Sydney Boys' High School

MOORE PARK, SURRY HILLS

2014
Year 11 Yearly

Mathematics Accelerated

General Instructions:

- Reading Time – 5 Minutes.
- Working time – 2 Hours.
- Write using black or blue pen.
- Board approved calculators maybe used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for untidy or badly arranged work.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 125

- Attempt all Questions
- The mark value of each question is shown on the right hand side.
- Each Question is to be answered in a **NEW** writing booklet, clearly labelled Question 1, Question 2 and so on.
- Ensure that the **graph sheet** for Question 4 is inside the booklet for Question 4.

Examiner: *Mr E. Choy*

Note: This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Standard Integrals

$$\int x^n \, dx \quad = \frac{1}{n+1} x^{n+1}, \qquad n \neq -1; \; x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} \, dx \quad = \ln x, \qquad x > 0$$

$$\int e^{ax} \, dx \quad = \frac{1}{a} e^{ax}, \qquad a \neq 0$$

$$\int \cos ax \, dx \quad = \frac{1}{a} \sin ax, \qquad a \neq 0$$

$$\int \sin ax \, dx \quad = -\frac{1}{a} \cos ax, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx \quad = \frac{1}{a} \tan ax, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx \quad = \frac{1}{a} \sec ax, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx \quad = \frac{1}{a} \tan^{-1} \frac{x}{a}, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx \quad = \sin^{-1} \frac{x}{a}, \qquad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx \quad = \ln\left(x + \sqrt{x^2 - a^2}\right), \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx \quad = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \qquad x > 0$

Question 1 (30 Marks) - Start a NEW writing booklet.

Marks

(a) Find an approximation to $e^3 - 2e^2$ (correct to 2 d.p.) **1**

(b) Find x if $\log_2 x = 4$. **1**

(c) For each of the following, write down the value of k:

(i) $\log_e\left(\frac{1}{x^2}\right) = k\log_e x$ **2**

(ii) $\log_e 3 - \frac{1}{2}\log_e 4 = \log_e k$ **2**

(iii) $k = \sqrt{\log_{11}(\log_e e)}$ **2**

(d) Solve $\sin x = \frac{\sqrt{3}}{2}$, for $0 \le x \le 2\pi$. **2**

(e) Sketch the graph of $y = \cos 2x$, for $0 \le x \le \pi$. **2**

(f) Given that m is a positive number find the smallest and the largest of the following numbers **2**

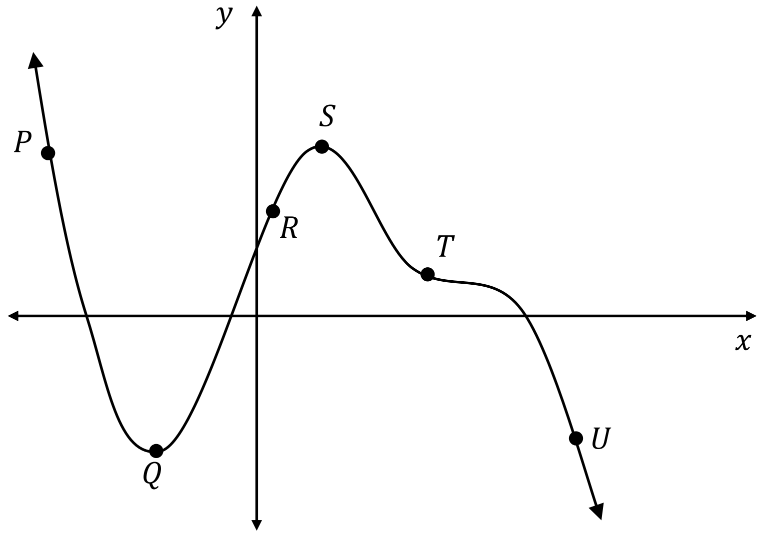
$$e^{\frac{m}{2}}, e^m, e^{-m}, e^{-\frac{m}{2}}$$

(g) **(i)** Find $\int_1^a \frac{1}{x}\,dx$, where $a > 1$. **2**

(ii) Hence find the value of a when $\int_1^a \frac{1}{x}\,dx = 1$. **2**

(h) The line $y = mx$ is a tangent to the curve $y = e^{2x}$. Find m. **2**

(i)



A curve $y = f(x)$ is sketched. Six points on the graph are labelled: P, Q, R, S, T and U.

Points R and T are points of inflexion. Point Q is a relative minimum turning point and Point S is a relative maximum turning point.

State at which of the points P, Q, R, S, T or U.

(i) $f'(x) = 0$ and $f''(x) < 0$. **1**

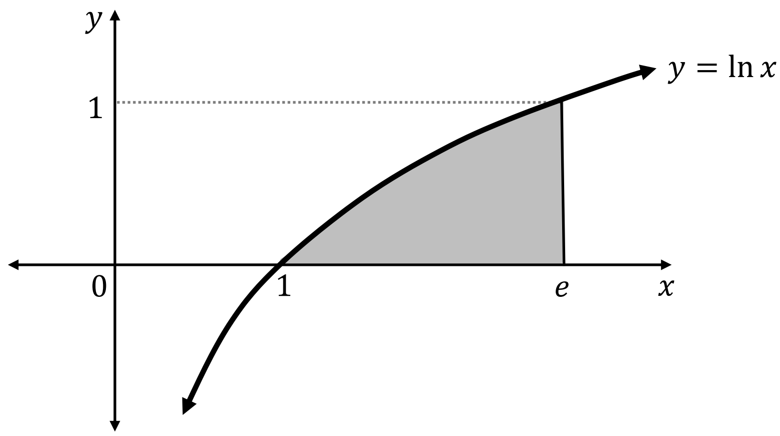
(ii) $f'(x) \neq 0$ and $f''(x) = 0$. **1**

(iii) $f'(x) < 0$ and $f''(x) < 0$. **1**

(iv) $f'(x) = 0$ and $f''(x) = 0$. **1**

(j) If $\dfrac{dy}{dx} = e^{2x-1}$, and $y = 5$ when $x = \dfrac{1}{2}$, find the exact value of y when $x = 1$. **2**

(k) **4**



The graph of $y = \ln x$ is sketched above and the shaded area is rotated about the $y -$ axis. Find the exact volume of the resulting solid.

Question 2 (30 Marks) - Start a NEW writing booklet.

