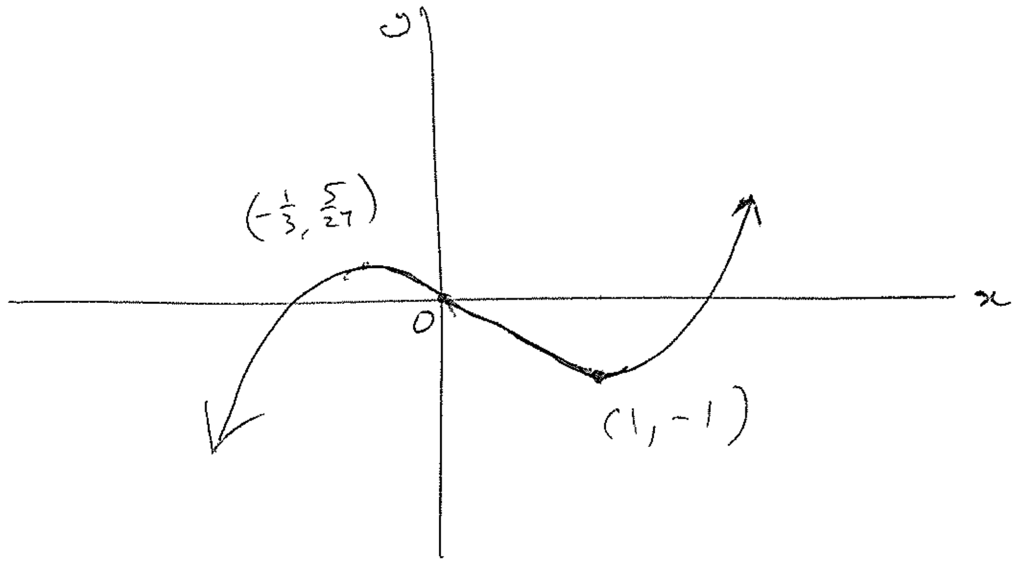
$x < -\frac{1}{3}$, $x > 1$



(β) $x = -\frac{1}{3}$. (from graph of derivative in (α))

ii). when $x = -\frac{1}{3}$ $\quad x = 1 \quad x = 0 \quad y=0$

$y = \frac{5}{27} \quad y = -1. \quad y = 0$



b) $f(x) = x^3 + x^2 - 16x + 7$

$f'(x) = 3x^2 + 2x - 16$

let $f'(x) = 0$.

$3x^2 + 2x - 16 = 0$

$\times$ -48
$+$ 2

$\dfrac{(3x+8)(3x-6)}{3} = 0$

$(3x+8)(x-2) = 0$.

$x = 2, -\frac{8}{3}$.

$f(-5) = -13$

$f(-\frac{8}{3}) = 37\frac{22}{27}$.

$f(2) = -13$

$\underline{f(5) = 77}$ $\therefore$ maximum value is 77.

c) $V = 2000 - 50t$

i) when $t=0$.

$V = 2000 - 50(0)$

$V = 2000$ m^3.

ii) $\dfrac{dV}{dt} = -50$

Ice is melting at a constant rate of $50\ m^3/h$.

iii) when $V=0$

$2000 - 50t = 0$

$50t = 2000$

$t = \dfrac{2000}{50}$

$t = 40$ hours

d) $x(t) = 3 + \ln(t+1)$

i) when $t=0$

$x(0) = 3 + \ln((0)+1)$

$= 3$ (since $\ln 1 = 0$)

$\therefore$ particle starts 3m to right of O.

ii) $v = \frac{dx}{dt} = \frac{1}{t+1} \neq 0$

$\therefore$ particle is never at rest.

iii) $v = \frac{1}{t+1}$

$v = (t+1)^{-1}$

$\frac{dv}{dt} = -(t+1)^{-2} \cdot 1$

$\frac{dv}{dt} = -\frac{1}{(t+1)^2}$

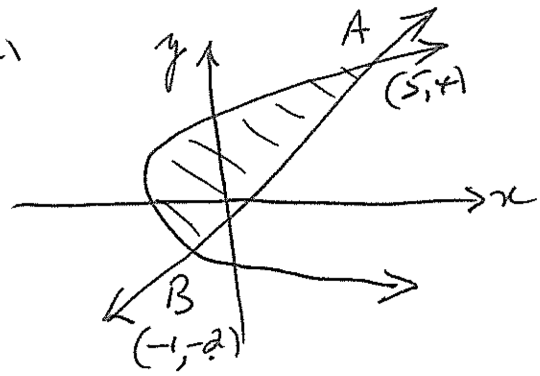
as $t \to \infty$

$-\frac{1}{(t+1)^2} \to 0^-$.

