Name: .... **Teacher:** ....

Fort Street High School

2011 Assessment task 3

Mathematics

Time allowed: 90 minutes

Outcomes Assessed	Questions	Marks
Demonstrates the ability to manipulate and simplify numeric and algebraic expressions.	1	
Solves problems involving inequalities, indices and logs.	2	
Synthesises mathematical understanding and applies appropriate techniques to solve problems involving linear functions and parabolas.	3,4	
Manipulates algebraic expressions to solve problems with the locus and the parabola.	5,6	

Question	1	2	3	4	5	6	Total (60)	%
Marks								

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question.
  Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each Question is to be started in a new booklet.

Question 1. (10 marks) **Marks**

a) Evaluate $-7-|-5|$ **1**

b) Factorise x^3-27 **1**

c) Find the values of *a* and *b* if $\dfrac{5}{1+\sqrt{5}}=a+b\sqrt{5}$ **2**

d) Solve $|2-3x|<5$ **2**

e) Simplify without negative indices $\dfrac{1+x^{-1}}{x-x^{-1}}$ **2**

f) Simplify $\sqrt{72}-\sqrt{50}$ **2**

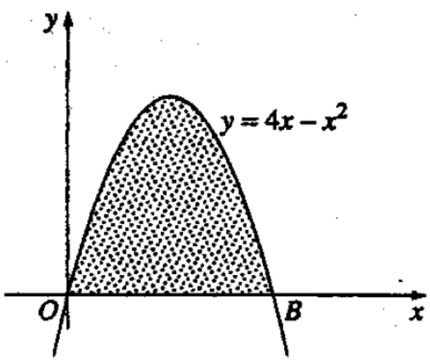
Question 2. **(10 marks)** *Start a separate booklet* **Marks**

a) Evaluate $\log_3 11$ correct to 2 significant figures. **1**

b) Simplify $\dfrac{3\log_2 8}{\log_3 9}$ **2**

c) Solve the equation $2(\log_2 x)^2 - 7\log_2 x + 3 = 0$ **3**

d) The diagram shows the graph of the function $y = 4x - x^2$

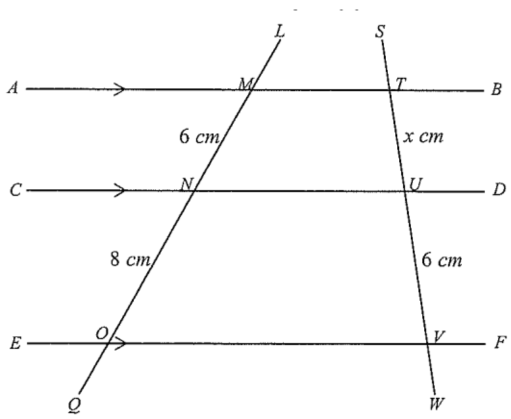


i) Find the range of the function. **2**

ii) Write down a pair of inequalities that specify the shaded region. **2**

Question 3. **(10 marks)** **Marks**

a) Find the value of x giving a reason for your answer **3**



b) i) Sketch the parabola $x^2 = 12y$ clearly showing the focus and directrix. **2**

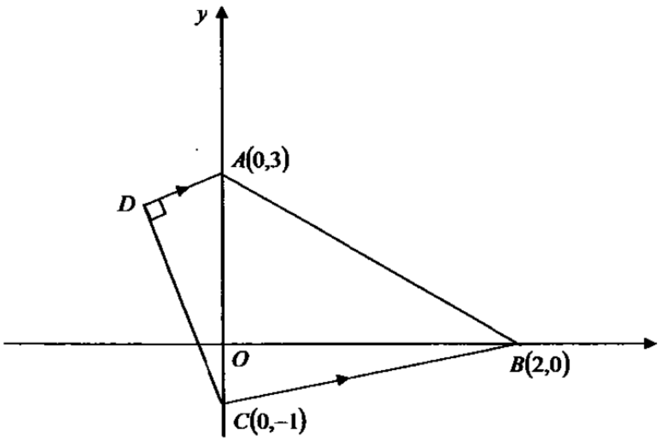
ii) Explain why an end point of the latus rectum is given by $(6,3)$. **1**

iii) A line, *l*, through the focus of the parabola makes an angle of inclination of 45° with the x- axis. Find the equation of the line, *l*, in general form. **2**

iv) Hence find the shortest distance from $(6,3)$ to the line *l*. **2**
Give your answer in exact form.