Marks

(a) Differentiate:

(i) e^{-5x} **1**

(ii) $3\ \sin x + \sqrt[3]{x}$ **2**

(iii) $\ln(3x - 1)$ **1**

(iv) $(x^2 + 3)^4$ **2**

(b) Find:

(i) $\int e^{2x+5}\ dx$ **2**

(ii) $\int \frac{x}{2x^2 + 4}\ dx$ **2**

(iii) $\int_e^{e^2} \left(1 + \frac{1}{x}\right)\ dx$ **2**

(c) Find the equation of the normal to the curve $y = 2\cos 2x$ at the point $\left(\frac{\pi}{4}, 0\right)$. **2**

(d) **(i)** Find $\frac{d}{dx}\ (x \ln x)$ **1**

(ii) Hence find $\int_1^e \ln x\ dx$. **2**

(e) Show that

(i) $\dfrac{1}{x-3} - \dfrac{1}{x+3} = \dfrac{6}{x^2-9}$ **1**

(ii) Hence find $\displaystyle\int \frac{dx}{x^2-9}$. **2**

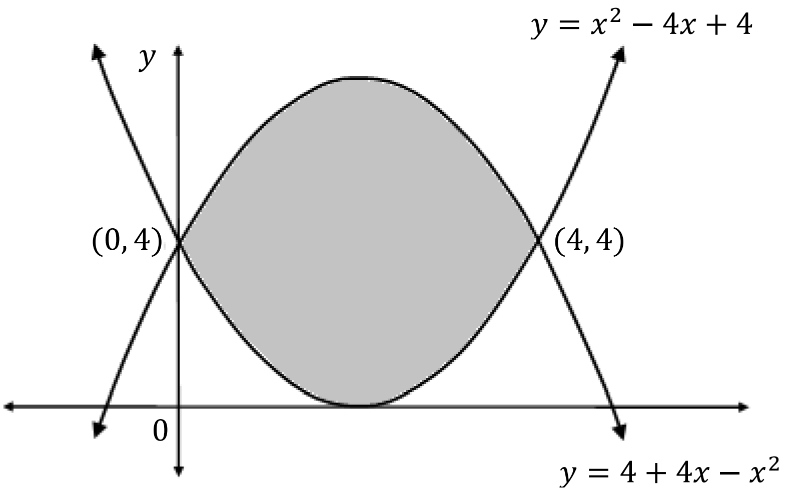
(f) A certain parabola has focus $(3, 7)$ and directrix $y = 1$.

(i) Find the coordinates of the vertex. **1**

(ii) Write down the focal length. **1**

(iii) Write the equation of the parabola in the form $(x-h)^2 = 4a(y-k)$. **1**

(g) **2**

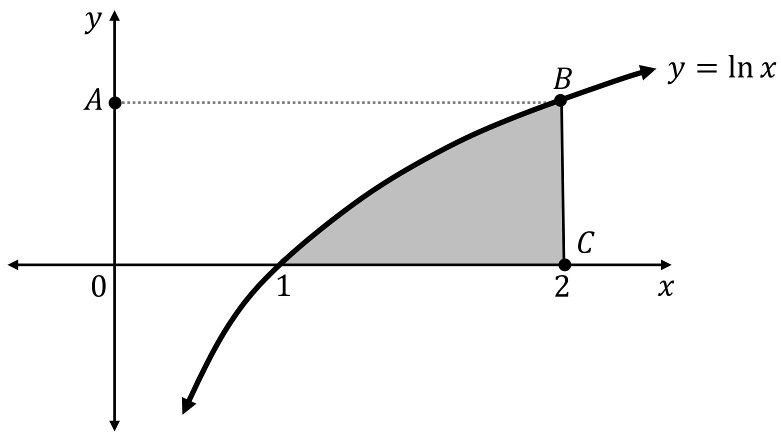


The diagram shows the region bounded by two parabolas:

$$y = x^2 - 4x + 4 \text{ and}$$
$$y = 4 + 4x - x^2.$$

Calculate the area of this region.

(h) **5**



(i) Find an expression for the area of the rectangle $OABC$.

(ii) Calculate the area bounded by the curve $y = \ln x$, both axes and the line $y = \ln 2$.

(iii) Show that the area of the shaded region is equal to $2 \ln 2 - 1$ square units.

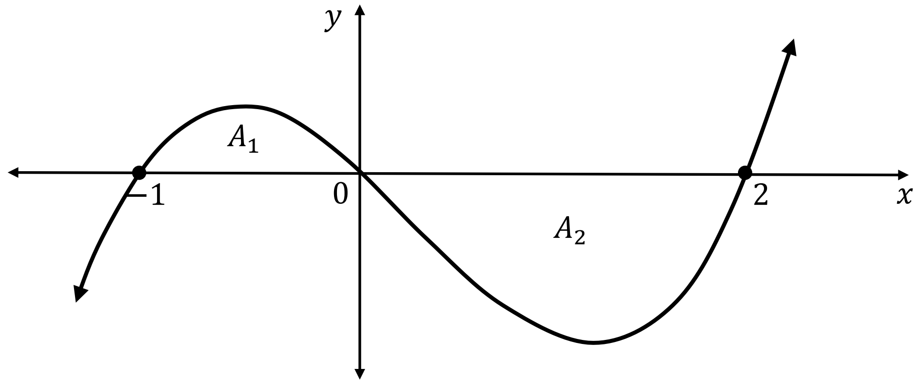
Question 3 (20 Marks) - Start a NEW writing booklet.

Marks

(a) **(i)** On the same set of axes, sketch the curves $y = e^x$ and $y = e^{-x}$ for the domain $-2 \leq x \leq 2$. **1**

(ii) Find the gradient of the tangents to the curves at the point $(0, 1)$. Hence find the angle between the tangents at this point. **2**

(b)



The diagram above is the graph of the function $y = x(x+1)(x-2)$

(i) Find the area A_1. **1**

(ii) Find the area A_2. **1**

(iii) Evaluate the definite integral **1**

$$\int_{-1}^{2} x(x+1)(x-2)\ dx$$

(iv) Why is the answer to **(iii)** not equal to the sum $A_1 + A_2$ of the two areas? **1**

(c) (i) Sketch the curve $y = \ln(x+1)$ for $-1 < x \leq 3$. **1**

(ii) The volume of the solid of revolution formed when the section of the curve $y = \ln(x+1)$ from $x = 0$ to $x = 2$ is rotated around the $x-$axis is given by **2**

$$V = \pi \int_0^2 [\ln(x+1)]^2 \ dx$$

Use Simpson's rule with three function values to approximate this integral. (Leave your answer correct to two decimal places).

