SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2016

Year 11 Yearly Examination

Mathematics Accelerated

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Leave your answers in the simplest exact form, unless otherwise stated.
- In Questions 11-14, show relevant mathematical reasoning and/or calculations
- A reference sheet has been provided

Total Marks – 70 marks

Section I Pages 3-6

10 Marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II Pages 8-15

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Examiner: *E. Choy*

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Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 Which of the following is equal to $\dfrac{\sin(\pi-\alpha)}{\sin(\frac{\pi}{2}-\alpha)}$?

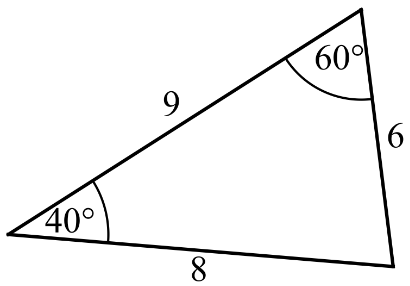
(A) $\cos(\frac{\pi}{2}-\alpha)$

(B) $\sin(\frac{\pi}{2}+\alpha)$

(C) $\frac{\pi}{2}+\alpha$

(D) $\tan\alpha$

2



NOT TO SCALE

In reference to the above diagram, which of the following statements is correct?

(A) $6^2 = 9^2 + 8^2 + 2\times 9\times 8\times \cos 40°$

(B) $9^2 = 6^2 + 8^2 - 2\times 9\times 6\times \cos 80°$

(C) $\dfrac{8}{\sin 60°} = \dfrac{9}{\sin 80°}$

(D) $\dfrac{6}{\sin 40°} = \dfrac{8}{\sin 80°}$

3 A deck of cards consists of 5 yellow and 5 green cards. Two cards are selected at random with replacement. What is the probability of choosing two cards of the same colour?

(A) $\dfrac{1}{50}$

(B) $\dfrac{2}{9}$

(C) $\dfrac{4}{9}$

(D) $\dfrac{1}{2}$

4 Consider the geometric series

$$1+\left(5-\sqrt{a}\right)+\left(5-\sqrt{a}\right)^2+\left(5-\sqrt{a}\right)^3+\ldots$$

If this series has a limiting sum which of the following statements is correct?

(A) $16 < a < 36$ and $a \neq 25$

(B) $4 < a < 6$ and $a \neq 25$

(C) $16 < a < 49$ and $a \neq 25$

(D) $a > 16$ and $a \neq 25$

5 What is a solution to the following equation?

$$\ln\left(\frac{1}{x^2}\right) = -4$$

(A) e^2

(B) e

(C) 2

(D) $\frac{1}{2}$

6 Which of the following is a correct statement when comparing the graph of $y = \cos x$ with that of $y = \cos\frac{1}{2}x$?

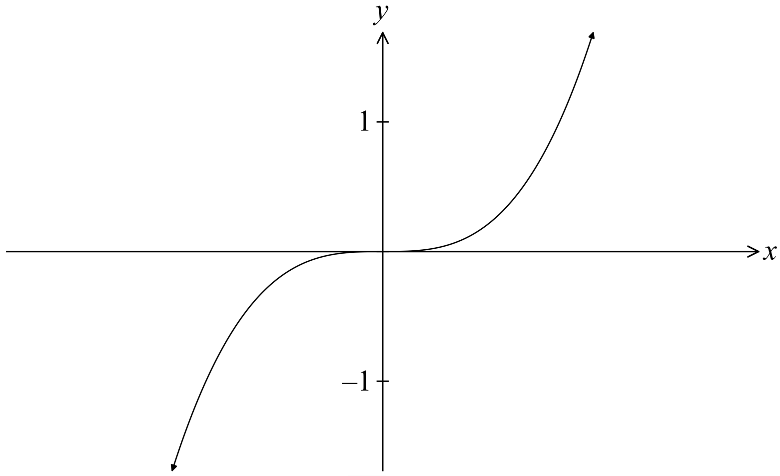
(A) The graph of $y = \cos\frac{1}{2}x$ has half the amplitude, but the same period.

(B) The graph of $y = \cos\frac{1}{2}x$ has same amplitude, but the half the period.

(C) The graph of $y = \cos\frac{1}{2}x$ has double the amplitude, but the same period.

(D) The graph of $y = \cos\frac{1}{2}x$ has same amplitude, but double the period.

7 The diagram below shows the graph of the function $y = 2x^3$.
Which expression gives the area between the curve and the y-axis from $y = -1$ to $y = 1$?