$\therefore$ acceleration approaches zero (as does velocity).

QUESTION THREE.

(a)



Firstly we need to find A & B

Solve $y = x - 1$ — (1)

& $y^2 = 2x + 6$ — (2)

$x^2 - 2x + 1 = 2x + 6.$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0.$

$x = -1, 5$

$\therefore y = -2, 4.$

NB if $y^2 = 2x + 6$

$x = \frac{1}{2}y^2 - 3$

$$\text{Area} = \int_{-2}^{4} \left[(y+1) - \left(\tfrac{1}{2}y^2 - 3\right) \right] dy.$$

(✓✓✓✓)

$$= \int_{-2}^{4} \left(y + 4 - \tfrac{1}{2}y^2 \right) dy$$

$$= \left[\frac{y^2}{2} + 4y - \frac{1}{6}y^3 \right]_{-2}^{4}$$

$$= \frac{16}{2} + 16 - \frac{64}{6} - \left(\frac{4}{2} - 8 + \frac{8}{6} \right)$$

$$= 8 + 16 - 10\tfrac{2}{3} - 2 + 8 - 1\tfrac{1}{3}.$$

$$= 30 - 12$$

$$= 18 \, u^2$$

(b) (i) $D = D_0 e^{kt}$

now $D = 2D_0$ when $t = 5$

$\therefore 2D_0 = D_0 e^{5k}$

$2 = e^{5k}$

$5k = \ln 2$

$k = \frac{1}{5} \ln 2.$

$\doteqdot 0.139$

(✓✓)

Q3 (CONTD)

(ii) $D = 50e^{0.139 \times 13}$

$= 304.6!$ (✓)

(iii) $3D_0 = D_0 e^{0.139t}$

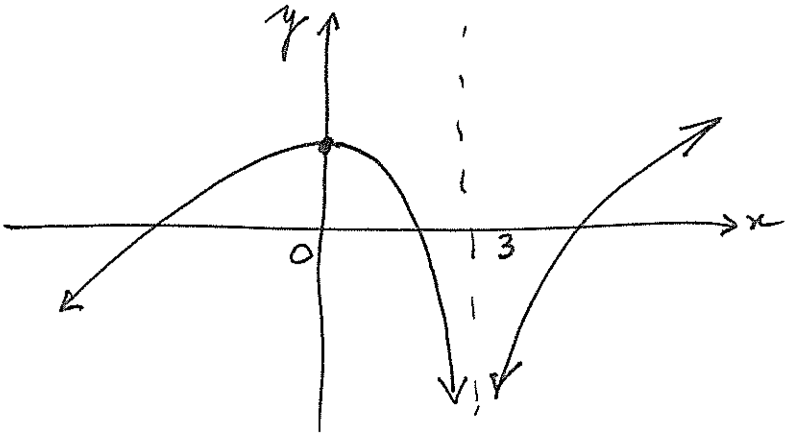
$3 = e^{0.139t}$ (✓✓)