(d) Consider the function $y = f(x) = \sqrt{x}\,(1-x)$.

(i) Find the domain of the function. **1**

(ii) Show that **2**

$$\frac{dy}{dx} = \frac{1-3x}{2\sqrt{x}}$$

(iii) Hence or otherwise find $\dfrac{d^2y}{dx^2}$ **2**

(iv) Find the equation of the tangent at $x = 0$. **1**

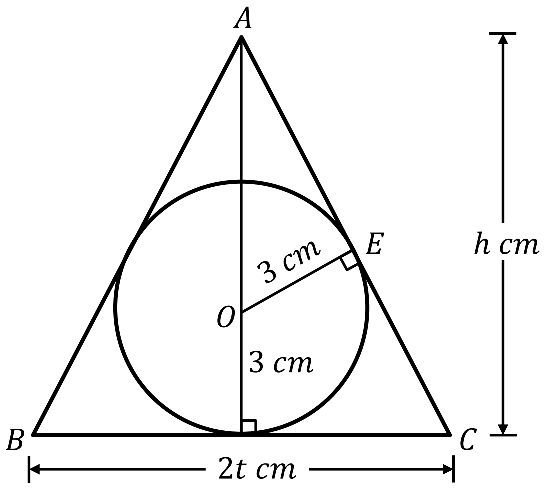
(v) Show that the curve $y = \sqrt{x}\,(1-x)$ has only one stationary point and determine its nature. **2**

(vi) Show that this curve has no point of inflexion. **1**

(vii) Determine the concavity of the curve for $x > 0$. **1**

Question 4 (20 Marks) - Start a NEW writing booklet.

Marks



$Figure$ (1)

ABC is a variable isosceles triangle with $AB = AC$ such that the radius of its inscribed circle is $3\ cm$. The height AD and the base BC of ΔABC are $h\ cm$ and $2t\ cm$ respectively, where $h > 6$ ($see\ Figure$ (1)). Let $p\ cm$ be the perimeter of ΔABC.

(a) Show that **3**

$$t^2 = \frac{9h}{h-6}$$

(b) Show that **3**

$$p = \frac{2h^{\frac{3}{2}}}{\sqrt{(h-6)}}$$

(c) Find:

(i) The range of the values of h for which $\dfrac{dp}{dh}$ is positive. **3**

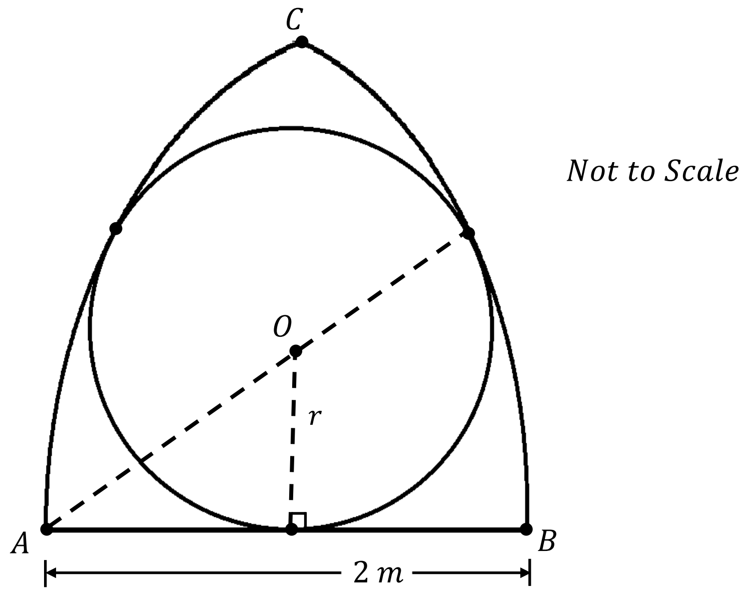
(ii) The minimum value of p. **3**

(d) **(i)** In $Figure$ (2), on the separate sheet provided. Sketch the graph of p against h for $h > 6$. **2**

(ii) Hence write down the range of values of p for which two different isosceles triangle whose inscribed circles are of radii $3\ cm$ can have the same perimeter $p\ cm$. **2**

(e) The diagram ABC shows a design of a Gothic Window. The length AB is 2 metres. AC is a circular arc with centre B and radius AB. BC is also a circular arc with centre A and radius AB.

A circle with centre O and a radius r is constructed in the window so that it touches the three sides of the window.



(i) Find the distance AO in terms of r. **1**

(ii) Show that $r = 0.75\ m$ **3**

Question 5 (25 Marks) - Start a NEW writing booklet.

Marks

(a) A parachutist jumps out of an aircraft from a great height and sometime later opens the parachute. The speed at any time t seconds from when the parachute opens is v metres per second, where

$$v = 8 + 22e^{-0.07t}$$

(i) Find the speed of the parachutist when the parachute opens. **2**

(ii) State the limiting speed that the parachutist would approach. **2**

(iii) Find the value of t, to the nearest $\frac{1}{10}th$ of a second, when the parachutist is travelling at 20 m/s. **2**

(iv) Find an expression for the acceleration t seconds after the parachute is opened. **2**

(v) Explain what happens to the acceleration of the parachutist. **2**

(b) A tank is emptied of water by a continuously opening valve so that t minutes after the valve begins to open, the volume V litres of water in the tank is given by

$$V = a + bt^2$$

where a and b are constants.

Initially the tank contains 500 litres and it takes 5 minutes to empty it.

(i) Find the value of a and b. **2**

(ii) Find the rate at which the volume is changing after 2.5 minutes. **2**

(iii) Find the rate at which the volume is changing when the tank is half empty (to the nearest litre/minute). **2**

(c) The rate of decay of a radioactive material is such that when the mass is present is m,

$$\frac{dm}{dt} = -km$$

where $k > 0$ is a constant.

(i) Verify the function $M = M_0e^{-kt}$ (M_0 constant) satisfies this equation. **2**

(ii) If the half – life of the radioactive material is T, prove that **2**

$$k = \frac{\ln 2}{T}$$

(iii) A substance contains two types of radioactive material A and B with a half – life T_1 (for A) and T_2 (for B) respectively ($T_1 > T_2$). **5**

Initially the mass of B is twice that of A.