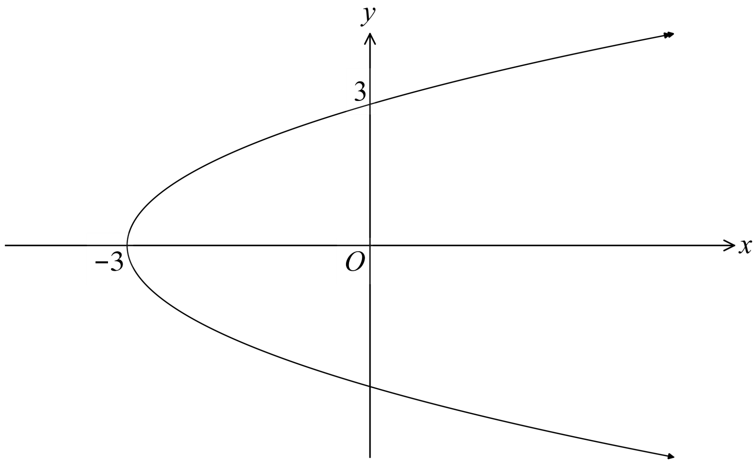
(A) $\int_{-1}^{1} 2y^3 \, dy$

(B) $2\int_{0}^{1} 2y^3 \, dy$

(C) $\left| \int_{-1}^{1} \left(\frac{y}{2}\right)^{\frac{1}{3}} dy \right|$

(D) $2\int_{0}^{1} \left(\frac{y}{2}\right)^{\frac{1}{3}} dy$

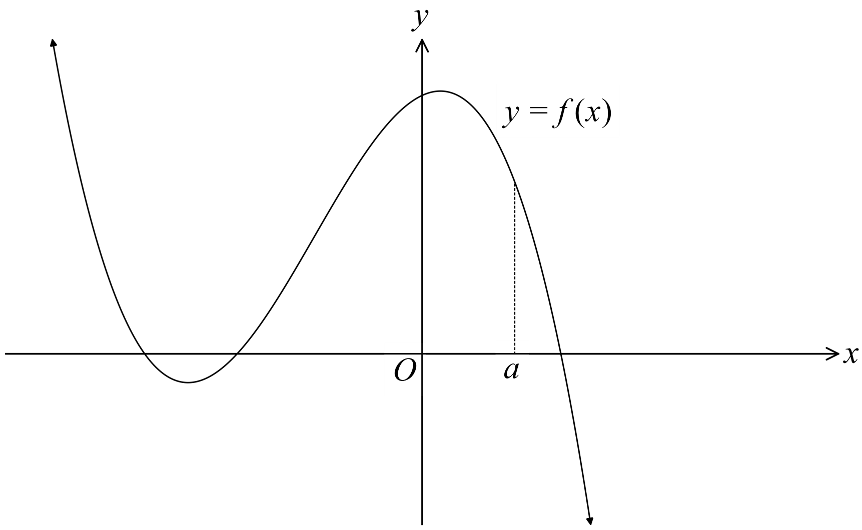
8



A parabola passes through the point (0, 3) and has its vertex at (–3, 0).
What is the equation of the parabola?

(A) $y^2 = -3(x + 3)$
(B) $y^2 = 3(x + 3)$
(C) $x^2 = -3(y + 3)$
(D) $x^2 = 3(y + 3)$

9



For the graph above, which of the following statements is true for $x = a$?

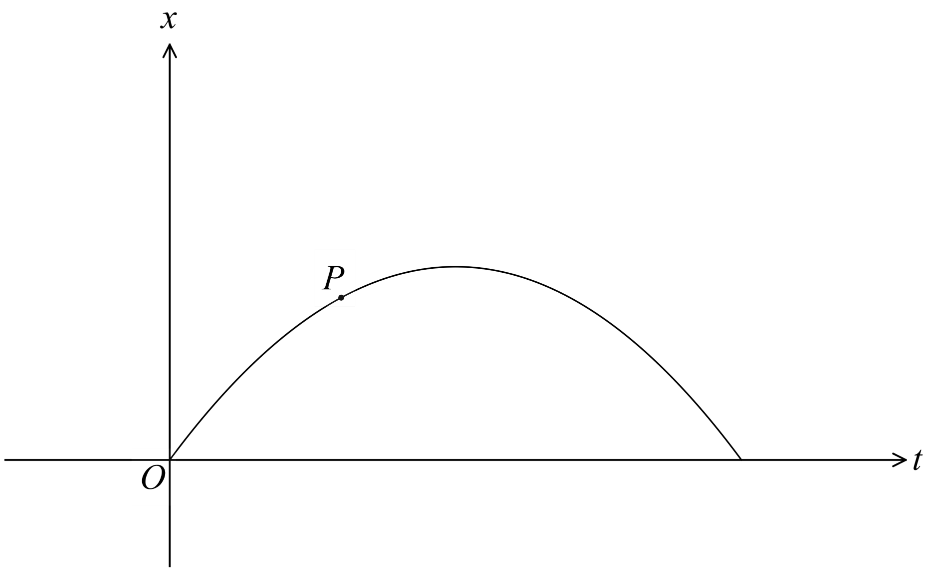
(A) $f'(a) > 0$ and $f''(a) > 0$

(B) $f'(a) < 0$ and $f''(a) > 0$

(C) $f'(a) > 0$ and $f''(a) < 0$

(D) $f'(a) < 0$ and $f''(a) < 0$

10 The graph shows the displacement-time graph for a particle moving on a straight line.



Which statement describes the motion of the particle at point P?

(A) Velocity is positive and acceleration is negative.

(B) Velocity is negative and acceleration is negative.

(C) Velocity is positive and acceleration is positive.

(D) Velocity is negative and acceleration is positive.

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Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) A parabola has the equation

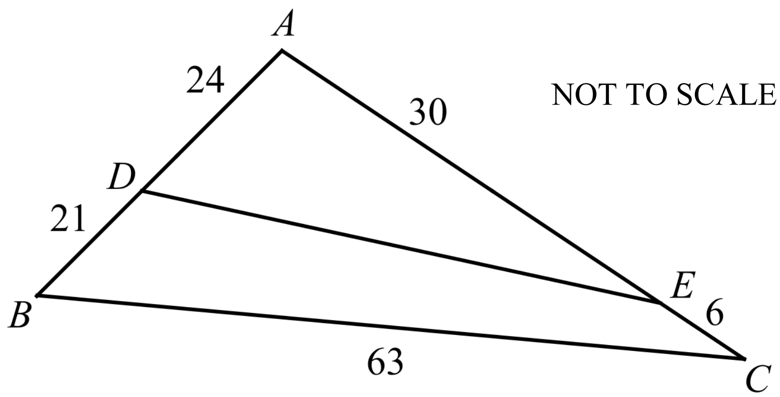
$$8y = x^2 - 6x + 17$$

By first writing the equation in the form $(x - h)^2 = 4a(y - k)$, find

(i) the focal length. **1**

(ii) the equation of the directrix **1**

(b) In the diagram below, $AD = 24$ cm, $DB = 21$ cm, $AE = 30$ cm, $EC = 6$ cm and $BC = 63$ cm



(i) Prove $\Delta ADE \parallel\!| \Delta ACB$. **2**

(ii) Find the length of DE. **1**

Question 11 continues on page 9

Question 11 (continued)

(c) Differentiate the following with respect to x:

(i) $x\cos x$. **2**

(ii) $\dfrac{\ln x}{x^3}$, **2**

(d) (i) If $y = e^{2x^3}$, find $\dfrac{dy}{dx}$. **1**

(ii) Hence evaluate $\displaystyle\int_0^1 x^2 e^{2x^3}\,dx$. **2**

(e) (i) Show that $(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$ **1**

(ii) If α and β are the roots of $2x^2 - 3x - 1 = 0$, find the value of $\alpha^3 + \beta^3$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the function $f(x) = \left(2\sqrt{x} - 1\right)^2$