Question 4. **(10 marks)** **Marks**

a) In the diagram, points A, B and C have coordinates $(0,3),(2,0)$ and $(0,-1)$ respectively. $AD \parallel BC$ and $AD \perp CD$.



NOT TO SCALE

Copy this diagram into your answer booklet.

(i) Show that the gradient of the line BC is equal to $\frac{1}{2}$. **1**

(ii) Show that the equation of the line AD is $x-2y+6=0$. **2**

(iii) Find the equation of line CD. Give your answer in general form. **2**

(iv) By solving simultaneously the equations from (ii) and (iii), find the coordinates of point D. **2**

(v) Find the area of the quadrilateral $ABCD$. **3**

Question 5. (10 marks) **Marks**

a) Let α and β be the roots of the equation $x^2 - 5x + 2 = 0$.

Find the values of

i) $\alpha + \beta$ **1**

ii) $\alpha\beta$ **1**

iii) $(\alpha + 1)(\beta + 1)$ **2**

b) The quadratic equation $P(x)$ is given by $P(x) = x^2 - 2(k-3)x + (k-1)$.

i) Find the values of k for which $P(x) = 0$ has equal roots. **2**

ii) Find the range of values of k for which the quadratic function is positive definite. **2**

c) Find the value of m if one root of the function $y = x^2 - 2mx + (m+3)$ is three times the other. **2**

Question 6. **(10 marks)** **Marks**

a) If $ax\left(x-b\right)+\left(x-c\right)\equiv 2x^2-x-1$, find the values of *a*, *b* and *c*. **2**

b) A point *P*(*x*,*y*) moves so that its distance from the point *A*(6,0) is twice its distance from the point *B*(0,6) . **3**

Find the equation of the locus and describe it geometrically.

c) A parabola has the equation $x=12+4y-y^2$.

i) Find the coordinates of the vertex **2**

ii) Find the focal length **1**

iii) Find the equation of the directrix **1**

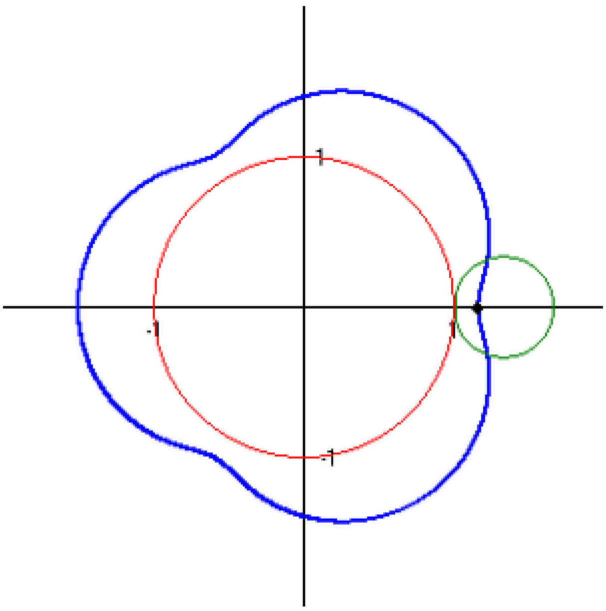
iv) Hence sketch $x=12+4y-y^2$ **1**

Fort Street High School

2011 Assessment task 3

Mathematics

Solutions



Question 1. (10 marks)

a) Evaluate $-7-|-5|$

Solution

$$-7-|-5| = -7-5$$

$$= -12$$

Marking Guideline:

1 For correct response

b) Factorise x^3-27

Solution

$$x^3-27 = (x-3)(x^2+3x+9)$$

Marking Guideline:

1 For correct response

c) Find the values of *a* and *b* if $\frac{5}{1+\sqrt{5}} = a+b\sqrt{5}$

Solution

$$\frac{5}{1+\sqrt{5}} = \frac{5}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$= \frac{5-5\sqrt{5}}{1-5}$$

$$= \frac{5-5\sqrt{5}}{-4}$$

$$= -\frac{5}{4}+\frac{5}{4}\sqrt{5}$$

$$\Rightarrow a=-\frac{5}{4}, b=\frac{5}{4}$$

Marking Guideline:

2 For correct response

1 For finding $\frac{5-5\sqrt{5}}{1-5}$

d) Solve $|2-3x|<5$

Solution

$$(2-3x)<5 \qquad or \qquad (2-3x)>-5$$

$$-3x<3 \qquad\qquad -3x>-7$$

$$x>-1 \qquad\qquad x<\frac{7}{3}$$

Marking Guideline:

2 For correct response

1 For one correct answer

e) Simplify without negative indices $\dfrac{1+x^{-1}}{x-x^{-1}}$

Solution

$$\frac{1+x^{-1}}{x-x^{-1}} = \frac{1+\frac{1}{x}}{x-\frac{1}{x}}$$

$$= \frac{\frac{x}{x}+\frac{1}{x}}{\frac{x^2}{x}-\frac{1}{x}}$$

$$= \frac{\frac{x+1}{x}}{\frac{x^2-1}{x}}$$

$$= \frac{x+1}{x}\times\frac{x}{x^2-1}$$

$$= \frac{x+1}{x^2-1}$$

$$= \frac{x+1}{(x+1)(x-1)}$$

$$= \frac{1}{x-1}$$

Marking Guideline:

2 For correct response

1 For finding $\dfrac{\frac{x+1}{x}}{\frac{x^2-1}{x}}$

f) Simplify $\sqrt{72}-\sqrt{50}$

Solution

$$\begin{aligned}\sqrt{72}-\sqrt{50} &= \sqrt{36}\times\sqrt{2}-\sqrt{25}\times\sqrt{2}\\ &= 6\sqrt{2}-5\sqrt{2}\\ &= \sqrt{2}\end{aligned}$$

Marking Guideline:

2 For correct response

1 For finding $6\sqrt{2}-5\sqrt{2}$

Question 2. **(10 marks)** *Start a separate booklet* **Marks**

a) Evaluate $\log_3 11$ correct to 2 significant figures.

Solution

$$\log_3 11 = \frac{\log_{10} 11}{\log 3}$$

$$= 2.2$$

Correct to two significant figures.

Marking Guideline:

1 For correct response

b) Simplify $\frac{3\log_2 8}{\log_3 9}$

Solution

$$\frac{3\log_2 8}{\log_3 9} = \frac{3\log_2 2^3}{\log_3 3^2}$$

$$= \frac{9\log_2 2}{2\log_3 3}$$

$$= \frac{9}{2}$$

Marking Guideline:

2 For correct response

1 For correct use of one log law

c) Solve the equation $2(\log_2 x)^2 - 7\log_2 x + 3 = 0$

Solution

$$2(\log_2 x)^2 - 7\log_2 x + 3 = 0$$

let $m = \log_2 x$ then

$$2(m)^2 - 7m + 3 = 0$$

$$(2m-1)(m-3) = 0$$

$$2m - 1 = 0 \quad \text{or} \quad m - 3 = 0$$

$$m = \frac{1}{2} \qquad m = 3$$

but $m = \log_2 x$

so $\log_2 x = \frac{1}{2}$ or $\log_2 x = 3$

$x = 2^{\frac{1}{2}}$

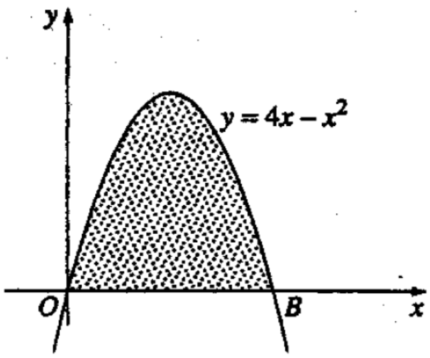
$= \sqrt{2}$

$x = 2^3$

$= 8$

> *Marking Guideline:*
>
> 3 For correct response or
>
> 2 For finding one correct answer or
>
> 1 For finding $(2m-1)(m-3)$

d) The diagram shows the graph of the function $y = 4x - x^2$



i) Find the range of the function.

Solution

The range will be all values of *y* less than (or equal to) the *y* ordinate of the vertex.

Finding the vertex:

$$x = \frac{-b}{2a}$$

$$= \frac{-(4)}{2(-1)}$$

$$= 2$$

When $x = 2$, $y = 4(2) - (2)^2$

$= 4$

Therefore the range is $y \le 4$

> *Marking Guideline:*
>
> 2 For correct response
>
> 1 For correct procedure with an error

ii) Write down a pair of inequalities that specify the shaded region.

Solution

The shaded region will be below the graph and above the line $y = 0$.