Prove that the substance will contain an equal mass of A and B after time t seconds where

$$t = \frac{T_1\,T_2}{T_1 - T_2} \text{ seconds.}$$

End of Paper

Maths Accel: 2014 Yearly

QUESTION ONE.

a) $e^3 - 2e^2 = 5 \cdot 3074$

$= 5 \cdot 31 \ (2dp).$

b) $\log_2 x = 4$

$x = 16.$

c) i) $\log_e \frac{1}{x^2}$

$= \log_e x^{-2}$

$= -2 \log_e x$

$\therefore K = -2$

ii) $\log 3 - \frac{1}{2} \log 4$

$= \log 3 - \log 4^{\frac{1}{2}}$

$= \log 3 - \log 2.$

$= \log\left(\frac{3}{2}\right)$

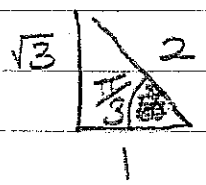
$\therefore K = \log_e\left(\frac{3}{2}\right)$

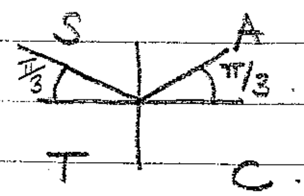
iii) $K = \sqrt{\log_{11} (\log_e e)}$

$= \sqrt{\log_{11} 1}$

$= 0$

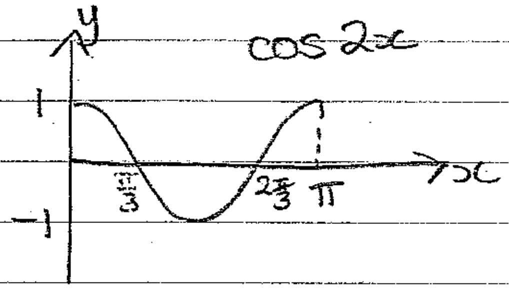
d) $\sin x = \frac{\sqrt{3}}{2}$





$x = \pi/3,\ 2\pi/3$

e)



$T = \frac{2\pi}{2} = \pi$

f) $e^{m/2}$ e^{m} $\frac{1}{e^{m}}$ $\frac{1}{e^{m/2}}$

smallest $= \frac{1}{e^{m}},\ \frac{1}{e^{m/2}},\ e^{m/2},\ e^{m}$

g) i) $\int_1^a \frac{1}{x} \cdot dx \qquad a > 1$

$= \left[\ln x\right]_1^a$

$= \ln a - \ln 1$

$= \ln a \qquad \ln 1 = 0$

ii) $\ln a = 1$

$\underline{\underline{a = e}}$

i) i) S
ii) R
iii) U
iv) T

(4)

j) $\frac{dy}{dx} = e^{2x-1}$

$\int \frac{dy}{dx}\,dx = \int e^{2x-1}\,dx$

$y = \frac{1}{2}e^{2x-1} + c$

$y=5$ $x=\frac{1}{2}$. $5 = \frac{1}{2}e^{2(1/2)-1} + c$

$5 = \frac{1}{2}e^{0} + c$

$9/2 = c$

$y = \frac{1}{2}e^{2x-1} + 9/2$

when $x=1$

$y = \frac{1}{2}e^{1} + 9/2$ (2)

$y = \frac{e+9}{2}$

h) $y = e^{2x}$

$\frac{dy}{dx} = 2e^{2x}$

$y - y_1 = m(x - x_1)$.

$y - y_1 = 2e^{2x}(x - x_1)$ $\quad x = \frac{1}{2} \quad y = e$

$y - e = 2e(x - \frac{1}{2})$

$y = 2ex$.

$m = 2e$ ✗

$V_1 = \pi \int_0^1 (e^{y})^2 \cdot dy$ $\qquad$ $y = \ln x$

$e^y = x$

$= \pi \left[\frac{1}{2} e^{2y} \right]_0^1$

$= \pi \left(\frac{1}{2} e^2 - \frac{1}{2} e^0 \right)$ $\qquad$ (4)

$\quad$ $\int_1^e$

$= \pi \; \frac{e^2}{2} - \frac{1}{2}$

$= \frac{\pi}{2} (e^2 - 1)$ units^3

V_2 Volume of whole Solid $= \pi r^2 h$

$= \pi e^2 1$

Required Volume =

Volume btwn y-axis + curve

$V_2 - V_1$

$= \pi e^2 - \frac{\pi}{2} e^2 + \frac{\pi}{2}$

$= \frac{\pi e^2 + \pi}{2}$

$= \frac{\pi}{2} (e^2 + 1)$ units^3 C

QUESTION 2.

(a) (i) $-5e^{-5x}$ [1]

(ii) $3\cos x + \frac{1}{3}x^{-\frac{2}{3}}$ [2]

$= 3\cos x + \frac{1}{3\sqrt[3]{x^2}}$

(iii) $\frac{3}{3x-1}$ [1]

(iv) $8x(x^2+3)^3$ [2]

(b) (i) $\frac{1}{2}e^{2x+5} + c$ [2]

(ii) $\frac{1}{4}\ln(2x^2+4) + c$ [2]

(iii) $\left[x + \ln x\right]_e^{e^2} = e^2 + 2 - (e+1)$ [2]

$= e^2 - e + 1.$

(c) $y = 2\cos 2x$

$y' = -4\sin 2x$

$\therefore m_T$ at $\left(\frac{\pi}{4}, 0\right) = -4\sin\frac{\pi}{2}$

$= -4$

$\therefore m_N = \frac{1}{4}$

$\therefore$ Equation of normal: $\frac{y-0}{x-\frac{\pi}{4}} = \frac{1}{4}$ [2]

$4y = x - \frac{\pi}{4}$

$x - 4y - \frac{\pi}{4} = 0$

Q2 (CONTD)

(d) (i) $\frac{d}{dx}(x\ln x) = x \times \frac{1}{x} + 1 \times \ln x$

$= 1 + \ln x.$ [1]

(ii) $\therefore \int_1^e (1+\ln x)\,dx = \left[x\ln x\right]_1^e$

$= e$

$\therefore \int_1^e x\,dx + \int_1^e \ln x\,dx = e$

$e - 1 + \int_1^e \ln x.dx = e$

$\therefore \int_1^e \ln x\,dx = 1.$ [2]

(e) (i) $LHS = \frac{1}{x-3} - \frac{1}{x+3}$

$= \frac{x+3-(x-3)}{(x-3)(x+3)}$

$= \frac{6}{x^2-9}$ [1]

$= RHS.$

(ii) $\int \frac{6\,dx}{x^2-9} = \int \left(\frac{1}{x-3} - \frac{1}{x+3}\right) dx$

