SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2013

Year 11
Accelerated Mathematics
Yearly

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Start each **NEW** section in a separate answer booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- A table of standard integrals is included.

Total Marks – 75

- All answers must be given in exact simplified form unless otherwise stated.
- *All necessary working should be shown in every question.*

Examiner: *P. Bigelow*

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \qquad n \neq -1;\ \ x \neq 0, \qquad \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x\,, \qquad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \qquad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \qquad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \qquad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \qquad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \qquad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \qquad a > 0, \qquad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \qquad x > 0$

Section A (15 marks) Start a **NEW** booklet.

(a) Find the derivative of the following **3**

(i) $e^{\tan x}$

(ii) $x \ln(x+2)$

(iii) $\cos(2x)^\circ$

(b) Find **(i)** $\displaystyle\int \frac{8\,dx}{1+4x}$ **3**

(ii) $\displaystyle\int \frac{6}{\operatorname{cosec} x}\,dx$

(iii) $\displaystyle\int_0^8 e^{\frac{x}{4}}\,dx$

(c) Find x if **(i)** $e^x = 3.14$ **2**

(ii) $\ln x = 1.67$

(d) For $f(x) = \ln\dfrac{1+x}{1-x}$ **2**

(i) Show that $f(x)$ is an ODD function.

(ii) Find $f'(x)$.

(e) A particle is moving on the x −axis. At time t its position x is given by **3**

$$x = t + 1 + \frac{1}{1+t} \quad , \quad x \text{ is in metres, } t \text{ is in seconds.}$$

Find **(i)** the initial position

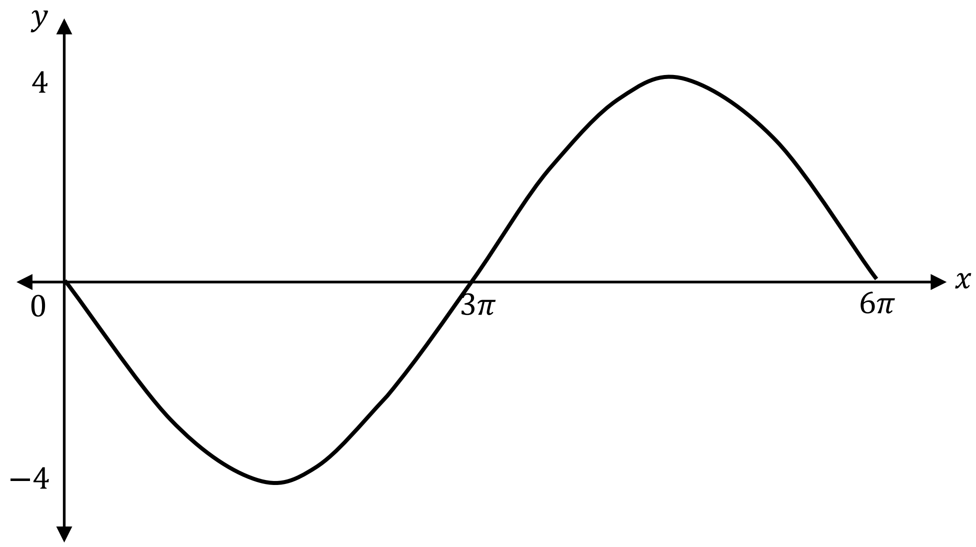
(ii) the velocity at $t = 2$.

(iii) the acceleration at time t.

(f) For what values of x is $f(x) = x^3 + 3x^2 + 12x - 7$ decreasing? **2**

Section B (15 marks) Start a **NEW** booklet.

(a) Write down an equation that best represents this sine function. **2**



(b) **(i)** Show that the curves $y = 9x - 6 - x^2$ and $y = 3x^2 - 3x - 6$ intersect at $x = 3$ and $x = 0$. **3**

(ii) Find the area between the two curves.

(c) Given that $4^x = e^{f(x)}$ **3**

(i) Find $f(x)$

(ii) Hence or otherwise find

($\boldsymbol{\alpha}$) $\dfrac{d}{dx}(4^x)$

($\boldsymbol{\beta}$) $\displaystyle\int 4^x \, dx$

(d) Find $f(x)$ if $f'(x) = \dfrac{e^{1-x}}{2}$ and $f(2) = \dfrac{1}{2e}$. **2**

(e) The mass M in grams of a radioactive substance may be expressed as **3**

$$M = Ae^{-kt}$$

where t is in years and k is a constant.

(i) At time $t = 0$, $M = 10$, find A.