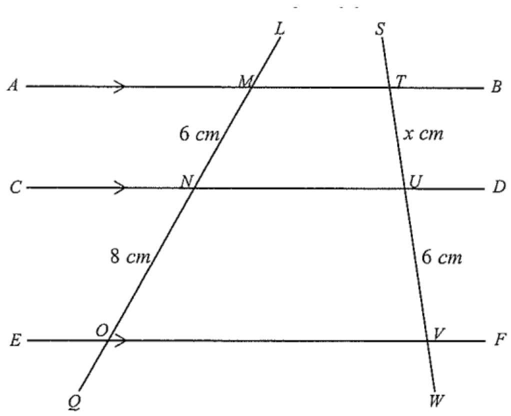
Therefore the inequality is $y \le 4x - x^2$ and $y \ge 0$

Marking Guideline:

2	For correct response
1	For correct procedure with an error

Question 3. **(10 marks)** **Marks**

a) Find the value of *x* giving a reason for your answer



Solution

Intercepts of parallel lines cut by transversals are in equal ratio.

$$\frac{x}{6}=\frac{6}{8}$$

$$x=\frac{36}{8}$$

$$=4\cdot5$$

Marking Guideline:

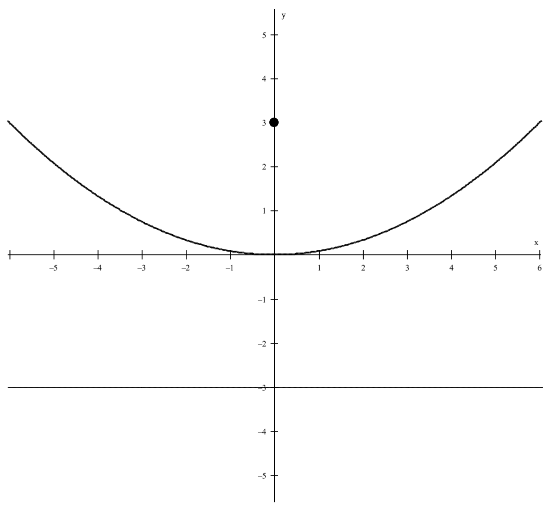
3 For correct response or

2 For correct value with no reason or

1 For finding $\frac{x}{6}=\frac{6}{8}$

b) i) Sketch the parabola $x^2=12y$ clearly showing the focus and directrix.

Solution



Marking Guideline:

2 For correct response

1 For correct shape

ii) Explain why an end point of the latus rectum is given by $(6,3)$.

Solution

Method 1.

The length of the latus rectum is given by $4a$.

Since $a=3$ the length would be $4\times3=12$ units.

Since the latus rectum is symmetrical about the axis, half the latus rectum would be 6 units.

This (half) length runs parallel to the *x* axis starting from the focus $(0,3)$. This places the end point at $(6,3)$

Method 2.

The equation of the latus rectum is $y=3$ and is a focal chord of the parabola.

Substituting $y=3$ into $x^2=12y$ to find the points of intersection

$$x^2=12(3)$$
$$x^2=36$$
$$x=\pm6$$

Therefore one end point would be $(6,3)$

> *Marking Guideline:*
>
> 1 For correct response

iii) A line, *l*, through the focus of the parabola makes an angle of inclination of 45° with the *x*- axis. Find the equation of the line, *l*, in general form.

Solution

The gradient of the line if found by
$$m=\tan\theta$$
$$=\tan 45^\circ$$
$$=1$$

The equation of the line is found by
$$y-y_1=m(x-x_1)$$
$$y-3=1(x-0)$$
$$x-y+3=0$$

> *Marking Guideline:*
>
> 2 For correct response
>
> 1 For finding the gradient

iv) Hence find the shortest distance from $(6,3)$ to the line *l*.

Give your answer in exact form.

