SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2011

Year 11 Yearly

Mathematics Accelerated

General Instruction

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Start each **NEW** question in a separate answer booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 80

- Attempt questions 1-5.

Examiner: *J. Chen*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

START A NEW ANSWER BOOKLET

QUESTION ONE [18 marks]

(a) **[5 marks]**

(i) Use the standard integrals to find,

$$\int \sec 2x \tan 2x \, . \, dx$$

(ii)

$$\int \frac{x-3}{x} . \, dx$$

(iii)

$$\int \tan x \, . \, dx$$

(b) Evaluate **[5 marks]**

(i)

$$\int_{-e}^{e} \sin(e-x) \, . \, dx$$

(ii)

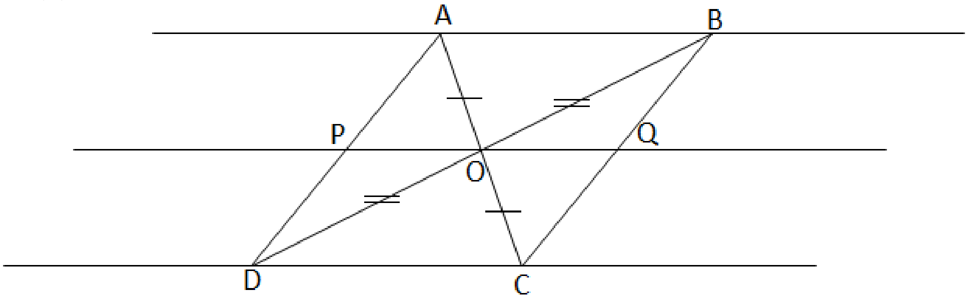
$$\int_{0}^{1} (2+e^{x}) . \, dx$$

(c) Differentiate the following with respect to x, **[3 marks]**

(i) $\tan(\sin x)$

(ii) $e^{x+\cos x}$

(d) **[5 marks]**



(i) Explain why ABCD is a parallelogram.

(ii) If PQ are the midpoints of AD and BC respectively, explain why AB||PQ||CD.

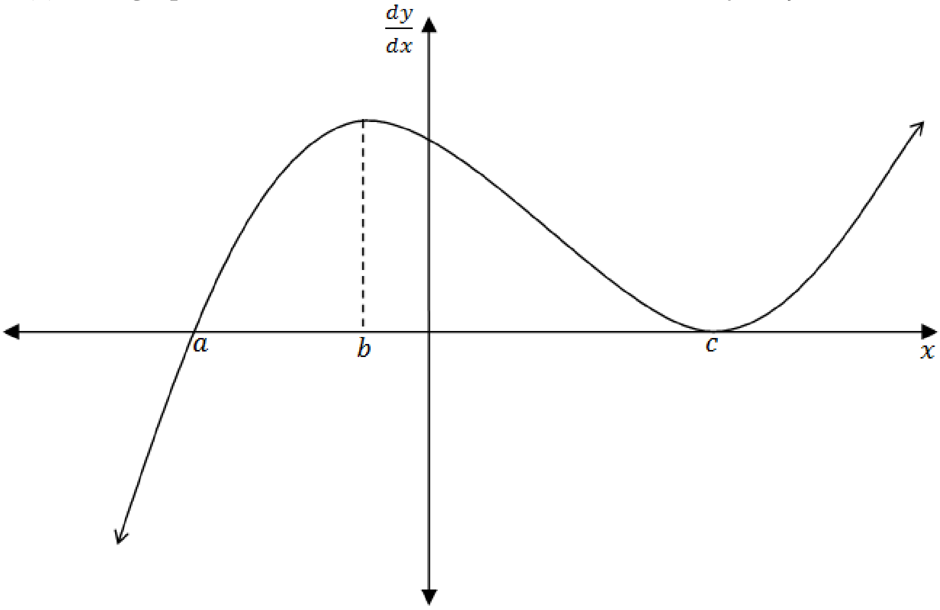
(iii) Prove that OP = OQ.

End of Question One

START A NEW ANSWER BOOKLET

QUESTION TWO [13 marks]

(a) The graph below shows the derivative of the curve $y = f(x)$. **[6 marks]**



(i) Explain why the curve $y = f(x)$ has stationary points at $x = a$ and $x = c$.