(ii) After 5 years the mass is 9 grams, find the mass after 20 years.

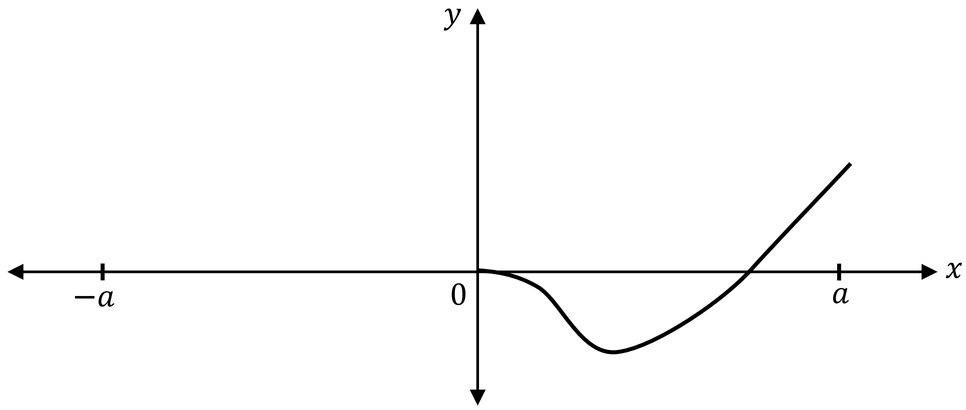
(f) Find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{2}{1 - \sin^2\theta} \; d\theta$ **2**

Section C (17 marks) Start a **NEW** booklet.

(a) For a given function $f(x)$, it is known that $f(0) = 3$ and $f(1) = 15$. Find in simplest exact form, the value of **2**

$$\int_0^1 \frac{f'(x)}{f(x)}\,dx$$

(b) The graph shows a function $f(x)$ for $0 \le x \le a$. It is known that the function is ODD and is stationary at (0,0). **4**



(i) Copy the graph and continue for $-a \le x \le 0$.

(ii) On a separate diagram, sketch $y = f'(x)$.

(c) The velocity of a particle moving in a straight line is given by: **6**

$$v = 1 - 2\cos t \text{ for } 0 \le t \le 2\pi.$$

where v is measured in metres per second and t is measured in seconds.

(i) At what times in the interval $0 \le t \le 2\pi$, is the particle at rest?

(ii) Sketch the graph of v against t for $0 \le t \le 2\pi$.

(iii) What is the maximum velocity of the particle in the interval $0 \le t \le 2\pi$?

(iv) Calculate the total distance travelled by the particle in the first π seconds.

(d) Solve $4^{1+x} = 6^{x-1}$ (giving your answer correct to three significant figures) **2**

(e) For $f(x) = x^4 + 4x$ **3**

(i) Find any stationary points and possible inflections.

(ii) Sketch the curve, showing essential features.

Section D (15 marks) Start a **NEW** booklet.

(a) $y = e^{\sqrt{x}}$ is rotated about the x axis between $x = 0$ and $x = 2$. Use Simpson's Rule with three function values to find the volume generated (answer correct to 3 significant figures). **3**

(b) Given $f(x) = e^{2x^2}$, find **2**

(i) $f'(1)$.

(ii) $f''(-1)$.

(c) A rainwater tank with a volume of 9 m^3 is installed in a house at 8 $a.m.$ Rain begins to fall and flows into the empty tank at a rate given by: **7**

$$\frac{dV}{dt} = \frac{36t}{t^2 + 20}$$

where t is the time in hours and V is the volume in cubic metres. At 8 $a.m.$, $t = 0$.

(i) Show that the volume of the water in the tank at time t, is given by:

$$V = 18 \ln\left(\frac{t^2 + 20}{20}\right), \quad t > 0.$$

(ii) Find the time when the tank will be completely full with water (to the nearest minute).

(iii) Later, when the tank is full and the rain has stopped, the owner turns on the pump which pumps the water out at a rate given by

$$\frac{dV}{dt} = \frac{t^2}{k}$$

The pump continues for 5 hours until the tank is empty. Find the value of k.

(d) **(i)** Find the minimum value of $x + 900x^{-1}$ where $x > 0$, giving reasons. **3**

(ii) A company runs a ship between ports A and B, d kilometres apart. The ship maintains a constant speed of v kilometres per hour. For a given v, the cost per hour of running the ship is $\$\,(9000 + 10v^2)$, find the value of v which minimises the cost of the trip.