Solution

Shortest distance is perpendicular distance:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|1(6) - 1(3) + 3|}{\sqrt{1^2 + (-1)^2}}$$

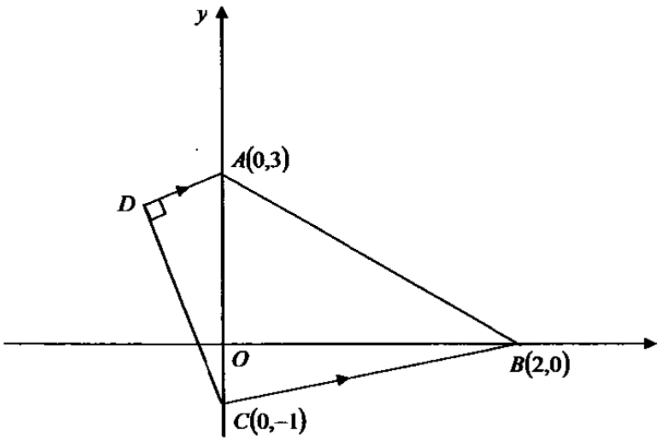
$$= \frac{6}{\sqrt{2}}$$

Marking Guideline:

2	For correct response
1	For correct procedure with arithmetic error

Question 4. **(10 marks)** **Marks**

a) In the diagram, points *A, B* and *C* have coordinates $(0,3),(2,0)$ and $(0,-1)$ respectively. $AD \| BC$ and $AD \perp CD$.



NOT TO SCALE

Copy this diagram into your answer booklet.

(i) Show that the gradient of the line *BC* is equal to $\frac{1}{2}$.

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{0 - -1}{2 - 0}$$

$$= \frac{1}{2}$$

> *Marking Guideline:*
>
> 1 For correct response

(ii) Show that the equation of the line *AD* is $x - 2y + 6 = 0$.

Solution

AD is parallel to *BC* $\Rightarrow$ $m_{AB} = m_{BC} = \frac{1}{2}$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{1}{2}(x - 0)$$

$$x - 2y + 6 = 0$$

> *Marking Guideline:*
>
> 2 For correct response
>
> 1 For not stating $m_{AB} = m_{BC} = \frac{1}{2}$

(iii) Find the equation of line CD. Give your answer in general form.

Solution

Since the line *CD* is perpendicular to *AD* the gradient of *CD* can be found using

$$m_1 m_2 = -1$$
$$\frac{1}{2} m_{CD} = -1$$
$$m_{CD} = -2$$

The equation of the line can now be found using the point gradient formula and point *C*

$$y - y_1 = m(x - x_1)$$
$$y - -1 = -2(x - 0)$$
$$2x + y + 1 = 0$$

Marking Guideline:

2 For correct response

1 For finding $m_{CD} = -2$

(iv) By solving simultaneously the equations from (ii) and (iii), find the coordinates of point D.

Solution

$$2x + y + 1 = 0 \quad ---1.$$
$$x - 2y + 6 = 0 \quad ---2.$$

1. ×2

$$4x + 2y + 2 = 0 \quad ---1a.$$
$$x - 2y + 6 = 0 \quad ---2.$$

1a. + 2

$$5x + 8 = 0$$
$$5x = -8$$
$$x = -\frac{8}{5}$$

Sub. $x = -\frac{8}{5}$ into 2.

$$-\frac{8}{5} - 2y + 6 = 0$$
$$8 + 10y - 30 = 0$$
$$10y = 22$$
$$y = \frac{22}{10}$$
$$= 2 \cdot 2$$

Therefore point *D* is $(-1 \cdot 6, 2 \cdot 2)$

Marking Guideline:

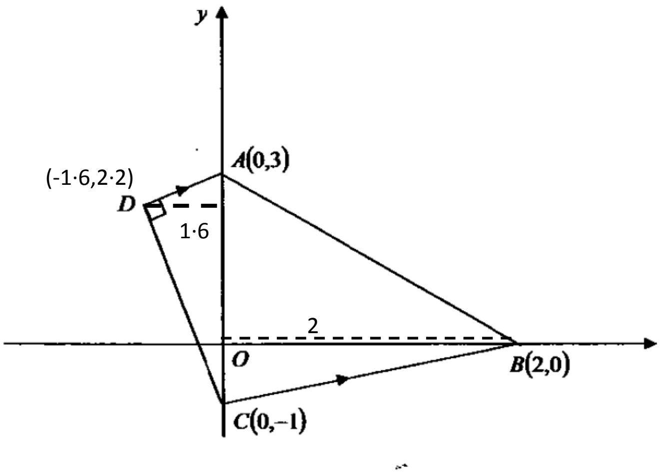
2 For correct response

1 For correct procedure with one error

(v) Find the area of the quadrilateral *ABCD.*

Solution

Break the quadrilateral into two triangles *ACD* & *ABC*



NOT TO SCALE

$$\text{Area of triangle } ABC = \frac{1}{2}bh$$
$$= \frac{1}{2} \times OB \times AC$$
$$= \frac{1}{2} \times 2 \times 4$$
$$= 4 \ \ u^2$$

$$\text{Area of triangle } ACD = \frac{1}{2}bh$$
$$= \frac{1}{2} \times AD \times AC$$
$$= \frac{1}{2} \times 1 \cdot 6 \times 4$$
$$= 3 \cdot 2 \ \ u^2$$