$= \ln(x-3) - \ln(x+3) + c$

$= \ln \frac{x-3}{x+3} + c$

$\therefore \int \frac{dx}{x^2-9} = \frac{1}{6} \ln \frac{x-3}{x+3} + c_1$ [2]

4(e) (CONTD)

(f) (i) (3, 4) [1]

(ii) 3 [1]

(iii) $(x-3)^2 = 12(y-4)$. [1]

(g) $$A = \int_0^4 \left[(4+4x-x^2)-(x^2-4x+4)\right]dx.$$

$$= \int_0^4 (8x-2x^2)\,dx.$$

$$= \left[4x^2 - \frac{2x^3}{3}\right]_0^4$$

$$= 64 - \frac{2}{3}\times 64$$

$$= \frac{64}{3}\ u^2.$$ [2]

(h) (i) $2\times \ln 2 = \ln 4.\ u^2$ [1]

(ii) $$\int_0^{\ln 2} e^y\,dy = \left[e^y\right]_0^{\ln 2}$$

$$= e^{\ln 2} - e^0$$

$$= 2 - 1$$

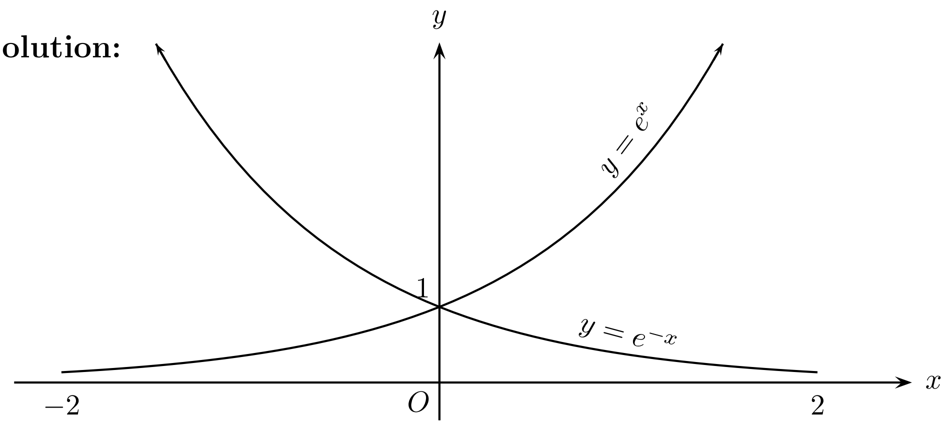
$$= 1\ u^2$$ [3]

(iii) $(\ln 4 - 1) = (2\ln 2 - 1)\,u^2.$ [1]

2014 Accelerated Mathematics Trial HSC:
Solutions— Question 3

3. (a) (i) Onthe same set of axes, sketch the curves $y = e^x$ and $y = e^{-x}$ for the domain $-2 \leqslant x \leqslant 2$. [1]

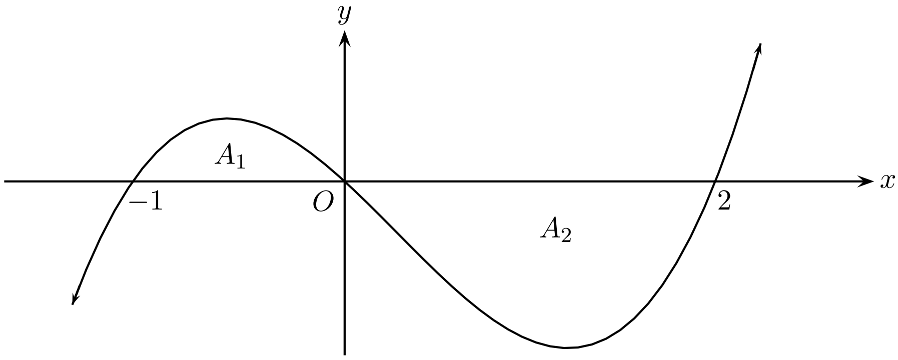
Solution:



(ii) Find the gradients of the tangents to the curves at the point $(0, 1)$. Hence find the angle between the tangents at this point. [2]

Solution:
$$\begin{aligned} y &= e^x, & y &= e^{-x}, \\ y' &= e^x, & y' &= -e^{-x}, \\ &= 1 \text{ at } (0,1). & &= -1 \text{ at } (0,1). \end{aligned}$$
Now $-1 \times 1 = -1$, so the tangents are orthogonal.

(b)



The diagram above is the graph of the function $y = x(x + 1)(x - 2)$.

(i) Find the area A_1. [1]

Solution:
$$\begin{aligned} \text{Area } A_1 &= \int_{-1}^{0} \left(x^3 - x^2 - 2x\right) dx, \\ &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0}, \\ &= 0 - \left(\tfrac{1}{4} + \tfrac{1}{3} - 1\right), \\ &= \frac{5}{12}. \end{aligned}$$

(ii) Find the area A_2. [1]

Solution:
$$\begin{aligned} \text{Area } A_2 &= \left| \int_0^2 \left(x^3 - x^2 - 2x\right) dx \right|, \\ &= \left| \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^{0} \right|, \\ &= \left| \left(4 - \tfrac{8}{3} - 4\right) - 0 \right|, \\ &= \frac{8}{3}. \end{aligned}$$

(iii) Evaluate the definite integral [1]

$$\int_{-1}^{2} x(x+1)(x-2)\,dx.$$

Solution: Method 1—
From (i) and (ii),
$$\begin{aligned} \int_{-1}^{2} \left(x^3 - x^2 - 2x\right) dx &= \frac{5}{12} - \frac{8}{3}, \\ &= -\frac{9}{4}. \end{aligned}$$

Solution: Method 2—
$$\begin{aligned} \int_{-1}^{2} \left(x^3 - x^2 - 2x\right) dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^{2}, \\ &= \left(4 - \tfrac{8}{3} - 4\right) - \left(\tfrac{1}{4} + \tfrac{1}{3} - 1\right), \\ &= -\frac{9}{4}. \end{aligned}$$