Section E (13 marks) Start a **NEW** booklet.

(a) A horizontal line $y = a, a > 1$ is drawn to cross the two graphs $y = e^x$ and $y = \frac{1}{2}e^x$ at points C and D. Show that the distance CD is constant (i.e. independent of where it is drawn). **3**

(b) A sheep grazing in a paddock is tethered to a stake by a rope 30 m long. If the stake is 15 m from a long fence, find the area to the nearest square metre, over which the sheep can graze. **4**

(c) Given $f(x) = \sin x + \sin^2 x$. **6**

(i) Show that $f(x)$ has stationary points when $\cos x = 0$ or $\sin x = -\frac{1}{2}$.

(ii) Find the co-ordinates of the stationary points for $0 \leq x \leq 2\pi$.

(iii) Sketch $y = f(x)$ for $0 \leq x \leq 2\pi$.

End of Exam

YR 11 ACCELERATED YEARLY.

a) i) $\frac{d}{dx}(e^{\tan x}) = \sec^2 \cancel{x}\, e^{\tan x}$.

ii) $\frac{d}{dx}(x\ln(x+2)) = \frac{d}{dx}(u \cdot v)$.

$u = x \qquad v = \ln(x+2)$.

$u' = 1 \qquad v' = \frac{1}{x+2}$.

$= vu' + uv'$.

$= \ln(x+2) + \frac{x}{x+2}$.

iii) $\cos 2x^\circ = \cos 2x \times \frac{\pi}{180}$

$\frac{d}{dx}\left(\cos \frac{2\pi x}{180}\right) = -\frac{\pi}{90} \sin\left(\frac{\pi x}{90}\right)$.

b) i) $\int \frac{2}{1+4x} \, dx = 2 \int \frac{4}{1+4x} \, dx$.

$= 2\ln(1+4x) + C$.

ii) $\int \frac{6}{\operatorname{cosec} x} \, dx = 6 \int \sin x \, dx$.

$= -6\cos x + C$.

iii) $\int_0^8 e^{x/4} \, dx = \left[4\cancel{x}e^{x/4}\right]_0^8$

$= 4e^2 - 4e^0$.

$= 4(e^2 - 1)$.

c) i) $e^{x} = 3.14.$

$$\ln e^{x} = \ln 3.14.$$
$$x = \ln 3.14.$$

ii) $\ln x = 1.67.$

$$e^{\ln x} = e^{1.67}.$$
$$x = e^{1.67}.$$

d) i) $f(-x) = \ln\left(\frac{1-x}{1+x}\right)$

$$= \ln(1-x) - \ln(1+x).$$
$$= -\left(\ln(x+1) - \ln(1-x)\right).$$
$$= -\ln\left(\frac{x+1}{x-1}\right)$$
$$= -f(x).$$

$\therefore$ $f(x)$ is odd.

ii) $f'(x) = f'\left(\ln(1+x) - \ln(1-x)\right).$

$$= \frac{1}{x+1} - \frac{-1}{1-x}.$$
$$= \frac{1-x+(x+1)}{x}$$
$$= \frac{2}{1-x^{2}}.$$

e) $x = t + 1 + \frac{1}{1+t}$.

i) $t = 0$ $\quad x = 1 + \frac{1}{1} = 2$ to the right.

ii) $V = \frac{dx}{dt} = 1 - \frac{1}{(1+t)^2}$.

$t = 2$ $\quad V = 1 - \frac{1}{3^2} = \frac{8}{9}\ ms^{-1}$.

iii) $\ddot{x} = \frac{dv}{dt} = \frac{+2}{(1+t)^3}$.

f) $y = x^3 + 3x^2 + 12x - 7$.

$y' = 3x^2 + 6x + 12$.

decreasing when $y' < 0$.

$y' = 3(x^2 + 2x + 4) \leq 0$.

$(x^2 + 2x + 1) + 3 \leq 0$.

$(x+1)^2 + 3 \leq 0$.

which has no solutions.

$\therefore$ there are no values for x for which y is decreasing.

y is an increasing fn.