(i) State the domain of the function. **1**

(ii) Show that $f'(x) = \dfrac{2\left(2\sqrt{x} - 1\right)}{\sqrt{x}}$. **2**

(iii) Find any stationary point(s) and determine their nature. **2**

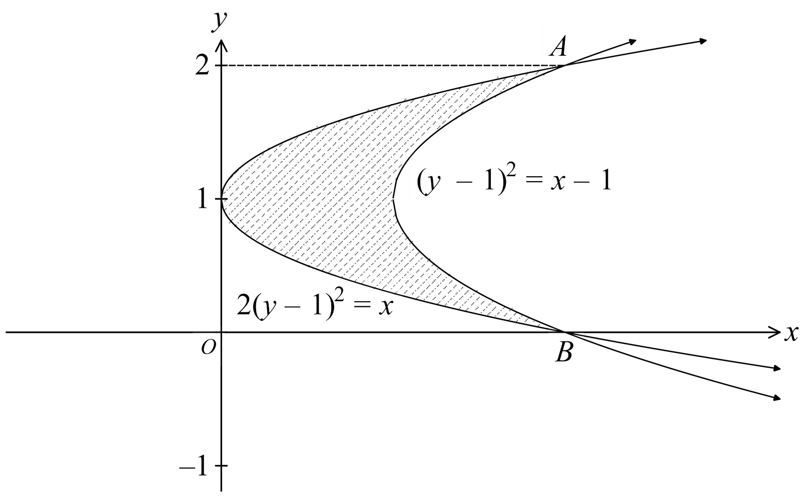
(iv) If $k = \left(2\sqrt{x} - 1\right)^2$, find the values of k for which there are two solutions. **1**

(b) Use Simpson's Rule with 3 function values to find an approximation for **2**

$$\int_2^6 \ln x \, dx.$$

Give your answer correct to 3 significant figures.

(c) The diagram below shows the graphs of $2(y - 1)^2 = x$ and $(y - 1)^2 = x - 1$. **2**
The two graphs intersect at the points A and B.



Calculate the shaded area.

Question 12 continues on page 11

Question 12 (continued)

(d) A particle is moving along a straight line so that its acceleration is given by **2**

$$\ddot{x} = \frac{1}{2t+1} \text{ m/s}^2$$

where x is the particle's displacement, measured in metres, from the origin O, and the time, t, is measured in seconds

The velocity of the particle is $\ln 9$ m/s when $t = 4$.

Find the velocity when $t = 5$.
Leave your answer in exact form.

(e) The quadratic equation

$$x^2 + Lx + M = 0$$

has one root twice the other.

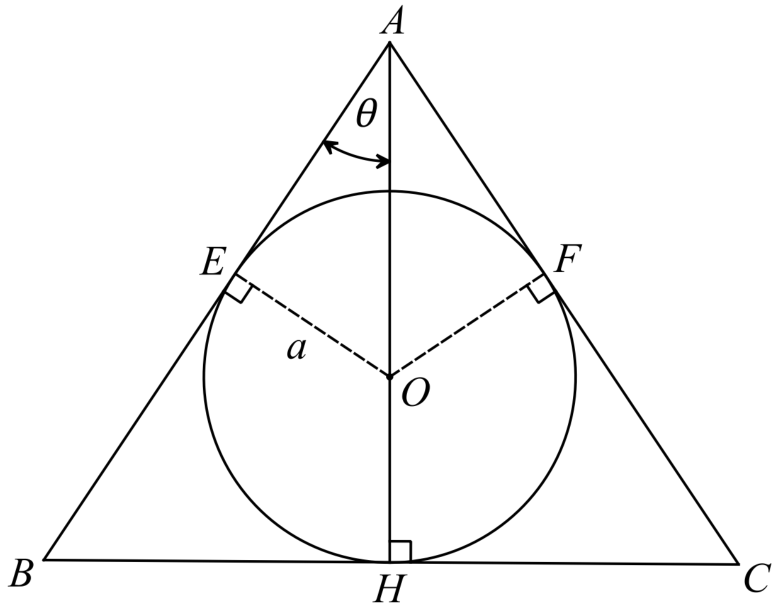
(i) Prove $M = \dfrac{2L^2}{9}$. **2**

(ii) Prove the roots are rational whenever L is rational. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a)



The above figure shows a circle, with centre O and radius a, inscribed in triangle ABC.
ΔABC is isosceles with $AB = AC$.
Let $\angle OAB = \theta$.

(i) Show that $OA = \dfrac{a}{\sin\theta}$ **1**

(ii) Find, in terms of a and θ, the height AH of ΔABC. **1**

(iii) Hence show that the area of ΔABC is $\dfrac{a^2(1+\sin\theta)^2}{\sin\theta\cos\theta}$. **2**

(b) A paediatrician, Dr Yohc, proposes that the ability of a child to memorise information during the first four years can be modelled by the function

$$f(x) = 1 + x\ln x,\ 0 < x \le 4.$$

This means that the ability to memorise at age x years is $f(x)$.

(i) During which month is the ability to memorise at a minimum in the first four years? **2**

(ii) When is the ability to memorise at a maximum during the first four years? Justify your answer with appropriate working. **2**

Question 13 continues on page 13

Question 13 (continued)

(c) 100 white and 100 black marbles are mixed together.

Some are placed in container A and the rest are placed in container B.

The probability of selecting a white marble from container A is $\frac{2}{3}$.

If a white marble is now taken from container A and placed in container B then the probability of selecting a black marble from container B is also $\frac{2}{3}$.

(i) The number of white and black marbles, originally in A, are w and b respectively. **1**

Show that $w = 2b$.

(ii) Show that $\dfrac{100-b}{201-w-b} = \dfrac{2}{3}$. **2**

(iii) Find the number of each colour originally in container A. **1**

(d) The curve $y = 4\ln x$, between $x = 1$ and $x = e$, is rotated about the y-axis.