(ii) What type of stationary point is at $x = a$ and why?

(iii) What type of stationary point is at $x = c$ and why?

(iv) Sketch a possible graph of $y = f(x)$.

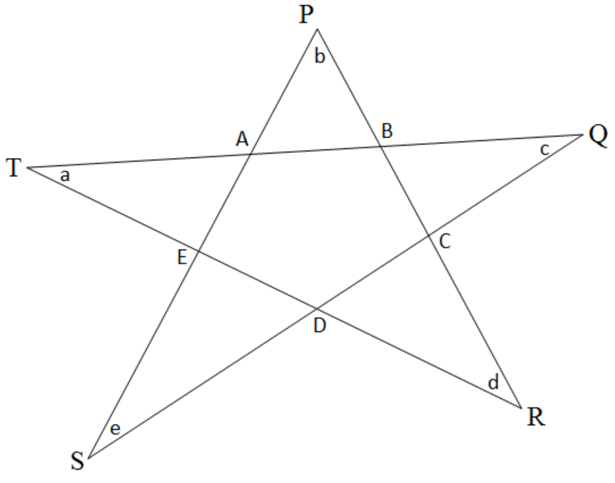
(b) **[4 marks]**

(i) Differentiate xe^x.

(ii) Hence, evaluate

$$\int_0^1 xe^x \,.dx$$

(c) Determine the value of a + b + c + d + e, giving reasons. **[3 marks]**

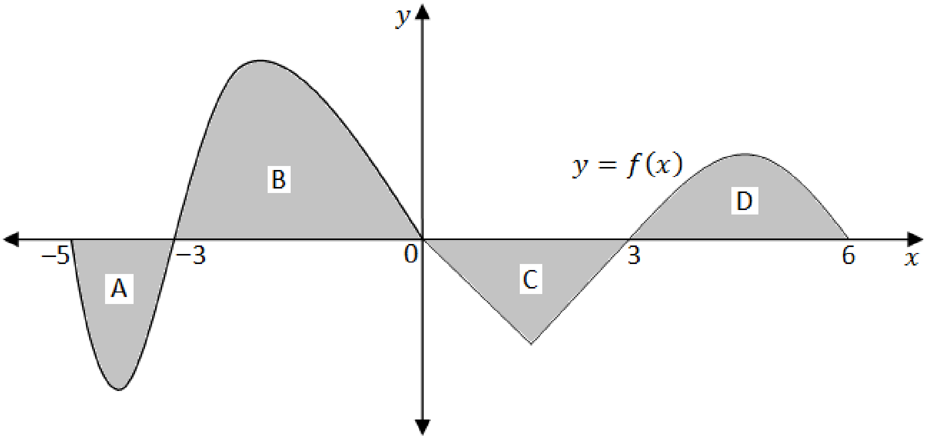


End of Question Two

START A NEW ANSWER BOOKLET

QUESTION THREE [17 marks]

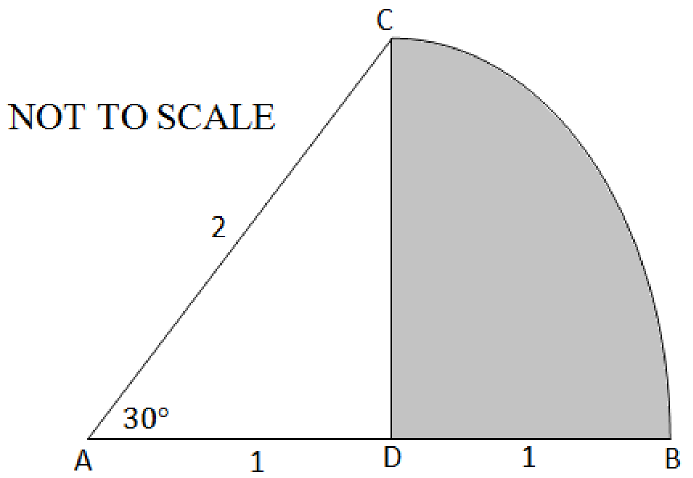
(a) In the diagram, the shaded area A is 5 cm^2, the shaded area B is 8 cm^2, the shaded area C is 7 cm^2 and the shaded area D is 6 cm^2. **[1 mark]**