Year 11 Yearly

Section B

(a) $y = -4\sin\frac{1}{3}x$ (2) $T = 6\pi = \frac{2\pi}{n}$

$\underline{n = \frac{1}{3}}$

(b) (i) $y = 9x - 6 - x^2$ (1) and $y = 3x^2 - 3x - 6$ (2)

Sub (1) in (2) $\Rightarrow$ $3x^2 - 3x - 6 = 9x - 6 - x^2$

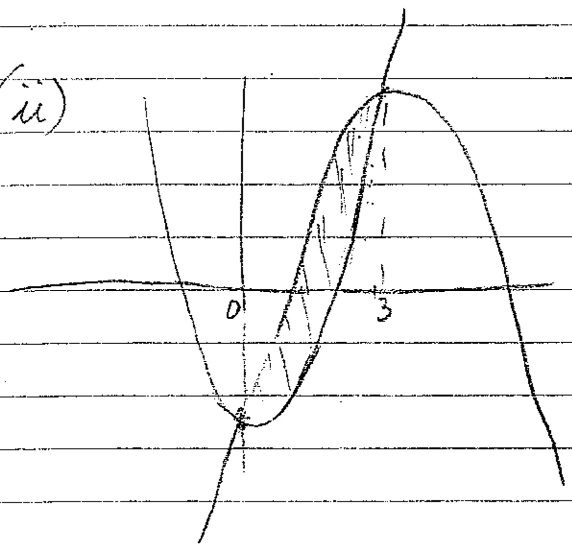
$4x^2 - 12x = 0$

$4x(x-3) = 0$

$\underline{x=0}$ or $\underline{x=3}$ are Pts of intersectio

$(0,-6)$ and $(3,12)$

(ii)



$$\text{Area} = \int_0^3 (9x-6-x^2) - (3x^2-3x-6)\,dx$$

$$= \int_0^3 (12x - 4x^2)\,dx$$ (3)

$$= \left[6x^2 - \frac{4x^3}{3}\right]_0^3$$

$$= (54-36) - (0)$$

$$= 18 \text{ sq. units}$$

(c) $4^x = e^{f(x)}$

(i) $\ln 4^x = \ln e^{f(x)}$

$\underline{x.(\ln 4) = f(x)}$.

(ii) (α) $\frac{d}{dx}(4^x) = (\ln 4)4^x$

(β) $\int 4^x\,dx = \frac{4^x}{\ln 4} + c$

(d) $f'(x) = \frac{1}{2}e^{1-x}$

$f(x) = \frac{1}{2}\frac{e^{1-x}}{-1} + c$

$f(x) = -\frac{1}{2}e^{1-x} + c$ ✓

Also $f(2) = \frac{1}{2e} \Rightarrow \frac{1}{2e} = -\frac{1}{2}e^{-1} + c$

$\frac{1}{2e} = -\frac{1}{2e} + c$

$\underline{c = \frac{1}{e}}$ ✓

2

$\therefore f(x) = \underline{\underline{-\frac{1}{2}e^{1-x} + \frac{1}{e}}}$

(e) $M = Ae^{-kt}$

(i) When $t=0, M=10 \Rightarrow 10 = Ae^{0}$

$\Rightarrow \underline{A = 10}$ ✓

(ii) $\therefore M = 10e^{-kt}$

$t=5, m=9 \Rightarrow 9 = 10e^{-5k}$

$0.9 = e^{-5k}$

$\ln 0.9 = -5k$

$k = \frac{\ln 0.9}{-5} \doteq 0.0210721$ ✓

Then $M = 10e^{\left(\frac{\ln 0.9}{5}\right)t}$

3

When $t=20$, $m = 10e^{4\ln 0.9}$

$\underline{M = 6.561g}$. ✓

(f) $\int_0^{\frac{\pi}{4}} \frac{2}{1-\sin^2\theta}\, d\theta$

$= \int_0^{\pi/4} 2\sec^2\theta \, d\theta$

$= \left[2\tan\theta\right]_0^{\pi/4}$

$= (2) - (0)$

$= 2$ square units.