(i) Show that the volume formed is given by **2**

$$V = \pi \int_0^4 e^{\frac{y}{2}}\, dy$$

(ii) Hence, find the volume of the solid in exact form. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The number of bacteria, N, in a culture increases at a rate proportional to the number present i.e.

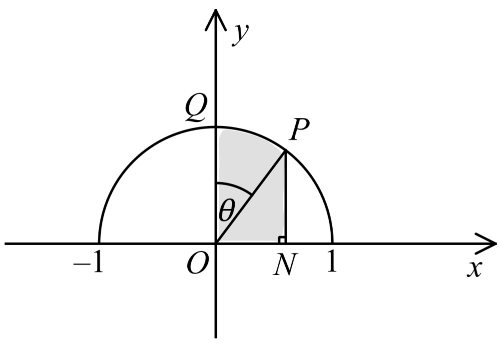
$$\frac{dN}{dt} = kN \text{, for some constant } k,$$

where t is the time in hours.
Initially, there are 2000 bacteria in the culture.
After 10 hours there are 12 000 bacteria in the culture.

(i) Show that the growth rate k is $\frac{\ln 6}{10}$. **1**

(ii) Find how long it will take for the culture to increase to 30 000 bacteria. Leave your answer correct to the nearest hour. **2**

(b)



The above diagram shows a semi-circle with centre (0, 0) and radius 1 unit.
The semi-circle cuts the y-axis at $Q(0, 1)$.
PN is parallel to the y-axis and cuts the semi-circle at P and the x-axis at $N(a, 0)$, where $0 < a < 1$.

Also $\angle POQ = \theta$.

(i) Show that the shaded area is **2**

$$\frac{a}{2}\sqrt{1-a^2} + \frac{\theta}{2}$$

(ii) Hence find **2**

$$\int_{0.8}^{1} \sqrt{1-x^2}\, dx\,,$$

correct to 3 decimal places.

Question 14 continues on page 15

Question 14 (continued)

(c) The size y of a population at time t is given by

$$y = \frac{N}{1 + Ke^{-2t}},$$

for $t \geq 0$, where $N > 0$ and $K > 1$ are constants.

(i) Write down the value of y at $t = 0$. **1**

(ii) What limiting value does the population size approach for large values of t? **1**

(iii) Find $\frac{dy}{dt}$, and hence show that the population is always increasing. **2**

(iv) Show that $\frac{dy}{dt} = \frac{2}{N} y(N - y)$ **2**

(v) Using part (iv), or otherwise, show that the population is increasing most rapidly when **2**

$$y = \frac{N}{2}.$$

End of paper

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SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2016

Year 11 Yearly Examination

Mathematics Accelerated

Sample Solutions

Question	Teacher
Q11	BD
Q12	PSP
Q13	AF
Q14	AMG

MC Answers

1.	D	**2.**	C	**3.**	D	**4.**	A	**5.**	A
6.	D	**7.**	D	**8.**	B	**9.**	D	**10.**	A

2U Accelerated Y11 Yearly 2016 Multiple choice solutions

Mean (out of 10): 9.32

1. $\dfrac{\sin(\pi-\alpha)}{\sin(\frac{\pi}{2}-\alpha)}$

$= \dfrac{\sin\alpha}{\cos\alpha}$

$= \tan\alpha$ D

A	2
B	2
C	0
D	108

2. A ×

B ×

C ✓

D × C

A	1
B	1
C	107
D	3

3. $\frac{1}{2}$ D

A	0
B	2
C	11
D	99

4. $r = 5-\sqrt{a}$

$|r| < 1$

$\therefore |5-\sqrt{a}| < 1$

$\therefore -1 < 5-\sqrt{a} < 1$

$\therefore -6 < -\sqrt{a} < -4$

$\therefore 4 < \sqrt{a} < 6$

$\therefore 16 < a < 36$

$(a \neq 25)$ as series would be $1+0+0+\ldots$ which is not a geometric series

A

A	96
B	2
C	0
D	14

5. $\ln\left(\frac{1}{x^2}\right) = -4$

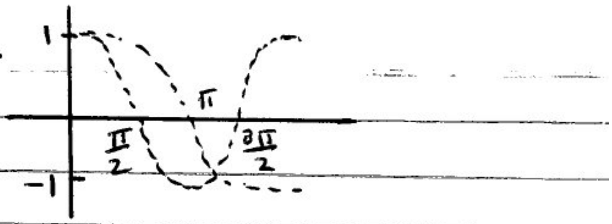
$\ln x^2 = 4$

$x^2 = e^4$

$x = \pm e^2$ A

A	112
B	0
C	0
D	0

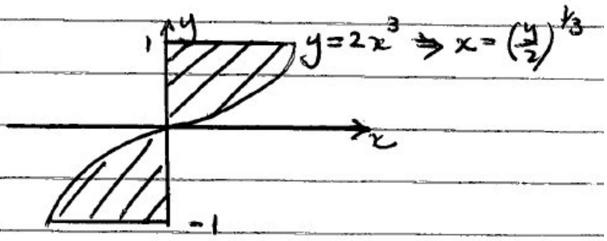
6.



A × (amplitude same)
B × (period is double)
C × (amplitude same, period double)
D ✓ D

A	0
B	7
C	1
D	104

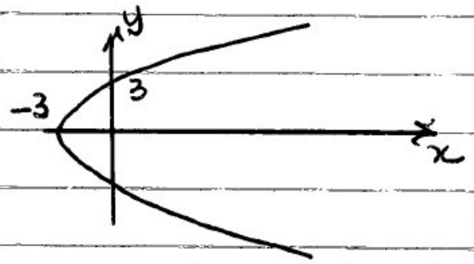
7.



$\text{Area} = 2\int_0^1 \left(\frac{y}{2}\right)^{1/3} dy$ D

A	0
B	3
C	6
D	103

8.