Find

$$\int_{-5}^{6} f(x)\,.\,dx$$

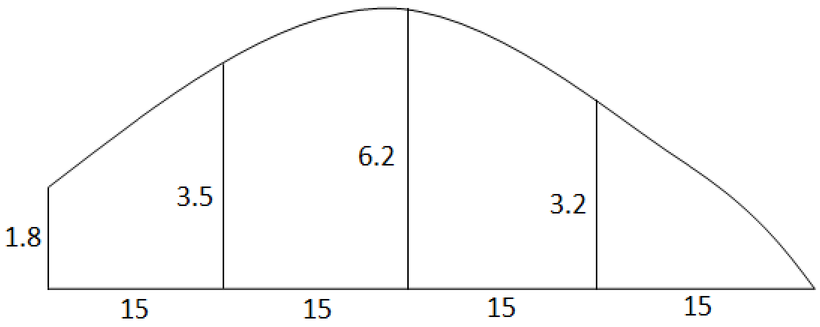
(b) In the diagram below, ABC is the sector of a circle with radius 2 cm, ∠CAB is 30° and AD = BD = 1 cm. **[6 marks]**



(i) Find the perimeter of the shaded region BCD correct to the nearest 2 decimal places.

(ii) Find the exact area of the shaded region BCD.

(c) The diagram below shows Mr. Smith's farm. All measurements are in metres. **[2 marks]**



Use Simpson's rule with 5 function values to approximate the area of the farm.

(d) **[4 marks]**

(i) Find the coordinates of the points of intersection of the two curves $y = x^2 - 2x + 1$ and $y = 4x - x^2 - 3$.

(ii) Calculate the area contained by the two curves between the points of intersection.

(e) The temperature of a cup of black coffee is given by $T = 100e^{-t/5}$ where t is the time in minutes. **[4 marks]**
If it is too hot to drink above 55℃ and too cold below 25℃.
Calculate the length of time during which the coffee is drinkable (to the nearest second).

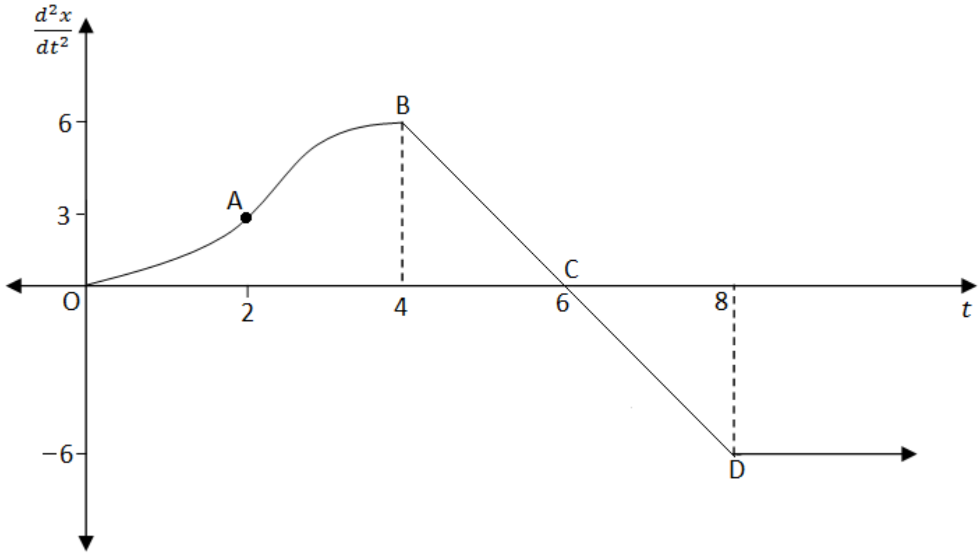
End of Question Three

START A NEW ANSWER BOOKLET

QUESTION FOUR [16 marks]

(a) A particle moves along the x-axis. Initially it is at rest at the origin. **[6 marks]**

The graph shows the acceleration, $\frac{d^2x}{dt^2}$, of the particle as a function of time t.



(i) Using Simpson's rule, estimate the velocity of the particle at $t = 4$.

(ii) Write down the time at which the velocity of the particle is a maximum.

(iii) Estimate the time at which the particle is furthest from the origin in the positive direction. Justify your answer.

(b) Consider the function $f(x) = (x^2 - 4)(x^2 - 2)$. **[10 marks]**

(i) Find the x intercepts of the curve.