2

Section C

a) $\displaystyle\int_0^1 \frac{f'(x)}{f(x)}\,dx$

$= \left[\ln f(x)\right]_0^1$

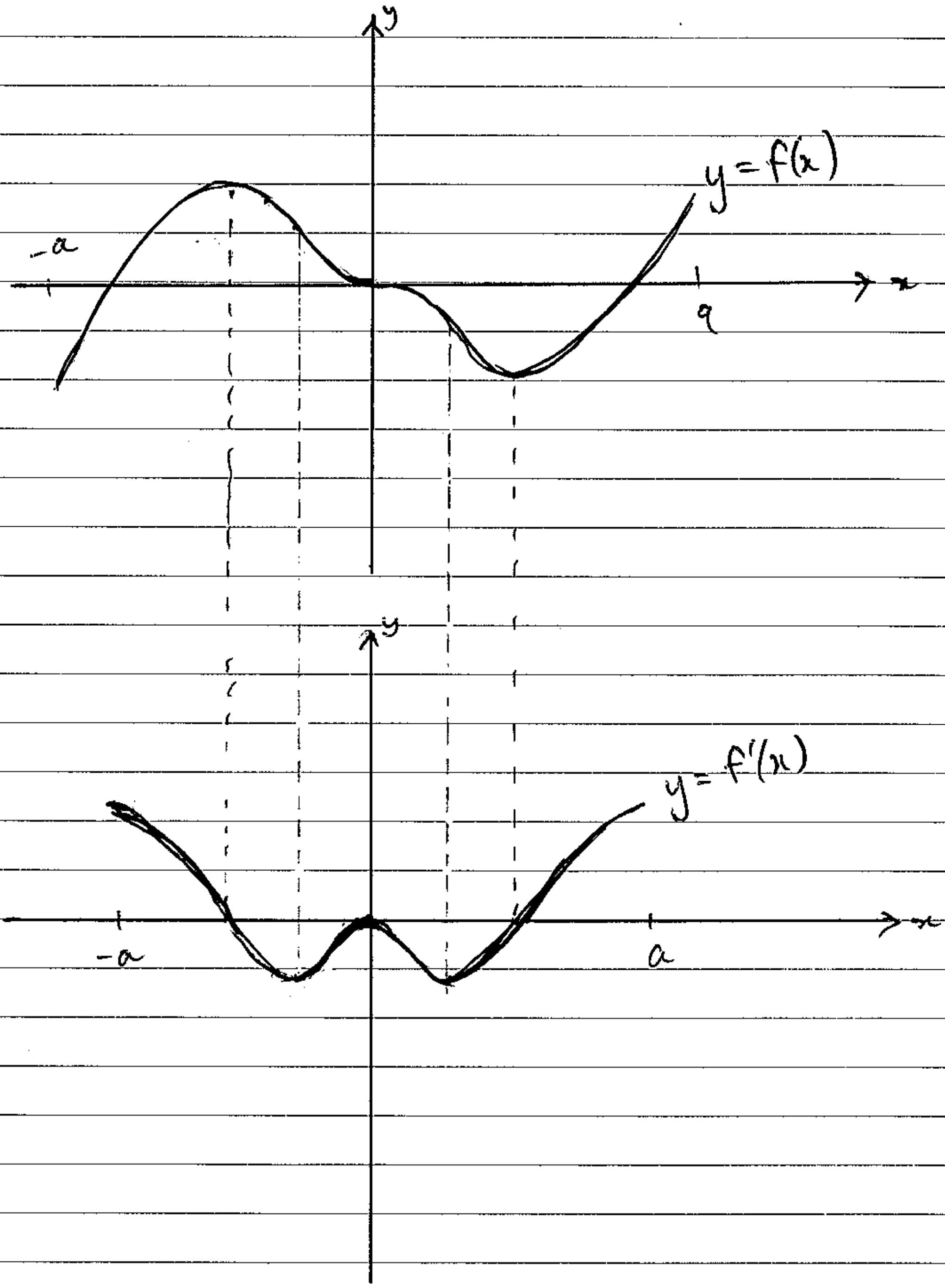
$= \ln f(1) - \ln f(0)$

$= \ln 15 - \ln 3$

$= \ln\left(\frac{15}{3}\right)$

$= \ln 5$

b)