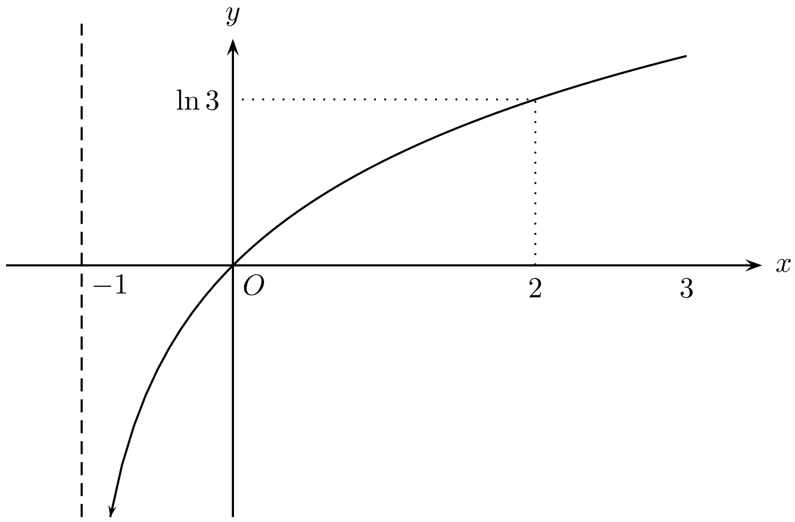
(iv) Why is the answer to (iii) not equal to the sum $A_1 + A_2$ of the two areas? [1]

Solution: When integrating, area below the x-axis counts as negative* so in part (ii) we had to take the absolute value to get the real area, whereas in part (iii) it remained negative.

*From the definition of integration as a sum of infinitesimally wide rectangles, $f(x) \times h$, if $f(x) < 0$ then the resultant bit of integral will be negative.

(c) (i) Sketch the curve $y = ln(x+1)$ for $-1 < x \leqslant 3$. $\boxed{1}$

Solution:



(ii) The volume of the solid of revolution formed when the section of the curve $y = ln(x+1)$ from $x = 0$ to $x = 2$ is rotated around the x-axis is given by $\boxed{2}$

$$V = \pi \int_0^2 \big[\ln(x+1)\big]^2 \, dx.$$

Use Simpson's rule with three function values to approximate this integral. (Leave your answer correct to two decimal places.)

Solution: $$\begin{aligned} \pi \int_0^2 \big[\ln(x+1)\big]^2 \, dx &\approx \frac{\pi}{3}\big[0^2 + 4(\ln 2)^2 + (\ln 3)^2\big], \\ &\approx 3.28 \text{ (to 2 dec. pl.)} \end{aligned}$$

(d) Consider the function $y = f(x) = \sqrt{x}(1-x)$.

(i) Find the domain of the function. $\boxed{1}$

Solution: $x \geqslant 0$.

(ii) Show that $\boxed{2}$

$$\frac{dy}{dx} = \frac{1-3x}{2\sqrt{x}}.$$

Solution: $$\begin{aligned} \frac{d}{dx}\big(x^{\frac{1}{2}}(1-x)\big) &= \tfrac{1}{2} \times x^{-\frac{1}{2}} \times (1-x) - x^{\frac{1}{2}}, \\ &= \frac{1-x-2x}{2x^{\frac{1}{2}}}, \\ &= \frac{1-3x}{2\sqrt{x}}. \end{aligned}$$

(iii) Hence or otherwise find $\dfrac{d^2y}{dx^2}$. [2]

> **Solution:** $$\frac{d^2y}{dx^2} = \frac{2\sqrt{x}(-3) - (1-3x)\frac{2}{2\sqrt{x}}}{4x},$$
> $$= \frac{-6x - 1 + 3x}{4x\sqrt{x}},$$
> $$= \frac{-3x-1}{4x\sqrt{x}}.$$

(iv) Find the equation of the tangent at $x = 0$. [1]

> **Solution:** As the curve is discontinuous when $x = 0$,
> there is no tangent at that point.
> **Acceptable solution:** When $x = 0,\ y = 0$ and y' is undefined.
> $\therefore$ Tangent is $x = 0$.

(v) Show that the curve $y = \sqrt{x}(1-x)$ has only one stationary point and determine its nature. [2]

> **Solution:** $\dfrac{dy}{dx} = 0$ when $x = \frac{1}{3}$,
> and $$\frac{d^2y}{dx^2} = \frac{-3 \times \frac{1}{3} - 1}{4 \times \frac{1}{3} \times \sqrt{\frac{1}{3}}} \approx -2.598 \text{ by calculator.}$$
> $\therefore$ There is a maximum at $\left(\dfrac{1}{3}, \dfrac{2\sqrt{3}}{9}\right)$.

(vi) Show that this curve has no point of inflexion. [1]

> **Solution:** If $\dfrac{d^2y}{dx^2} = 0,\quad -3x - 1 = 0,$
> $x = -\frac{1}{3}.$
> But $x \geqslant 0$ from part (i), so there can be no inflexion.

(vii) Determine the concavity of the curve for $x > 0$. [1]

> **Solution:** If concave downwards, $\dfrac{d^2y}{dx^2} < 0$.
> When $x > 0$, $-1 - 3x < 0$ and $4x\sqrt{x} > 0$; so $\dfrac{-3x-1}{4x\sqrt{x}} < 0$.
> Hence the curve is concave downwards for all $x > 0$.

Question 4:

a). In $\triangle AOE$ and $\triangle ADC$

$\angle OAE = \angle CAD$ (common)

$\angle AEO = \angle ADC$ (given)

$\therefore \triangle AOE \parallel\!\mid \triangle ADC$ (equiangular) ①

$$AE = \sqrt{(h-3)^2 - 9}$$

$$= \sqrt{h^2 - 6h + 9 - 9}$$

$$= \sqrt{h^2 - 6h}$$ ①

$$\frac{AE}{AD} = \frac{OE}{DC}$$ (corr. sides of similar $\triangle$s are prop.

$$\frac{\sqrt{h^2 - 6h}}{h} = \frac{3}{t}$$

$$\frac{h^2 - 6h}{h^2} = \frac{9}{t^2}$$

$$\frac{h-6}{h} = \frac{9}{t^2}$$

$$t^2 = \frac{9h}{h-6}$$ ①

b) $AC = AB = \sqrt{h^2 + t^2}$

$$p = 2\sqrt{h^2+t^2} + 2t$$

$$= 2\sqrt{h^2 + \frac{9h}{h-6}} + 2\sqrt{\frac{9h}{h-6}}$$

$$= 2\sqrt{\frac{h^3-6h^2+9h}{h-6}} + \frac{6\sqrt{h}}{\sqrt{h-6}}$$

$$= 2\sqrt{\frac{h(h-3)^2}{h-6}} + \frac{6\sqrt{h}}{\sqrt{h-6}}$$

$$= \frac{2\sqrt{h}(h-3)}{\sqrt{h-6}} + \frac{6\sqrt{h}}{\sqrt{h-6}}$$

$$= \frac{2h\sqrt{h}}{\sqrt{h-6}}$$

(3)

$$\therefore p = \frac{2h^{3/2}}{\sqrt{h-6}}$$

c) (i) $p = \dfrac{2h^{3/2}}{(h-6)}$

$$\frac{dp}{dh} = \frac{3h^{\frac{1}{2}}(h-6) - 2h^{3/2}}{(h-6)^2} \quad (1)$$

$$= \frac{h^{\frac{1}{2}}(3h-18-2h)}{(h-6)^2}$$

$$= \frac{\sqrt{h}\,(h-18)}{(h-6)^2} \quad (1)$$

$\boxed{\therefore\ h > 18}$ (1)