$x = A(y-3)(y+3)$

When $y=0$: $-3 = A \cdot -9$

$A = \frac{1}{3}$

$x = \frac{1}{3}(y^2-9)$

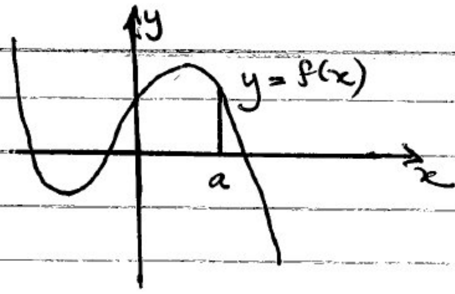
$3x = y^2-9$

$y^2 = 3x+9$

$y^2 = 3(x+3)$ B

A	0
B	108
C	0
D	4

9.



At $x=a$, $f'(x)<0$ and $f''(x)<0$

D

A	0
B	6
C	1
D	104

10. At P, $\dot{x}>0$ $\ddot{x}<0$ A

A	103
B	2
C	7
D	0

2U Accelerated Y11 2016 Q11 solutions

Mean (out of 15): 14.25

Q11 (a) $8y = x^2 - 6x + 17$

$x^2 - 6x + 9 = 8y - 8$

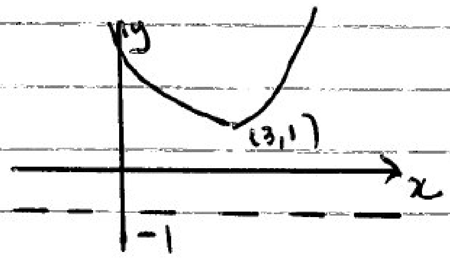
$(x-3)^2 = 4 \times 2 (y-1)$

(i) Focal length $= 2$

0	0.5	1	Mean
1	2	109	0.99

Most students were able to identify that the 8y term meant a focal length of 2.

(ii)

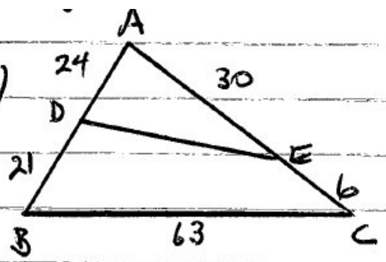


Directrix $y = -1$

0	0.5	1	Mean
10	7	95	0.89

Those who had problems did not realise that rewriting the equation allowed the vertex of the parabola to be identified and could then, using the focal length, determine the equation of the directrix.

(b) (i)



$\frac{AD}{AC} = \frac{24}{36} = \frac{2}{3}$

$\frac{AE}{AB} = \frac{30}{45} = \frac{2}{3}$

$\angle DAE = \angle CAB$ (common)

$\therefore \Delta ADE ||| \Delta ACB$ (2 pairs of sides in same ratio, included angles equal)

0	0.5	1	1.5	2	Mean
1	2	10	8	91	1.83

Most had a reasonable idea of the required process. However, some referred only to sides being in the same ratio (no mention of angles) or did not indicate that the angle involved had to be the included angle. The use of the abbreviation SAS was not penalised. However, this a dangerous technique to use as there is no clear indication about its acceptability in HSC examinations.

(ii) $\frac{DE}{CB} = \frac{DE}{63} = \frac{2}{3}$

$\therefore DE = 42$

0	0.5	1	Mean
3	10	99	0.93

Students who had problems did not use the ratio of corresponding sides, using instead the ratio of the two known sides in one of the triangles.

(c)(i) $\frac{d}{dx}(x \cos x) = \cos x . 1 + x(-\sin x)$

$= \cos x - x \sin x$

0	0.5	1	1.5	2	Mean
1	0	0	3	108	1.97

Caused problems for very few students. Loss of sign in derivative of cos x was the main problem.

(ii) $\frac{d}{dx}\left(\frac{\ln x}{x^3}\right) = \frac{x^3 . \frac{1}{x} - \ln x . 3x^2}{x^6}$

$= \frac{x^2(1 - 3\ln x)}{x^6}$

$= \frac{1 - 3\ln x}{x^4}$

0	0.5	1	1.5	2	Mean
0	0	4	6	102	1.94

Some did not divide out the common factor of x^2.

(d) (i) $y = e^{2x^3}$

$$\frac{dy}{dx} = e^{2x^3} \cdot 6x^2$$
$$= 6x^2 e^{2x^3}$$

0	0.5	1	Mean
1	0	111	0.99

Well done.

(ii) $\int_0^1 x^2 e^{2x^3} dx$

$$= \frac{1}{6}\int_0^1 6x^2 e^{2x^3} dx$$
$$= \frac{1}{6}\left[e^{2x^3}\right]_0^1$$
$$= \frac{1}{6}\{e^2 - 1\}$$

0	0.5	1	1.5	2	Mean
2	1	0	10	99	1.91

Most handled this well. Some assumed that substituting 0 into any function yields 0.

(e) (i) $(\alpha+\beta)^3 = (\alpha+\beta)(\alpha+\beta)^2$

$$= (\alpha+\beta)(\alpha^2+2\alpha\beta+\beta^2)$$
$$= \alpha^3 + 2\alpha^2\beta + \alpha\beta^2 + \alpha^2\beta + 2\alpha\beta^2 + \beta^3$$
$$= \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$

0	0.5	1	Mean
0	0	112	1

No problems.

(ii) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2$

$$= (\alpha+\beta)^3 - 3\alpha\beta(\beta+\alpha)$$
$$= \left(\frac{3}{2}\right)^3 - 3\cdot\left(-\frac{1}{2}\right)\cdot\frac{3}{2}$$
$$= \frac{27}{8} + \frac{9}{4}$$
$$= \frac{45}{8}$$

0	0.5	1	1.5	2	Mean
2	4	4	10	92	1.83

Most saw the reason for part (i) and applied it. Errors involved. Errors involved incorrect values for the sum and product of roots, loss of signs and incorrect evaluation.

Question 12 **SOLUTIONS**

(a)	Consider the function $f(x)=\left(2\sqrt{x}-1\right)^2$	
	(i) State the domain of the function.	**1**

$x \geq 0$

Comment: There are still students who are not convinced that $\sqrt{0}$ exists.