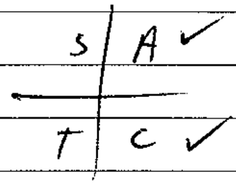


c) $v = 1 - 2\cos t \qquad 0 \le t \le 2\pi$

i) let $v = 0$

$1 - 2\cos t = 0$

$\cos t = \frac{1}{2}$

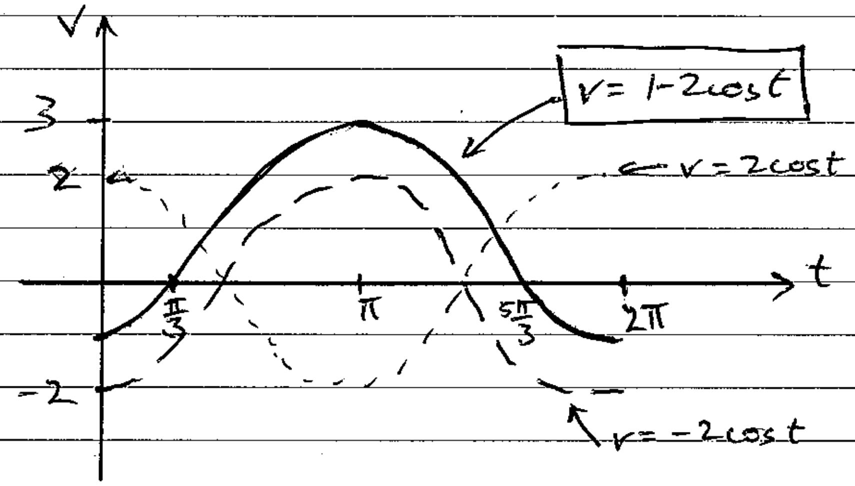


$\cos\alpha = \frac{1}{2}$

$\alpha = \frac{\pi}{3}$

$t = \frac{\pi}{3}, \frac{5\pi}{3}$ s.

ii)



iii) 3 m/s (as seen from graph)

iv) distance travelled $= -\int_0^{\frac{\pi}{3}} v\,dt + \int_{\frac{\pi}{3}}^{\pi} v\,dt$

$= -\int_0^{\frac{\pi}{3}} (1 - 2\cos t)\,dt + \int_{\frac{\pi}{3}}^{\pi} (1 - 2\cos t)\,dt$

$= -\left[t - 2\sin t\right]_0^{\frac{\pi}{3}} + \left[t - 2\sin t\right]_{\frac{\pi}{3}}^{\pi}$

$= -\left[\frac{\pi}{3} - 2\sin\frac{\pi}{3} - (0)\right] + \left[\pi - \cancel{2\sin\pi}_{0} - \left(\frac{\pi}{3} - 2\sin\frac{\pi}{3}\right)\right]$

$= -\left[\frac{\pi}{3} - 2\left(\frac{\sqrt{3}}{2}\right)\right] + \pi - \left(\frac{\pi}{3} - 2\left(\frac{\sqrt{3}}{2}\right)\right)$

$= \left(\frac{\pi}{3} + 2\sqrt{3}\right)$ m.

d) $4^{1+x} = 6^{x-1}$

$\ln 4^{1+x} = \ln 6^{x-1}$

$(1+x)\ln 4 = (x-1)\ln 6$

$\ln 4 + x\ln 4 = x\ln 6 - \ln 6$

$x\ln 6 - x\ln 4 = \ln 6 + \ln 4$

$x(\ln 6 - \ln 4) = \ln 6 + \ln 4$

$$x = \frac{\ln 6 + \ln 4}{\ln 6 - \ln 4}$$

$x \approx 7.84$

e) i) $f(x) = x^4 + 4x$
$f'(x) = 4x^3 + 4$
$f''(x) = 12x^2$

For stationary points $f'(x) = 0$

$4x^3 + 4 = 0$
$4(x^3 + 1) = 0$
$x^3 = -1$
$x = -1$
$f(-1) = (-1)^4 + 4(-1)$
$= -3$

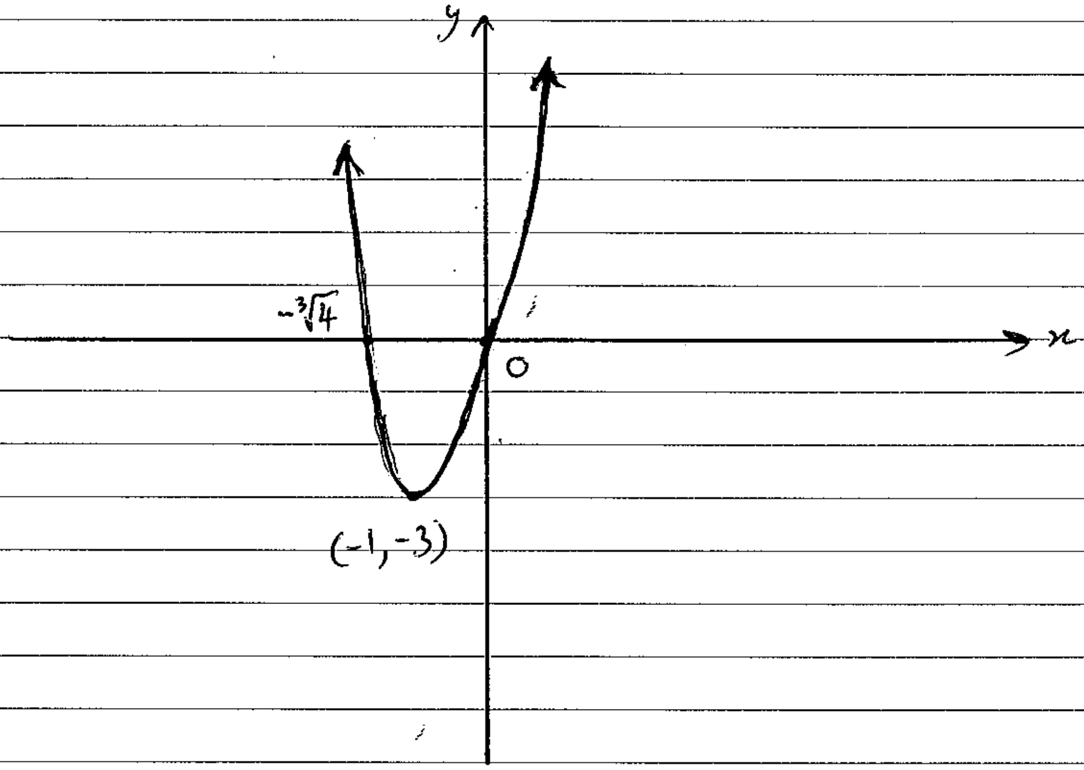
For possible inflexion points $f''(x) = 0$
$12x^2 = 0$
$x = 0$
$f(0) = 0$