(ii) Find the coordinates of the stationary points and determine their nature.

(iii) Find any points of inflexion.

(iv) Sketch $y = f(x)$, showing all critical points.

(i) Determine the values of x for which the function concaves up.

End of Question Four

START A NEW ANSWER BOOKLET

QUESTION FIVE [16 marks]

(a) A T-shirt company makes 500 shirts per month. At \$30 each, they can sell all the shirts. If the price of each shirt is increased by \$3, then this will result in a 5 shirt reduction in sales for each \$3 increment. Also, the company has fixed costs of \$6500 per month. **[6 marks]**

(i) Let the number of \$3 increments be x, prove that the monthly profit P, in dollars, is given by $P = 8500 + 1350x - 15x^2$.

(ii) Find how many shirts would be sold and the price that should be charged per shirt to ensure maximum monthly profit.

(b) Consider the function $f(x) = \frac{x}{\ln x}$, for $x > 1$. **[5 marks]**

(i) Show that the function $y = f(x)$ has a minimum point at $x = e$.

(ii) Hence, use (b) (i) to show that $x^e \leq e^x$ for $x > 1$.

(c) The region bounded by the curve $y = \log_3 x$, the line $y = 2$ and the x and y axes, is rotated about the y axis. **[5 marks]**

(i) Show that the volume of the solid of revolution formed is given by

$$V = \pi \int_0^2 9^y \, .dy$$

(ii) Hence evaluate the volume in exact simplified form.

End of Exam

2011 Accelerated Mathematics Yearly:
Solutions— Question 1

1. (a) (i) Use the standard integrals to find $\boxed{5}$

$$\int \sec 2x \tan 2x \, dx,$$

Solution: $\int \sec 2x \tan 2x \, dx = \frac{1}{2} \sec 2x + c.$

(ii) $\int \frac{x-3}{x} \, dx,$

Solution: $\int \left(1 - \frac{3}{x}\right) dx = x - 3 \ln x + c.$

(iii) $\int \tan x \, dx.$

Solution: $- \int \frac{-\sin x}{\cos x} \, dx = -\ln \cos x + c$ (or $\ln \sec x + c$).

(b) Evaluate $\boxed{5}$

(i) $\int_{-e}^{e} \sin(e - x) \, dx,$

Solution:
$$\begin{aligned} \int_{-e}^{e} \sin(e-x) \, dx &= \left[-^{-} \cos(e-x)\right]_{-e}^{e}, \\ &= 1 - \cos 2e. \end{aligned}$$

(ii) $\int_0^1 (2 + e^x) \, dx.$

Solution:
$$\begin{aligned} \int_0^1 (2+e^x) \, dx &= \left[2x + e^x\right]_0^1, \\ &= 2 + e - (0+1), \\ &= 1 + e. \end{aligned}$$

(c) Differentiate the following with respect to x: [3]

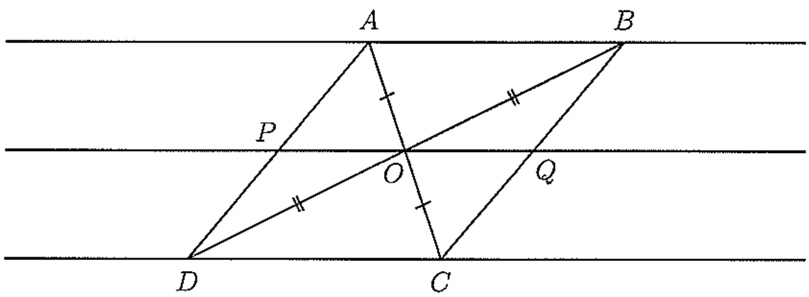
(i) $\tan(\sin x)$,

Solution:
$$\text{Put } y = \tan u, \qquad u = \sin x,$$
$$dy/du = \sec^2 u, \qquad du/dx = \cos x,$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx},$$
$$i.e., \ \frac{d}{dx}\big(\tan(\sin x)\big) = \cos x \sec^2(\sin x).$$

(ii) $e^{x+\cos x}$.

Solution:
$$\frac{d}{dx}\left(e^{x+\cos x}\right) = (1 + {}^{-}\sin x)e^{x+\cos x},$$
$$= (1 - \sin x)e^{x+\cos x}.$$

(d) [5]



(i) Explain why $ABCD$ is a parallelogram.