$0.139t = \ln 3$

$t = \dfrac{\ln 3}{0.139}$

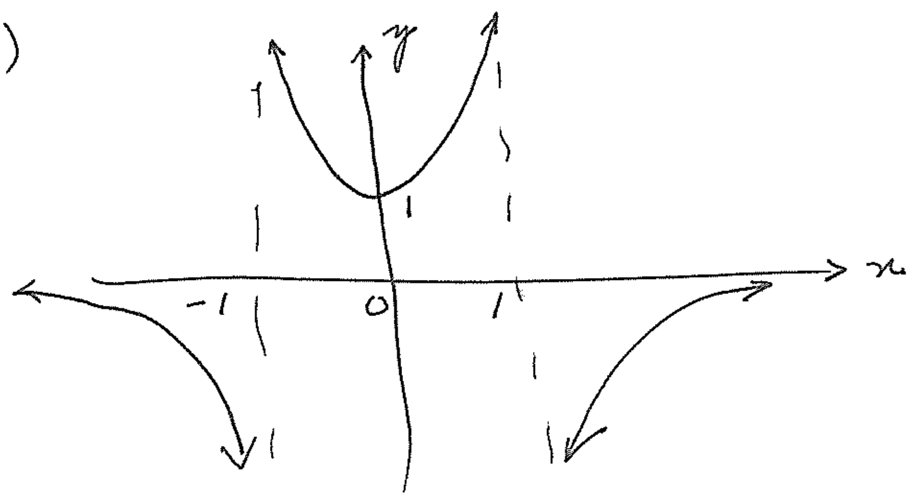
$= 7.903 \ldots$ yrs. (say 8)

(c)



(✓✓✓)

(d)

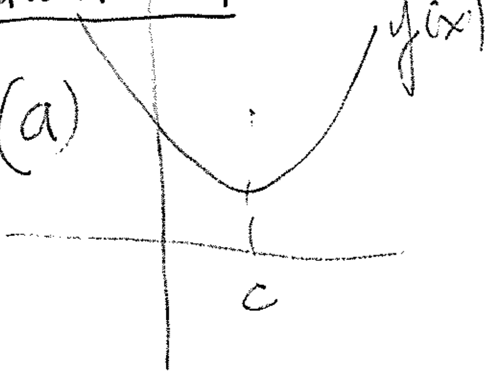


(✓✓✓)

Yr 11 Accel. Yearly 2010

Question 4

(a)



f has a min at c

$\therefore f'(c) = 0$

$f''(c) > 0$

Now $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(c) = -f'(c)$

$= 0$

$\therefore$ Stationary Point

$g''(c) = -f''(c)$

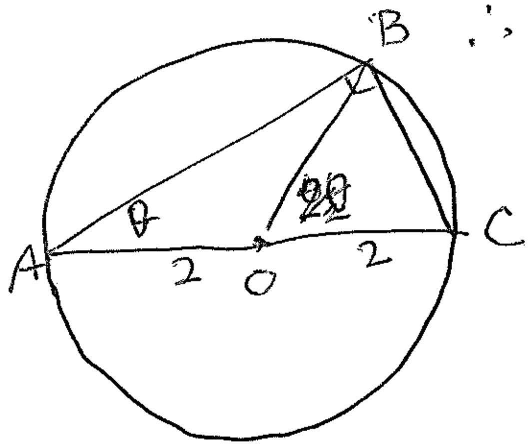
But $f''(c) > 0$

$\therefore -g''(c) > 0$

$\therefore g''(c) < 0$ [2]

$\therefore$ Maximum Point at c

(b)



(i) $\text{arc } BC = r \times 2\theta$

$= 2\theta \times 2$

$= 4\theta$ [1]

(ii) $\frac{AB}{4} = \cos\theta$

$AB = 4\cos\theta$

[1]

$S = \frac{D}{T} \Rightarrow T = \frac{D}{S}$

(iii) Time A to B $= \frac{4\cos\theta}{2}$

$= 2\cos\theta$ hrs

Time B to C (on arc)

$= \frac{4\theta}{4}$ [2]

$= \theta$ hrs.