- Stationary point at $(-1, -3)$
- possible inflexion point at $(0, 0)$

Note: $f''(x) = 12x^2$
> 0, for $x \neq 0$ ∪

$\therefore$ $(-1,-3)$ is a minimum turning point
$(0,0)$ is **not** an inflexion point.

ii)



For x-intercepts
let $f(x) = 0$

$x^4 + 4x = 0$

$x(x^3 + 4) = 0$

$x = 0, \quad x^3 = -4$

$x = -\sqrt[3]{4}$.

Section D

(a) $y = e^{\sqrt{x}}$

$\pi \int_0^2 e^{2\sqrt{x}} \, dx$

$\approx \frac{\pi}{3}\left(1 + 4(e^2) + e^{2\sqrt{2}}\right)$ 3

$= 49.7$ units^3.

(b) $f(x) = e^{2x^2}$

(i) $f'(x) = 4xe^{2x^2}$

$f'(1) = 4e^2$. 2

(ii) $f''(x) = 4xe^{2x^2} \times 4x + 4e^{2x^2}$.

$= 4e^{2x^2}(4x^2 + 1)$

$f''(-1) = 4e^2(5)$

$= 20e^2$

(c) $\frac{dV}{dt} = \frac{36t}{t^2+20}$.

$= 18 \frac{2t}{t^2+20}$.

$V = 18\ln(t^2+20) + C$

when $t=0$ $V=0$.

$0 = 18\ln(20) + C$.

$C = -18\ln(20)$

So $V = 18\ln(t^2+20) - 18\ln(20)$

$V = 18\ln\left(\frac{t^2+20}{20}\right)$. 3

(ii) Find t when $V=9$.

$9 = 18\ln\left(\frac{t^2+20}{20}\right)$.

$\frac{t^2+20}{20} = e^{\frac{1}{2}}$

$t^2+20 = 20e^{\frac{1}{2}}$ 2

$t = \pm\sqrt{20e^{\frac{1}{2}}-20}$

$t =$ 3hr 36 mins. since $t>0$

11:36 am.

(iii) $\frac{dv}{dt} = \frac{t^2}{k}$.

$V = \frac{t^3}{3k} + C$.

when $t=0$ $v=9$.

$9 = C$.

So $V = \frac{t^3}{3k} + 9$.

Find k When $V=0$, $t=5$.

$0 = \frac{5^3}{3k} + 9$.

$\frac{5^3}{3k} = -9$ 2

$5^3 = -3^3 k$.

$k = -\left(\frac{5}{3}\right)^3$

$= -\frac{125}{27}$.

(d) (i) Let $f(x) = x + 900x^{-1}$

$f'(x) = 1 - 900x^{-2}$

$f''(x) = \frac{1800}{x^3}$

Stat Pts $f'(x) = 0$.

$1 = \frac{900}{x^2}$.

$x = \pm 30$. $\quad x > 0$

So $x = 30$.

Nature

$f''(30) = \frac{1800}{30^3} > 0$ minima

$\therefore$ minimum value is $f(30) = 60$.

(ii)

Cost = (Cost per hour) x (hours).