(ii) $\frac{dp}{dh} = 0$ for stat. points

$$\frac{\sqrt{h}(h-18)}{(h-6)^2} = 0$$

$h = 0, 18$ but $h > 6$

$\therefore h = 18$ (1)

h	18^-	18	18^+
$\frac{dp}{dh}$	$-$	0	$+$

(1)

when $h = 18$

$$p = \frac{2(18)^{3/2}}{12}$$

$\boxed{\therefore p = 9\sqrt{2}}$ (1)

$p = \dfrac{2h^{3/2}}{\sqrt{h-6}}$

$$\frac{dp}{dh} = \frac{3h^{1/2}\sqrt{h-6} - \frac{1}{2}(h-6)^{-\frac{1}{2}} \cdot 2h^{3/2}}{h-6} \quad (1)$$

$$= \frac{h^{1/2}(h-6)^{-\frac{1}{2}}(3h-18-h)}{h-6}$$

$$= \frac{\sqrt{h}\,(h-6)^{-\frac{1}{2}}(2h-18)}{h-6}$$

$$= \frac{2\sqrt{h}\,(h-9)}{(h-6)^{3/2}} \quad (1)$$

$\boxed{\therefore\ h > 9}$ (1)

$\frac{dp}{dh} = 0$ for stat. points

$$\frac{2\sqrt{h}(h-9)}{(h-6)^{3/2}} = 0$$

$h = 0, 9$ but $h > 6$

$\therefore h = 9$ (1)

h	9^-	9	9^+
$\frac{dp}{dh}$	$-$	0	$+$

(1)

when $h = 9$

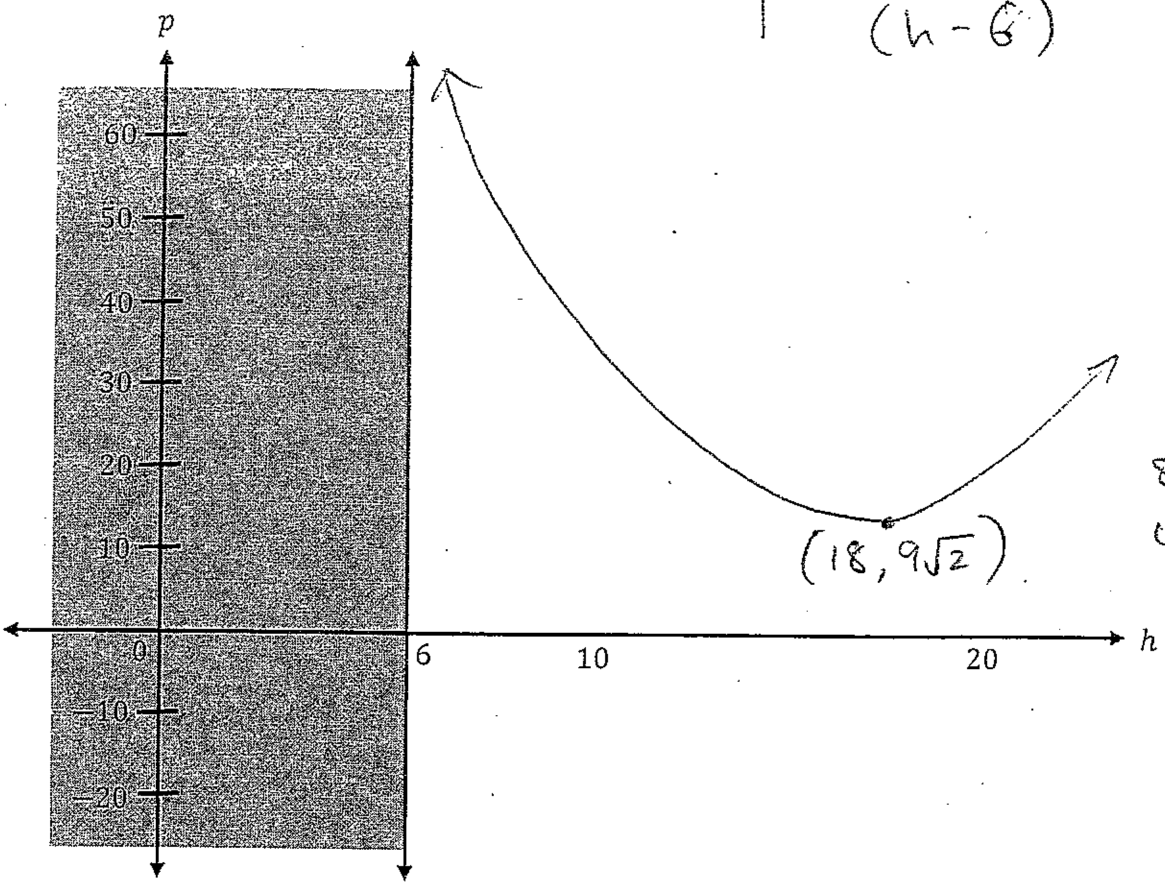
$$p = \frac{2(9)^{3/2}}{\sqrt{9-6}}$$

$\boxed{\therefore p = 18\sqrt{3}}$ (1)