(ii) Show that $f'(x)=\dfrac{2\left(2\sqrt{x}-1\right)}{\sqrt{x}}$.	**2**

$$f(x)=\left(2x^{\frac{1}{2}}-1\right)^2$$

$$f'(x)=2\left(2x^{\frac{1}{2}}-1\right)\left(x^{-\frac{1}{2}}\right)$$

$$=\frac{2\left(2\sqrt{x}-1\right)}{\sqrt{x}}$$

Comment: This is a "Show that" questions and as a result, students have to show something to get the full marks.
An answer which was simpliy a minor re-arrangement of the required answer did not score full marks.

(iii) Find any stationary point(s) and determine their nature.	**2**

Stationary points occur when $f'(x)=0$

$$\therefore \frac{2\left(2\sqrt{x}-1\right)}{\sqrt{x}}=0$$

$$\therefore 2\sqrt{x}-1=0$$

$$\therefore \sqrt{x}=\frac{1}{2}$$

$$\therefore x=\frac{1}{4}$$

$\therefore$ the stationary point is at $\left(\dfrac{1}{4}, 0\right)$.

Question 12 (continued)

(a) (iii) continued

x	$\frac{1}{9}$	$\frac{1}{4}$	1
$f'(x)$	-2	0	2
	\	–	/

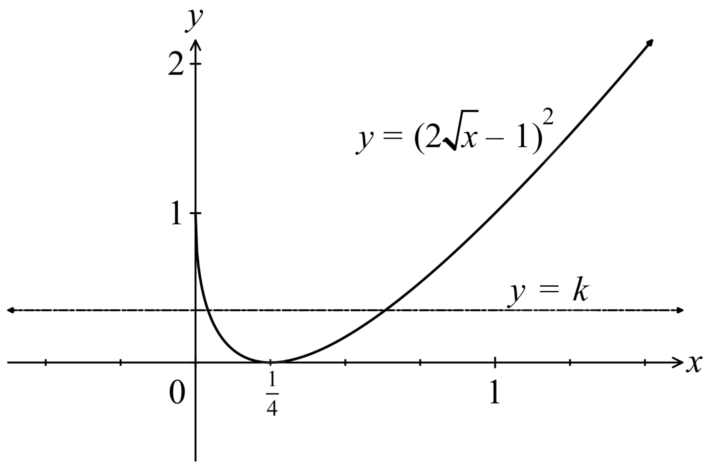
$\therefore \left(\frac{1}{4}, 0\right)$ is a (rel.) minimum turning point.

Comment: Students should show the value of their calculations.
There are times when it is essential that they do so and good practice would dictate that they do it always.

Some students calculated the second derivative, which was straight forward if they recognised that $f'(x) = 4 - \frac{2}{\sqrt{x}}$.

(iv) If $k = \left(2\sqrt{x} - 1\right)^2$, find the values of k for which there are two solutions. **1**

Intersecting the graphs of $y = k$ and $y = \left(2\sqrt{x} - 1\right)^2$ gets $k = \left(2\sqrt{x} - 1\right)^2$.
Using the information from parts (i) – (iii):



So when $0 < k \leq 1$, there will be two solutions to $k = \left(2\sqrt{x} - 1\right)^2$.

Comment: Students who did not start by sketching the graph were unable to justify getting any marks.
Parts (i) to (iii) were setting the student up to sketch!

Question 12 (continued)

(b)	Use Simpson's Rule with 3 function values to find an approximation for $$\int_2^6 \ln x \, dx.$$ Give your answer correct to 3 significant figures.	**2**

3 function values means 2 sub-intervals i.e. $h = \dfrac{6-2}{2} = 2$

x	2	4	6
y	ln2	ln4	ln6
w (weight)	1	4	1

$$\int_2^6 \ln x \, dx \doteqdot \frac{h}{3} \times \left(1 \times \ln 2 + 4 \times \ln 4 + 1 \times \ln 6\right)$$

$$= \frac{2}{3} \times \left(\ln 2 + 4 \ln 4 + \ln 6\right)$$

$$\doteqdot 5.35$$

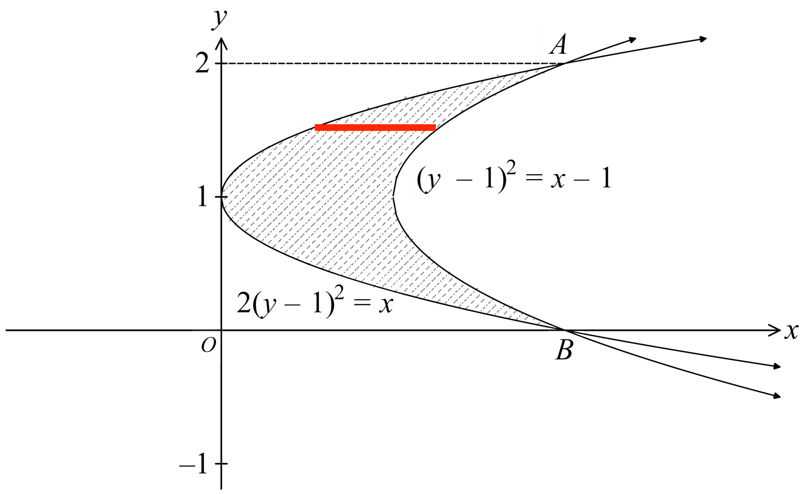
Comment: This was generally well done, apart from calculator problems with some students.

The formula for Simpson's Rule is on the reference sheet. As a result, any student who could not quote/use the formula correctly did not receive any marks. (Page 3, third section down.)

Question 12 (continued)

(c) The diagram below shows the graphs of $2(y-1)^2 = x$ and $(y-1)^2 = x-1$. **2**

The two graphs intersect at the points A and B.



Calculate the shaded area.

Where do the graphs intersect? $A(2, 2)$ and $B(0, 2)$

Note that the graphs have been given with x as subject. A big clue AND gift!
By considering the horizontal distance between the two graphs, the area can be calculated

$$\text{Area} = \int_0^2 \left[(y-1)^2 + 1 - 2(y-1)^2 \right] dy$$

$$= \int_0^2 \left[1 - (y-1)^2 \right] dy$$

$$= \left[y - \frac{1}{3}(y-1)^3 \right]_0^2$$

$$= (2 - \tfrac{1}{3} \times 1) - (0 - \tfrac{1}{3} \times (-1))$$

$$= 1\tfrac{1}{3}$$

Comment: Generally done well as most students saw the clue in the question i.e. x was essentially the subject of both graphs.
Students who tried to calculate the area wrt the x-axis could not get anywhere.