$S = \frac{D}{T}$

$T = \frac{D}{S} = \frac{d}{v}$ hours

$C(v) = (9000 + 10v^2)\left(\frac{d}{v}\right)$.

$= \frac{9000d}{v} + 10dv$.

$C'(v) = -\frac{9000d}{v^2} + 10d.$

$C''(v) = \frac{18000d}{v^3}$

Stat Pt $C'(v) = 0.$

$10d = \frac{9000d}{v^2}.$

$v = \pm 30. \quad v > 0$

2

so $v = 30.$

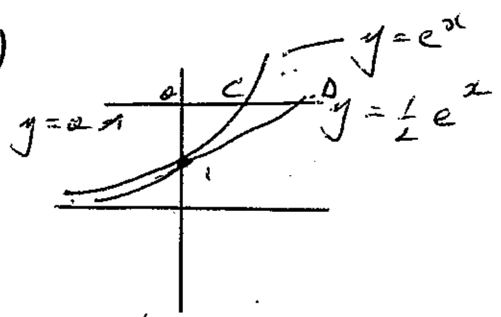
Nature

$C''(30) = \frac{2}{3} > 0$ minima.

30 km/h.

SECTION E

a)



$y = e^x \Rightarrow x = \ln y$

$= \ln a$ at $y = a$

$2y = e^x \Rightarrow x = \ln 2y$

$= \ln 2a$ at $y = a$

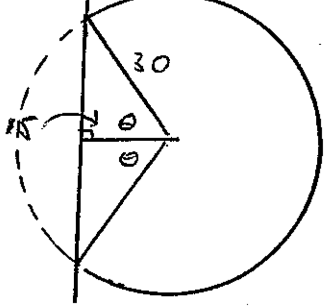
Hence $DC = \ln 2a - \ln a$

$= \ln \frac{2a}{a}$

$\boxed{= \ln 2}$

Which is independent of a

b)



$\cos\theta = \frac{1}{2}, \theta = \frac{\pi}{3}$

$2\theta = \frac{2\pi}{3}, r = 30$

Area consists of Circle − Segment

$A = \pi r^2 - (\frac{1}{2}r^2 2\theta - \frac{1}{2}r^2 \sin 2\theta)$

$= 900\pi - (\frac{1}{2} \times 900 \times 2\frac{\pi}{3} - \frac{1}{2} \times 900 \frac{\sqrt{3}}{2})$

$= 900\pi - 300\pi + 225\sqrt{3}$

$600\pi + 225\sqrt{3}$

$\boxed{= 2275 \ m^2}$

c) $= \sin x + \sin^2 x, \ 0 \le x \le 2\pi$

$= \sin x(1 + \sin x)$

$f(x) = 0$ when $\sin x = 0$

ie $x = 0, \pi, 2\pi$

$f(x) = 0$ when $\sin x = -1$

ie $x = \frac{3\pi}{2}$

i)

$f'(x) = \cos x + 2\sin x \cos x$

$= \cos x(1 + 2\sin x)$

$f'(x) = 0$ when $\cos x = 0$

$\sin x = -\frac{1}{2}$

ii)

If $\cos x = 0 \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$

When $x = \frac{\pi}{2} \quad f(x) = 2$

When $x = \frac{3\pi}{2}, \ f(x) = 0$

If $\sin x = -\frac{1}{2}, x = \frac{7\pi}{6}, \frac{11\pi}{6}$

When $x = \frac{7\pi}{6}, f(x) = -\frac{1}{4}$

When $x = \frac{11\pi}{6} \quad f(x) = -\frac{1}{4}$

The FOUR stationary points are $(\frac{\pi}{2}, 2)$ $(\frac{3\pi}{2}, 0)$ $(\frac{7\pi}{6}, -\frac{1}{4})$ $(\frac{11\pi}{6}, -\frac{1}{4})$

NOW $f''(x) = -\sin x + 2(-\sin^2 x + \cos^2 x)$

$= -\sin x + 2(1 - 2\sin^2 x)$

$= 2 - \sin x - 4\sin^2 x$

When $x = \frac{\pi}{2}, \ f''(x) = 2 - 1 - 4 = -3$ MAX

When $x = \frac{3\pi}{2}, \ f''(x) = 2 + 1 - 4 = -1$ MA

When $x = \frac{7\pi}{6}, \ f''(x) = 2 + \frac{1}{2} - 1 = 1\frac{1}{2}$ MIN

When $x = \frac{11\pi}{6}, \ f''(x) = 2 + \frac{1}{2} - 1 = 1\frac{1}{2}$ MI