(iv) Aim to minimise travel time

$T = \theta + 2\cos\theta$

$\frac{dT}{d\theta} = 1 - 2\sin\theta, \quad \frac{d^2T}{d\theta^2} = -2\cos\theta$

For min, $\frac{dT}{d\theta} = 0, \frac{d^2T}{d\theta^2} > 0$

$1 - 2\sin\theta = 0$

$\sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

But $0 \leq \theta \leq \pi/2$

$\therefore \theta = \frac{\pi}{6}$ (or $30°$) [2]

$\frac{d^2T}{d\theta^2}\left(\frac{\pi}{6}\right) = -2 \times \frac{\sqrt{3}}{2}$

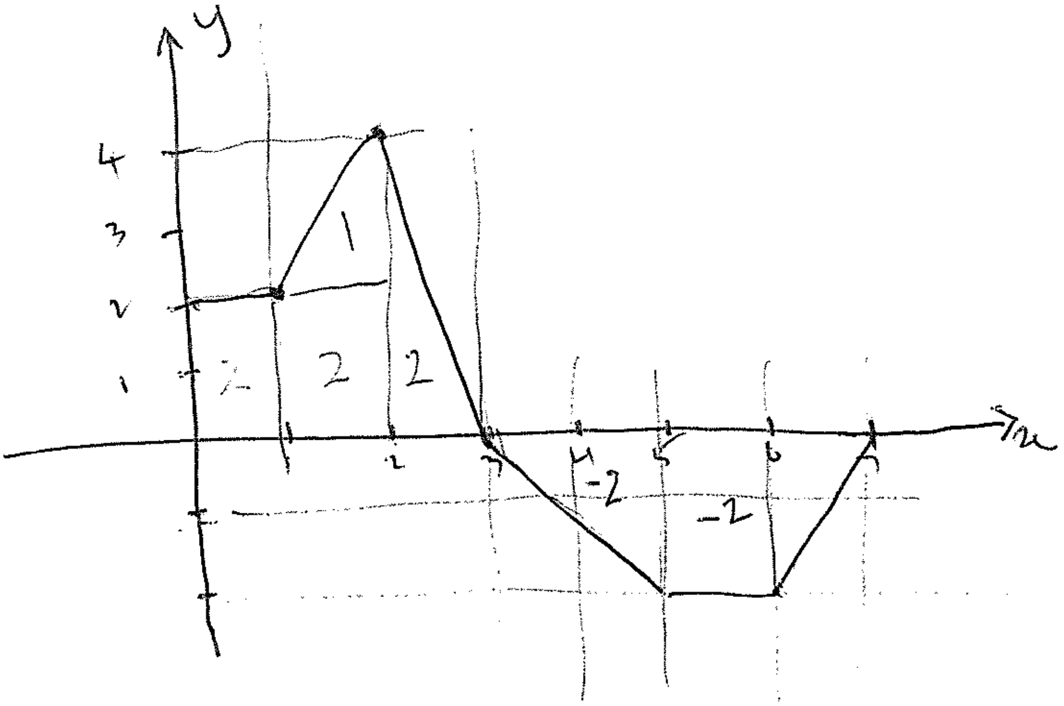
$= -\sqrt{3}$ $\therefore$ Not a minimum.

Boundaries: $\theta = 0$ $\quad$ $\theta = \pi/2$ $\quad$ $\therefore$ $90°$ to diameter.

$T = 2$ hrs $\quad$ $T = \frac{\pi}{2}$ hrs

$\doteq 1.57$ hrs $\therefore$ Walk all the way.

(c) $g(x) = \int_0^x f(t)\,dt$



(i) $g(0) = 0$, $g(3) = 7$, $g(6) = 7 - 4$
$= 3$ [3]

(ii) g decreases in $3 < x < 7$ [1]

(iii) g has max at $x = 3$. [1]