Question 12 (continued)

(d) A particle is moving along a straight line so that its acceleration is given by **2**

$$\ddot{x} = \frac{1}{2t+1} \text{ m/s}^2$$

where x is the particle's displacement, measured in metres, from the origin O, and the time, t, is measured in seconds

The velocity of the particle is $\ln 9$ m/s when $t = 4$.

Find the velocity when $t = 5$.
Leave your answer in exact form.

$t = 4$, $v = \ln 9$

$$\frac{dv}{dt} = \frac{1}{2t+1}$$

$$\therefore v = \int \frac{1}{2t+1}\,dt$$

$$= \frac{1}{2}\int \frac{2}{2t+1}\,dt$$

$$= \frac{1}{2}\ln|\,2t+1\,| + C$$

Substitute $t = 4$, $v = \ln 9$

$$\therefore \ln 9 = \frac{1}{2}\ln 9 + C$$

$$\therefore C = \frac{1}{2}\ln 9$$

$$\therefore v = \frac{1}{2}\ln|\,2t+1\,| + \frac{1}{2}\ln 9$$

$$t = 5 \Rightarrow v = \frac{1}{2}\ln 11 + \frac{1}{2}\ln 9 = \frac{1}{2}\ln 99$$

Comment: Generally well done, except for presenting the final answer.
This revealed that many students don't know their logarithm rules.

It is surprising that given that the "formula" for integrating functions like this is in the reference sheet, many students could not get the right coefficient for their primitive. (Page 3 second column, 3rd formula down.)

But for the 2 marks available for this questions, many students would have been penalised as the instructions clearly state on the front cover that the simplest answer is expected.

Question 12 (continued)

(e) The quadratic equation

$$x^2 + Lx + M = 0$$

has one root twice the other.

(i) Prove $M = \dfrac{2L^2}{9}$. **2**

Let the roots be α and β with $2\alpha = \beta$

$\therefore 2\alpha + \alpha = -L$ [sum of roots]

$\therefore \alpha = -\tfrac{1}{3}L$ −(1)

Also $2\alpha \times \alpha = M$ [product of roots]

$\therefore 2\alpha^2 = M$ −(2)

Substitute (1) into (2)

$\therefore 2(-\tfrac{1}{3}L)^2 = M$

$\therefore \tfrac{2}{9}L^2 = M$

Comment: Generally well done.

(ii) Prove the roots are rational whenever L is rational. **1**

A rational number is a number that can be written in the form $\dfrac{p}{q}$ where p and q are integers.

$\therefore$ From part (i) the roots are $-\tfrac{1}{3}L$ and $-\tfrac{2}{3}L$, which are rational if L is rational.

Comment: Students were penalised if they did not communicate what they understood by rational numbers as it became clear that that many students did not understand what rational numbers are.

Students who tried to use an argument about the discriminant being positive, clearly did not understand the difference between real and rational numbers and could not get anywhere.

A common mistake was that if the sum and product of two numbers is rational then the two numbers must be rational.

End of Q12 Solutions

13) a) i) $\sin\theta = \dfrac{OE}{OA}$

$$\sin\theta = \frac{a}{OA}$$

$$OA = \frac{a}{\sin\theta}$$

ii) $AH = OA + OH$

$$= \frac{a}{\sin\theta} + a$$

$$= \frac{a(1+\sin\theta)}{\sin\theta}$$

iii) $\tan\theta = \dfrac{BH}{AH}$

$$BH = AH\tan\theta$$

$$= \frac{a(1+\sin\theta)}{\sin\theta} \cdot \frac{\sin\theta}{\cos\theta}$$

$$= \frac{a(1+\sin\theta)}{\cos\theta}$$

$$A = \frac{1}{2} BC . AH$$

$$= BH . AH$$

$$= \frac{a(1+\sin\theta)}{\cos\theta} \cdot \frac{a(1+\sin\theta)}{\sin\theta}$$

$$= \frac{a^2(1+\sin\theta)^2}{\sin\theta\cos\theta}$$

COMMENT:

This question was done well by students with some difficulty in part (iii) experienced from some students

b) i) $f(x) = 1 + x\ln x, \quad 0 < x \leq 4$

$f'(x) = x \cdot \frac{1}{x} + 1 \cdot \ln x$

$= 1 + \ln x$

For stationary points $f'(x) = 0$

$1 + \ln x = 0$

$\ln x = -1$

$x = e^{-1}$

$\approx 0.367879\ldots$ years

$= 4.41455\ldots$ months

$f''(x) = \frac{1}{x}$

$f''\left(\frac{1}{e}\right) = \frac{1}{\left(\frac{1}{e}\right)}$

$= e$

> 0

$\therefore$ Minimum during 5^{th} month

ii) $f''(x) > 0$ over the domain $(0 < x \leq 4)$

$\therefore$ maximum will occur at an endpoint of domain

$f(4) = 1 + 4\ln 4$

≈ 6.545

as $x \to 0^+$, $f(x) \to 1^-$

$\therefore$ Maximum at the end of 4^{th} year.

COMMENT:

i) If students answered during 4^{th} month they lost $\frac{1}{2}$ mark. This was a common mistake

ii) Students were expected to justify their answer. Some students stated that $f(x)$ is increasing for $x > \frac{1}{e}$ (since only one stationary point).
Hence, $x = 4$ gives maximum for $f(x)$.
The problem is they neglected to test the other endpoint $(x \to 0^+)$ of the domain.