Question 4

(d) (i) *Figure* (2)

$$p = \frac{2h^{3/2}}{(h-6)}$$



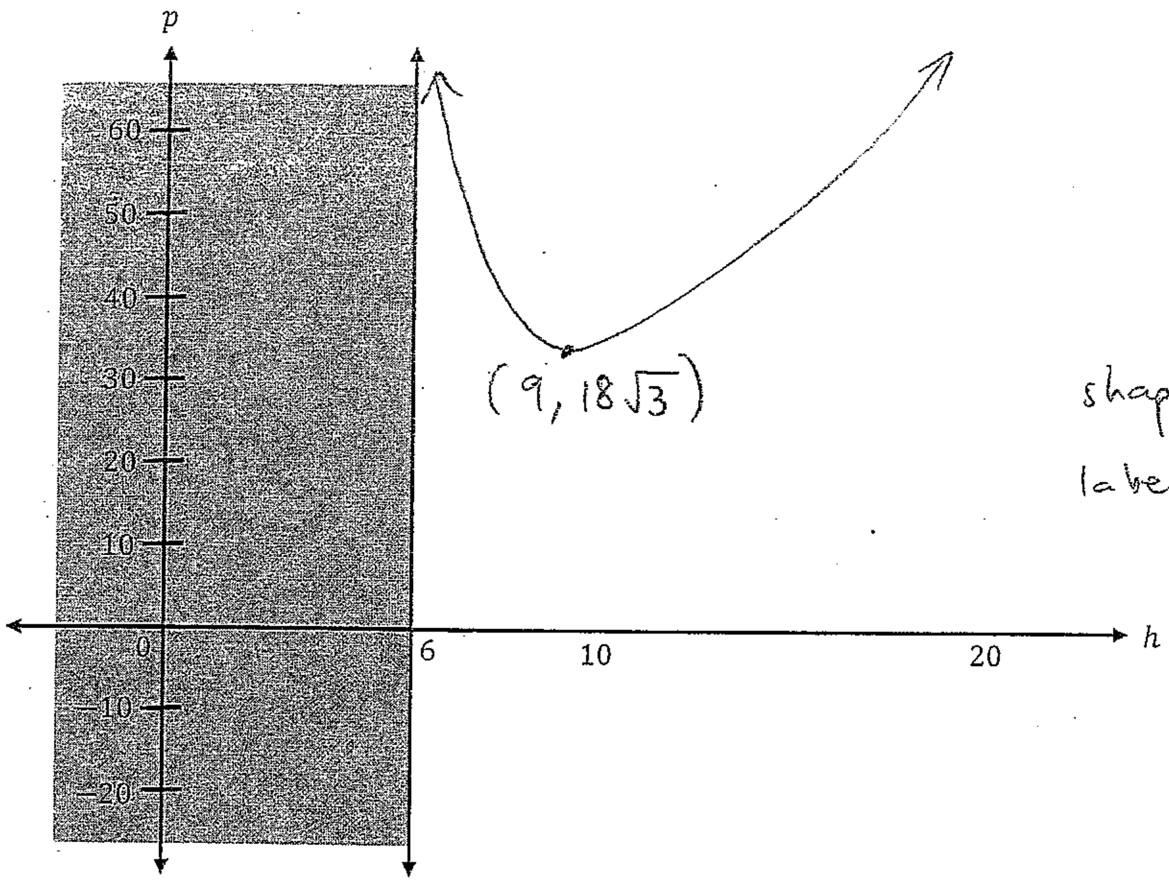
shape (1)
label point (1)

(ii) $p > 9\sqrt{2}$ (2)

Question 4

(d) (i) *Figure* (2)

$p = \frac{2h^{3/2}}{\sqrt{h-6}}$



shape (1)

label point (1)

(ii) $p > 18\sqrt{3}$ (2)

e) (i) $AO = \sqrt{r^2+1}$ or $2-r$ (1)

(ii) $\sqrt{r^2+1} + r = 2$ (1)

$\sqrt{r^2+1} = 2 - r$

$r^2+1 = (2-r)^2$ (1)

$r^2+1 = 4 - 4r + r^2$

$4r = 3$

$r = \frac{3}{4}$ (1)

Question 5

(a) $v = 8 + 22e^{-0.07t}$

(i) When $t = 0$ **2**

$v = 8 + 22e^{0}$

$= 30$ m/s

(ii) As $t \to \infty$ so $e^{-0.07t} \to 0$. **2**

Thus limiting speed is 8 m/s.

(iii) $20 = 8 + 22e^{-0.07t}$ **2**

$12 = 22e^{-0.07t}$

$\frac{6}{11} = e^{-0.07t}$

Taking natural logs:

$-0.07t = \ln\frac{6}{11}$

$t = \frac{\ln\frac{6}{11}}{-0.07}$

≈ 8.7 sec

(iv) Acceleration **2**

$\frac{dv}{dt} = 0 + 22(-0.07)e^{-0.07t}$

$a = -1.54e^{-0.07t}$

(v) As $t \to \infty$ so $e^{-0.07t} \to 0$. **2**

Thus $a \to 0$.

That is, acceleration approaches zero.

(b) Volume in tank $V = a + bt^2$.

(i) When $t = 0$, $V = 500$ **2**

$500 = a + 0$

$\therefore V = 500 + bt^2$

When $t = 5$, $V = 0$

$0 = 500 + b(5)^2$

$25b = -500$

$b = -20$

$\therefore V = 500 - 20t^2$

(ii) $\frac{dV}{dt} = -40t$ **2**

After 2.5 minutes:

$\frac{dV}{dt} = -40 \times 2\tfrac{1}{2}$

$= -100 \text{ L/min}^2$

(iii) When $V = 250$ L **2**

$250 = 500 - 20t^2$

$-250 = -20t^2$

$t^2 = 12\tfrac{1}{2}$

$t \approx 3.535$

$\frac{dV}{dt} = -40(3.535)$

≈ -141 L/min

(c) $\frac{dM}{dt} = -kM$

(i) $M = M_0 e^{-kt}$ **2**

$\frac{dM}{dt} = -kM_0 e^{-kt}$

$= -kM$

Thus $M = M_0 e^{-kt}$ satisfies the DE.

(ii) At half-life, $M = \tfrac{1}{2} M_0$ **2**

$\therefore \tfrac{1}{2} M_0 = M_0 e^{-kt}$

$\tfrac{1}{2} = e^{-kt}$

Taking natural logs:

$-kt = \ln \tfrac{1}{2}$

$k = \frac{\ln \tfrac{1}{2}}{-t}$

$= \frac{\ln 2}{t}$

(iii) $M_{oA} e^{-k_A t} = 2M_{0A} e^{-k_B t}$ **7**

Taking Natural logs:

$-k_A t = \ln 2 + (-k_B t)$

$t(k_B - k_A) = \ln 2$

$t = \frac{\ln 2}{k_B - k_A}$

$= \frac{\ln 2}{\frac{\ln 2}{T_2} - \frac{\ln 2}{T_1}}$

$$= \frac{1}{\frac{1}{T_2} - \frac{1}{T_1}}$$

$$\therefore t = \frac{T_1 T_2}{T_1 - T_2} \text{ as required.}$$