QUESTION 5

a) (i) $y = \sec x$

Sub $(0,1)$

$LHS = y$
$= 1$
$RHS = \sec x$
$= \sec 0$
$= 1$
$LHS = RHS$

Sub $(\frac{\pi}{3}, 2)$

$LHS = y$
$= 2$
$RHS = \sec \frac{\pi}{3}$
$= 2$
$LHS = RHS$

$y = \frac{3}{\pi}x + 1$

Sub $(0,1)$

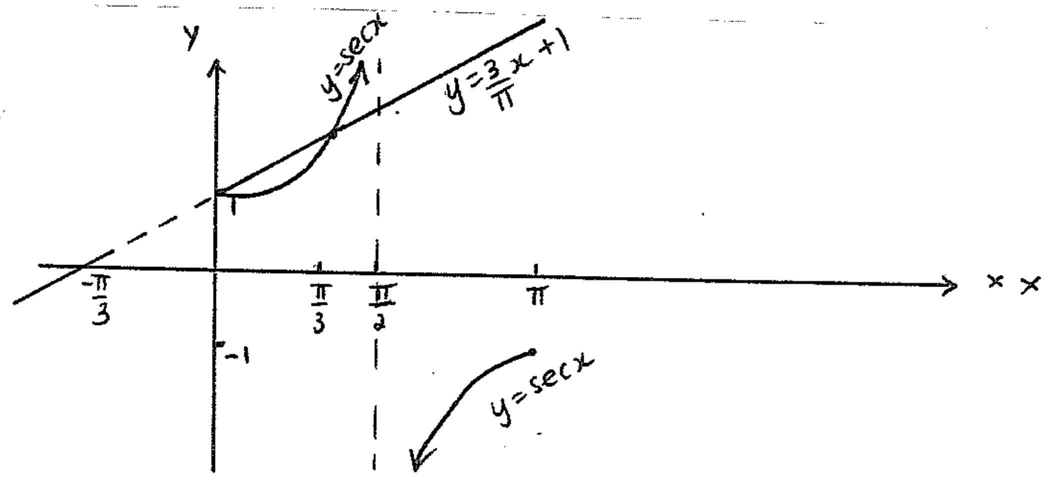
$LHS = y$
$= 1$
$RHS = \frac{3}{\pi}x + 1$
$= 0 + 1$
$= 1$
$LHS = RHS$. $\therefore (0,1)$ is a point of intersection.

Sub $\frac{\pi}{3}, 2$

$LHS = y$
$= 2$
$RHS = \frac{3}{\pi}\left(\frac{\pi}{3}\right) + 1$
$= 1 + 1$
$= 2$
$LHS = RHS$ $\therefore (\frac{\pi}{3}, 2)$ is a point of intersection.

2

a)(ii)



2

- Full marks - 2marks
  - needed to show <u>straight</u> line graph and $y = \sec x$
- 1 mark - only one graph drawn correctly.

a) (iii) $V = \pi \int_0^{\pi/3} \sec^2 x \, dx$

$= \pi \left[\tan x \right]_0^{\pi/3}$

$= \pi \left[\tan \frac{\pi}{3} - 0 \right]$

$= \pi \sqrt{3}$ c.units.

2

a) (iv) $V = \pi \int_0^{\pi/3} \left(\frac{3}{\pi} x + 1 \right)^2 dx \quad - \pi\sqrt{3}$

$= \pi \left[\frac{\left(\frac{3}{\pi} x + 1 \right)^3}{3 \cdot \frac{3}{\pi}} \right]_0^{\pi/3} - \pi\sqrt{3}$

$= \pi \left[\frac{\pi}{9} \left(\frac{3}{\pi} x + 1 \right)^3 \right]_0^{\pi/3} - \pi\sqrt{3}$

$= \pi \left[\frac{\pi}{9} \left(\frac{3}{\pi} \cdot \frac{\pi}{3} + 1 \right)^3 - \frac{\pi}{9} \right] - \pi\sqrt{3}$

$= \pi \left[\quad \frac{7\pi}{9} \right] - \pi\sqrt{3}$

$= \frac{7\pi^2}{9} - \pi\sqrt{3}$ c.units

OR

$V = \pi \int_0^{\pi/3} \left(\frac{3}{\pi} x + 1 \right)^2 dx \quad - \pi\sqrt{3}$

$= \pi \int_0^{\pi/3} \left(\frac{9}{\pi^2} x^2 + \frac{6x}{\pi} + 1 \right) dx - \pi\sqrt{3}$

$= \pi \left[\frac{9}{\pi^2} \cdot \frac{x^3}{3} + \frac{6}{\pi} \cdot \frac{x^2}{2} + x \right]_0^{\pi/3} - \pi\sqrt{3}$