Solution: Diagonals AC, BD bisect each other at O (data),
$\therefore ABCD$ is a parallelogram.

(ii) If P, Q are the midpoints of AD and BC respectively, explain why $AB \parallel PQ \parallel CD$.

Solution: P is the midpoint of AD (data),
O is the midpoint of AC ($AO = OC$, given),
$\therefore PO \parallel DC$ (midpoint theorem for $\triangle ADC$).
Similarly, $PO \parallel AB$ (midpoint theorem for $\triangle ADB$).
$\therefore AB \parallel PQ \parallel DC$.

(iii) Prove that $OP = OQ$.

Solution: $OP = \frac{1}{2}DC$ (midpoint theorem for $\triangle ADC$)
$OQ = \frac{1}{2}DC$ (midpoint theorem for $\triangle BDC$)
$\therefore OP = OQ$.

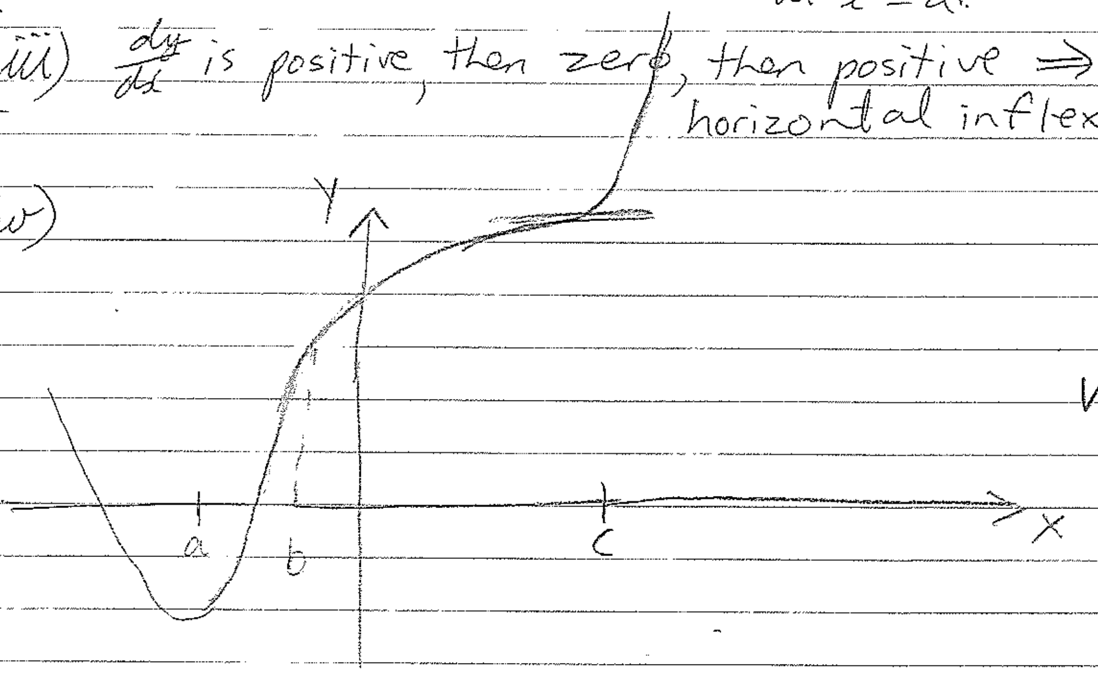
2(a)

(i) Stat pts at $x=a$ and $x=c$ because $\frac{dy}{dx}=0$ at these points. ✓

(ii) $\frac{dy}{dx}$ is increasing from negative to 0 to positive
∴ minimum ~~local~~ stat. point at $x=a$. ✓✓

(iii) $\frac{dy}{dx}$ is positive, then zero, then positive $\Rightarrow$ point of horizontal inflexion at $x=c$ ✓✓

(iv)



✓ (6)

(b)(i) $y = xe^x$

$\frac{dy}{dx} = x.e^x + e^x.1 = xe^x + e^x$ ✓ (1)