ALTERNATIVELY

$$\text{for } 0 < x \leq 1$$
$$\ln x \leq 0$$
$$x \ln x \leq 0$$
$$1 + x\ln x \leq 1$$
$$f(x) \leq 1$$

since $f(x)$ is increasing for $x > \frac{1}{e}$

$$f(4) = 1 + 4\ln 4$$
$$> 1$$

Maximum $f(x)$ when $x = 4$.

c)

	A		B		Total
white	w	$[w-1]$	$100-w$	$[101-w]$	100
black	b		$100-b$		100
Total	$w+b$	$[w+b-1]$	$200-w-b$	$[201-w-b]$	200

i) $$\frac{w}{b+w} = \frac{2}{3}$$

$$3w = 2b + 2w$$

$$w = 2b$$

ii) $$\frac{100-b}{201-w-b} = \frac{2}{3}$$ (from table)

iii) $$\frac{100-b}{201-(2b)-b} = \frac{2}{3}$$

$$300 - 3b = 402 - 6b$$

$$3b = 102$$

$$b = 34$$

$$w = 2(34)$$

$$= 68.$$

COMMENT:

This question was answered reasonably well by students.

d) i) $y = 4\ln x$

$\ln x = \frac{y}{4}$

$x = e^{\frac{y}{4}}$

when $x = 1$

$y = 4\ln 1$

$y = 0.$

when $x = e$

$y = 4\ln e$

$y = 4$

$V = \pi \int_a^b x^2 \, dy$

$V = \pi \int_0^4 (e^{\frac{y}{4}})^2 \, dy$

$V = \pi \int_0^4 e^{\frac{y}{2}} \, dy$

ii) $V = \pi \left[2e^{\frac{y}{2}} \right]_0^4$

$V = 2\pi \left[e^{\frac{(4)}{2}} - e^{\frac{(0)}{2}} \right]$

$V = 2\pi \left[e^2 - 1 \right]$ cubic units

COMMENT:

The majority of students gained full marks in this question.

(a) (i) $\dfrac{dN}{dt} = kN$

Hence $N = Ae^{kt}$
When t=0, N=2000
Thus $2000 = Ae^{0}$
$\therefore \quad A = 2000$
Now $N = 2000e^{kt}$

When t = 10, N = 12000

$$12000 = 2000e^{10k}$$
$$6 = e^{10k}$$
$$10k = \ln 6$$
$$k = \frac{\ln 6}{10}$$

[**Comment:** Very well answered, possibly 100% correct.]

(ii)

$$30000 = 2000e^{kt}$$
$$15 = e^{kt}$$
$$kt = \ln 15$$
$$t = \frac{\ln 15}{k}$$
$$= \ln 15 \times \frac{10}{\ln 6}$$
$$\simeq 15 \text{ hours}$$

[**Comment:** Very well answered. A few candidates made calculation errors.]

(b) (i) $ON = a,\ OP = 1,\ PN = \sqrt{1-a^2}$

$$\Delta OPN = \frac{1}{2} \times a \times \sqrt{1-a^2}$$
$$= \frac{a}{2}\sqrt{1-a^2}$$

Sector $OPQ = \tfrac{1}{2} r^2\theta$

$$= \frac{\theta}{2}$$

$\therefore$ Shaded Area $= \dfrac{a}{2}\sqrt{1-a^2} + \dfrac{\theta}{2}$ as required.

[**Comment:** Very well answered.]

(b) (continued)

(ii) $$\int_{0.8}^{1} \sqrt{1-x^2}\,dx = \frac{1}{4}\text{circle} - \text{shaded area } (a = 0.8)$$

$$= \frac{\pi}{4} - \left(\frac{0.8}{2}\sqrt{1-0.8^2} + \frac{\sin^{-1} 0.8}{2} \right)$$

$$\simeq 0.082 \text{ unit}^2$$

[**Comment:** Quite well answered. Some ignored the “hence” requirement, some failed to use radians, and some made simple algebraic errors.]

(c) $$y = \frac{N}{1+ke^{-2t}}$$

(i) $$y(0) = \frac{N}{1+k}$$

[**Comment:** Almost 100% correct.]

(ii) As $t \to \infty$, $e^{-2t} \to 0$
So $y \to N$

[**Comment:** Well answered, although some went for 1, or ∞.]

(iii) $$y' = \frac{-f'(t)}{(f(t))^2}$$

$$= \frac{N\left(-\left(-2ke^{-2t}\right)\right)}{\left(1+ke^{-2t}\right)^2}$$

$$= \frac{2kNe^{-2t}}{\left(1+ke^{-2t}\right)^2}$$

$$> 0 \quad \text{for all } t > 0$$

Since $k > 0$, $N > 0$
$\therefore$ Population is always increasing.

[**Comment:** Most found the correct derivative, but some failed to recognize N as a constant.]

(c) (continued)

(iv) RTP $\dfrac{dy}{dt} = \dfrac{2}{N} y(N - y)$

$$\text{RHS} = \frac{2}{N}\frac{N}{1+ke^{-2t}}\left(N - \frac{N}{1+ke^{-2t}}\right)$$
$$= \frac{2N}{1+ke^{-2t}} - \frac{2N}{(1+ke^{-2t})^2}$$
$$= \frac{2kNe^{-2t}}{(1+ke^{-2t})^2}$$
$$= \text{RHS} \qquad \text{QED}$$

[**Comment:** About half the candidates got this right. Some tried to fox their answer, including some who had got the derivative wrong.]

(v) Given $\dot{y} = \dfrac{2}{N} y(N - y)$ we seek greatest increase of pop'n.

That is we seek a maximum turning point in $\dot{y}$.

NB that $\dot{y}$ is a concave down quadratic in y.

$\therefore$ the maximum occurs at the vertex i.e. the midpoint of the vertices 0 and N.

$\therefore$ maximum of $\dot{y}$ occurs when $y = \dfrac{N}{2}$.

Alternatively

$$\frac{d\dot{y}}{dy} = \frac{2}{N}(N - 2y)$$
$$= 0 \text{ when } y = \frac{N}{2}$$
$$\frac{d^2\dot{y}}{dy^2} = \frac{-4}{N}$$
$$< 0 \text{ since } N > 0$$

Thus $\dot{y}$ has a relative maximum when $y = \dfrac{N}{2}$.

Hence y has a max rate of increase at this point.

[**Comment:** Of all those who attempted this, all but three used the alternative method, but most of these named the derivative wrongly. A very few properly employed implicit derivatives.]