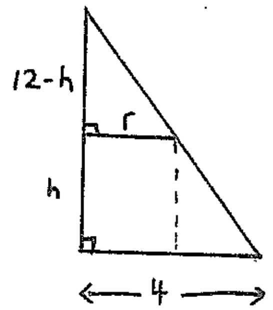
$= \pi \left[\frac{9}{\pi^2} \cdot \frac{1}{3} \cdot \frac{\pi^3}{27} + \frac{6}{\pi} \cdot \frac{1}{2} \cdot \frac{\pi^2}{9} + \frac{\pi}{3} \right] - \pi\sqrt{3}$

$= \pi \left[\frac{\pi}{9} + \frac{\pi}{3} + \frac{\pi}{3} \right] - \pi\sqrt{3}$

$= \frac{7\pi^2}{9} - \pi\sqrt{3}$

2

5.b)



Using similar Δ's

$$\frac{r}{4} = \frac{12-h}{12}$$ (corresponding sides in the same ratio)

$$12r = 48 - 4h$$

$$4h = 48 - 12r$$
$$h = 12 - 3r$$

$$\text{Volume of Cylinder} = \pi r^2 h$$
$$= \pi . r^2 (12 - 3r)$$
$$V = 12\pi r^2 - 3\pi r^3$$
$$V' = 24\pi r - 9\pi r^2$$
$$V'' = 24\pi - 18\pi r$$

Min/Max Vol. will occur at stat. pts : ie $V' = 0$

$$24\pi r - 9\pi r^2 = 0$$
$$3\pi r (8 - 3r) = 0$$
$$r = 0, \quad r = \frac{8}{3}$$

Test $r = \frac{8}{3}$

$$V'' = 24\pi - 18 \times \pi \times \frac{8}{3}$$
$$< 0$$

5

$\therefore$ max at $r = \frac{8}{3}$.

$$\text{Max } V = 12 \times \pi \times \left(\frac{8}{3}\right)^2 - 3\pi \left(\frac{8}{3}\right)^3$$
$$= \frac{768\pi}{9} - \frac{1536\pi}{27}$$
$$= \frac{256\pi}{9} \text{ c.units OR } 89.4 \text{ c.units}$$

OR $\frac{r}{4} = \frac{12-h}{12}$ (corresponding sides of similar Δ's in proportion)

$$r = \frac{48-4h}{12}$$
$$r = 4 - \frac{h}{3}$$

$$V \text{ of Cylinder} = \pi r^2 h$$
$$= \pi \left(4 - \frac{h}{3}\right)^2 . h$$
$$V = \frac{\pi}{9}\left(144h - 24h^2 + h^3\right)$$

$V' = \frac{\pi}{9}(144 - 48h + 3h^2)$

$V'' = \frac{\pi}{9}(-48 + 6h)$

Min/Max. Vol will occur when $V' = 0$

$144 - 48h + 3h^2 = 0$

$(h-12)(h-4) = 0$

$\therefore h = 12,\ h = 4$

Test $h=4$

$V'' < 0$

$\therefore$ max when $h=4$

Test $h=12$

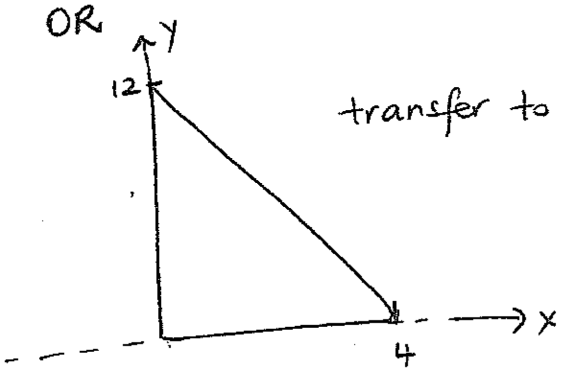
$V'' > 0$

$\therefore$ min.

$\therefore \max V = \frac{\pi}{9}(144 \times h - 24 \times 4^2 + 4^3)$

$= \frac{256\pi}{9}$ c.units.

OR



transfer to the number plane

$m = -3$

$b = 12.$

$y = -3x + 12.$

ie. $h = -3r + 12.$

Use this substitution as above.