(ii) $\int (xe^x + e^x)\,dx = xe^x + C$

$\Rightarrow \int xe^x\,dx = xe^x - \int e^x\,dx + C$ ✓

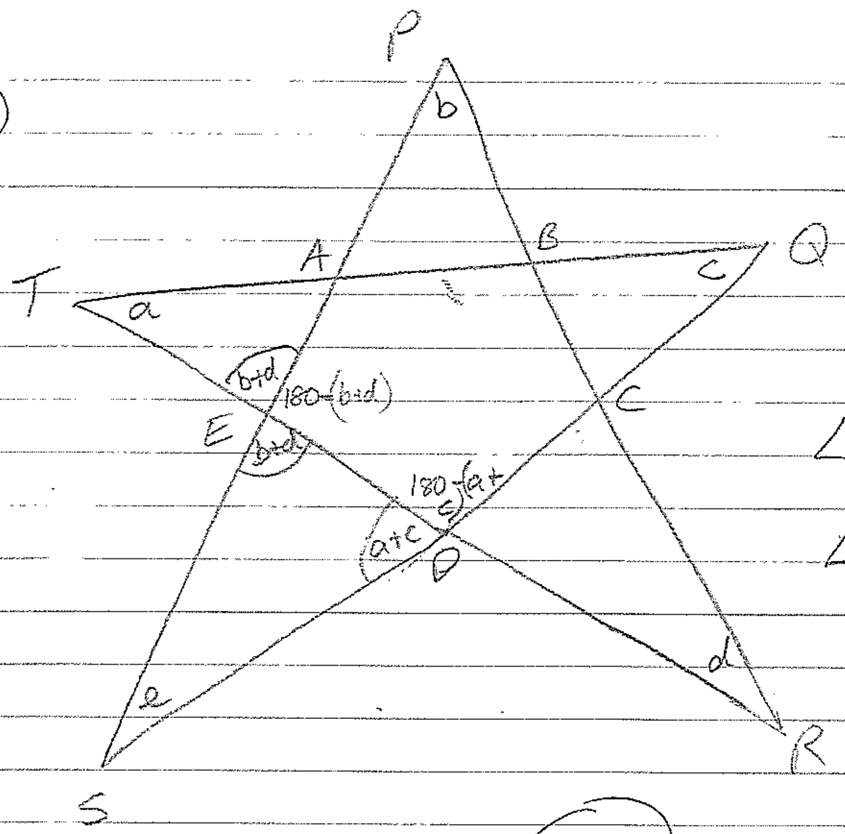
$\int xe^x\,dx = xe^x - e^x + C$. ✓

$\therefore \int_0^1 xe^x\,dx = \left[xe^x - e^x\right]_0^1$ (3)

$= (e - e) - (1 - 1)$

$= 1$ ✓

2(c)



$\angle CDE = 180 - (a+c)$ (Angles in Δ add to $180°$)

$\angle EDS = a+c$ (Angles on st line add to $180°$)

$\angle PER = 180 - (b+d)$ ($\angle$'s in Δ add to $180°$)

$\angle RES = b+d$ (Angles on st line add to $180°$)

3

$\therefore$ In ΔSED,

~~$\angle S + \angle E + \angle D = 180°$~~ (Angle sum of Δ)

$\Rightarrow e + b+d + a+c = 180°$

$\therefore a+b+c+d+e = 180°$

$1\frac{1}{2}$ marks if no reasons.

Yr 11 Accelerated Yearly 2011

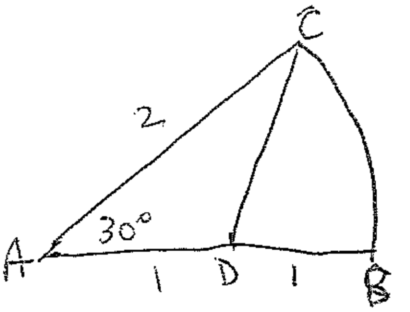
Question 3

(a) $\int_{-5}^{6} f(x)\,dx = -5+8-7+6$

$= -12+14$

$= 2$ [1]

(b)



(i) $BC = 2\times\frac{\pi}{6} = \frac{\pi}{3}$

$CD^2 = 1+4-2\times2\cos30^\circ$

$= 5-4\times\frac{\sqrt{3}}{2}$

$= 5-2\sqrt{3}$

$CD = \sqrt{5-2\sqrt{3}}$

$\therefore P = \frac{\pi}{3} + \sqrt{5-2\sqrt{3}} + 1$

$\doteqdot 3.29$ cm [3]

(ii) $A =$ Area(Sector) − Area $\triangle ADC$

$= \frac{1}{2}\times2^2\times\frac{\pi}{6} - \frac{1}{2}\times2\times1\times\sin30$

$= \frac{\pi}{3} - \frac{1}{2}$ cm^2 [3]

(c)

$A \doteqdot \frac{30}{6}\left[1.8+4\times3.5+6.2\right] + \frac{30}{6}\left[6.2+4\times3.2+0\right]$

$= 205$ m^2 [2]

(d) $y = x^2-2x+1$; $y = 4x-x^2-3$

At Intersections:

(i) $x^2-2x+1 = 4x-x^2-3$

$2x^2-6x+4=0$

$x^2-3x+2=0$

$(x-2)(x-1)=0$

$\therefore$ Intercepts at $x=1$, $x=2$

$y=0$, $y=1$

i.e. $(1,0)$, $(2,1)$ [2]

(ii) $A = \int_1^2 (-x^2+4x-3)-(x^2-2x+1)\,dx$

$= \int_1^2 (-2x^2+6x-4)\,dx$

$= \left[-\frac{2x^3}{3} + \frac{6x^2}{2} - 4x\right]_1^2$

$= \left(-\frac{2}{3}\times8+3\times4-8\right) - \left(-\frac{2}{3}+\frac{6}{2}-4\right)$

$= \frac{1}{3}$ unit^2 [2]

(e) $T = 100e^{-t/5}$

$\ln\left(\frac{T}{100}\right) = -t/5$

$\therefore t = -5\ln(T/100)$

$t_1 = -5\ln(0.55)$

$= 2'59''$

$t_2 = -5\ln(0.25)$

$= 6'56''$

$\therefore$ Drinkable time $= 3'57''$ [4]

$(= 237'')$

QUESTION 4

a

Let $\frac{d^2x}{dt^2} = f(t)$.

(i) $\int_a^b f(t)\,dt \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$

$\int_0^4 f(t)\,dt \approx \frac{2}{3}(0 + 12 + 6)$.

$= 12$ 2

(ii) Max velocity occurs when $f(t)=0$.

i.e. $t=6$. 1

(iii) Max displacement occurs when

$\frac{dx}{dt} = 0$. 3

i.e. $\int_0^{t_{max}} f(t)\,dt = 0$.

$\int_0^4 f(t)\,dt + \int_4^6 f(t)\,dt + \int_6^8 f(t)\,dt + \int_8^{t_{max}} f(t)\,dt = 0$.

$12 + \frac{1}{2}\times 2\times 6 - \frac{1}{2}\times 2\times 6 - \int_8^{t_{max}} f(t)\,dt = 0$.

$\int_8^{t_{max}} f(t)\,dt = 12$

$t_{max} = 10$.

(b) $f(x) = (x^2-4)(x^2-2)$.

(i) $(x^2-4)(x^2-2) = 0$

$x = \pm 2, \pm\sqrt{2}$. 2

(ii) $f'(x) = 2x(x^2-4) + 2x(x^2-2)$

$= 2x(2x^2-6)$.

$f'(x) = 0$

$x = 0 \quad x = \pm\sqrt{3}$.

$f''(x) = 2(2x^2-6) + 2x(4x)$.

$= 4x^2 - 12 + 8x^2$.

$= 12x^2 - 12$.

$= 12(x^2-1)$.

Nature

$f''(0) \leq 0$ maxima $(0, 8)$

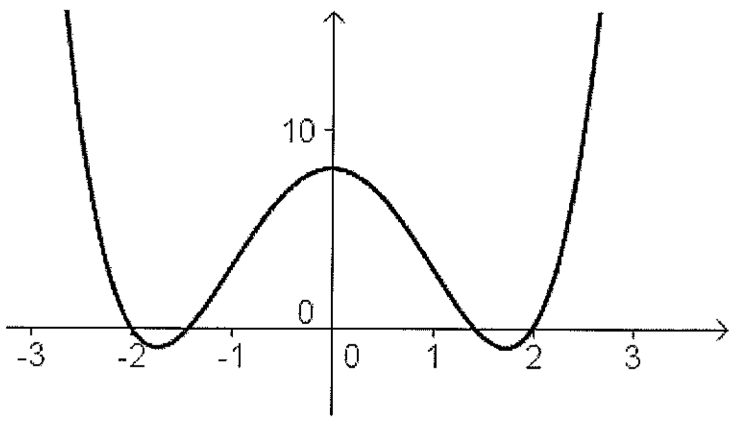
$f''(\sqrt{3}) > 0$ minima. $(\sqrt{3}, -1)$ 2

$f''(-\sqrt{3}) < 0$ maxima. $(-\sqrt{3}, -1)$.

(iii) $12(x^2-1) = 0$

$x = \pm 1$. $(1, 3)$ $(-1, 3)$ 2

(v) $x < -1$, $x > 1$. 2



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QUESTION FIVE

(a) (i) $P = (30+3x)(500-5x) - 6500$

$= 15(10+x)(100-x) - 6500$

$= 15(1000 + 90x - x^2) - 6500$

$= 15000 + 1350x - 15x^2 - 6500$

$= 8500 + 1350x - 15x^2$

(ii) $P' = 1350 - 30x$

$P'' = -30$

Let $P' = 0$.

$1350 - 30x = 0$

$x = 45$

∴ NO. OF ~~[illegible]~~ SHIRTS is $500 - 5 \times 45 = 275$

PRICE PER SHIRT

$= 30 + 3 \times 45$

$= \$165$

(NB $P'' < 0$ ∴ MAX.) IS OK.
If you use the 1'st derivative test make sure that numbers are used!!)

eg

1	2	3
+	0	−
/	—	\

∴ MAX. IS NOT GOOD ENOUGH

1	2	3
4.6	0	−3.1
/	—	\

∴ MAX is OK.

5 [CONTD]

(b)

$$f(x) = \frac{x}{\ln x}$$

(i)

$$f'(x) = \frac{\ln x - x \times \frac{1}{x}}{(\ln x)^2}$$

$$= \frac{\ln x - 1}{(\ln x)^2}$$

$$f''(x) = \frac{(\ln x)^2 \times \frac{1}{x} - (\ln x - 1) \times \frac{2 \ln x}{x}}{(\ln x)^4}$$

$$= \frac{(\ln x)^2 - 2(\ln x - 1)\ln x}{x(\ln x)^4}$$

$$= \frac{\ln x - 2(\ln x - 1)}{x(\ln x)^3}$$

[NB MUCH EASIER TO USE 1ST DERIVATIVE TEST WITH VALUES IN THE TABLE.]

$$f''(e) = \frac{\ln e - 2(\ln e - 1)}{e(\ln e)^3}$$

$$= \frac{1 - 2(1 - 0)}{e \cdot 1}$$

$$= \frac{1 - 2 \times 0}{e}$$

$$= \frac{1}{e}$$

$\therefore f''(e) > 0$ i.e. MIN. where

$$f'(x) = 0$$

i.e. $\ln x - 1 = 0$

$$x = e.$$

Q5 (CONT'D)

now $\frac{x}{\ln x} \geq e$ from (1).

$x \geq e \ln x$. (both sides positive because $x > 1 \therefore \ln x > 0$.)

$\therefore e^{x} \geq e^{e \ln x}$

$\therefore e^{x} \geq e^{\ln x^{e}}$ [e^{x} is an increasing function ie. if $a > b$, $e^{a} > e^{b}$]

$\therefore e^{x} \geq x^{e}$

ie. $x^{e} \leq e^{x}$.

[NB DON'T START WITH THE RESULT $x^{e} \leq e^{x}$]

(c) (i) $V = \pi \int_{0}^{2} x^{2}\, dy$

$= \pi \int_{0}^{2} 9^{y}\, dy$

[now $y = \log_{3} x$
$\therefore x = 3^{y}$
$x^{2} = (3^{y})^{2}$
$= 3^{2y}$
$= (3^{2})^{y}$
$= 9^{y}$]

(ii) $V = \frac{\pi}{\ln 9} \left[9^{y} \right]_{0}^{2}$

$= \frac{\pi}{\ln 9} (9^{2} - 9^{0})$

$= \frac{\pi}{\ln 9} (81 - 1)$

$= \frac{80\pi}{\ln 9}$

$= \frac{80\pi}{2 \ln 3}$

$= \frac{40\pi}{\ln 3}\ u^{3}$.