Total area of quadrilateral $ABCD = \ 7 \cdot 2 \ \ u^2$

Marking Guideline:

3	For correct response or
2	For correct procedure with one arithmetic error or
1	For finding area *ABC*

Question 5. **(10 marks)** **Marks**

a) Let α and β be the roots of the equation $x^2 - 5x + 2 = 0$.

Find the values of

i) $\alpha + \beta$

Solution

$$\begin{aligned}\alpha + \beta &= \frac{-b}{a} \\ &= \frac{-(-5)}{1} \\ &= 5\end{aligned}$$

Marking Guideline:

1 For correct response

ii) $\alpha\beta$

Solution

Marking Guideline:

1 For correct response

$$\begin{aligned}\alpha\beta &= \frac{c}{a} \\ &= \frac{2}{1} \\ &= 2\end{aligned}$$

iii) $(\alpha + 1)(\beta + 1)$

Marking Guideline:

2 For correct response

1 For expanding $(\alpha + 1)(\beta + 1)$

Solution

$$\begin{aligned}(\alpha + 1)(\beta + 1) &= \alpha\beta + \alpha + \beta + 1 \\ &= 2 + 5 + 1 \\ &= 8\end{aligned}$$

b) The quadratic equation $P(x)$ is given by $P(x) = x^2 - 2(k-3)x + (k-1)$.

i) Find the values of *k* for which $P(x) = 0$ has equal roots.

Solution

Method 1. The quadratic has equal roots when the discriminant is 0.

Note: $\Delta = b^2 - 4ac$

$$b^2 - 4ac = 0$$

$$\left[-2(k-3)\right]^2 - 4.1.(k-1) = 0$$

$$4(k-3)^2 - 4k + 4 = 0$$

$$4\left(k^2 - 6k + 9\right) - 4k + 4 = 0$$

$$k^2 - 6k + 9 + 1 = 0$$

$$k^2 - 7k + 10 = 0$$

$$(k-5)(k-2) = 0$$

$$k = 5 \quad or \quad k = 2$$

Method 2. Let the roots be α and α. Then

$$\alpha + \alpha = \frac{-b}{a} \qquad \alpha.\alpha = \frac{c}{a}$$

$$2\alpha = \frac{-\left[-2(k-3)\right]}{1} \qquad \alpha^2 = \frac{k-1}{1}$$

$$\alpha = k - 3 \qquad \alpha^2 = k - 1$$

sub α into α^2

$$(k-3)^2 = k - 1$$

$$k^2 - 6k + 9 = k - 1$$

$$k^2 - 7k + 10 = 0$$

$$(k-5)(k-2) = 0$$

$$k = 5 \quad or \quad k = 2$$

Marking Guideline:

2	For correct response
1	For correct procedure with one arithmetic error

ii) Find the range of values of k for which the quadratic function is positive definite.

Solution

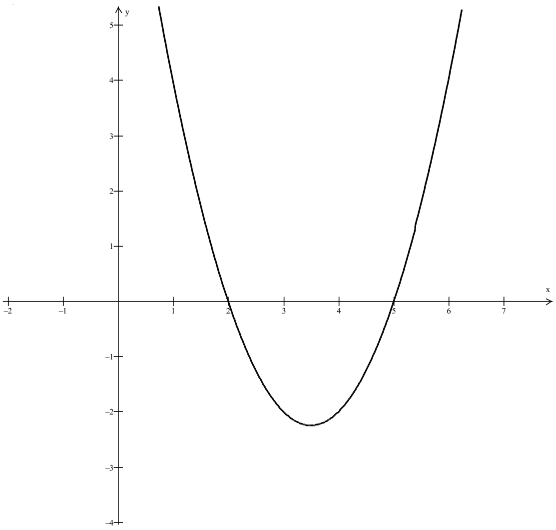
The quadratic is positive definite if the discriminant is less than 0.

Note: this is not the graph of $P(x)$ but $k^2 - 7k + 10$

From part i) the discriminant sipmlifies to $k^2 - 7k + 10$

$$k^2 - 7k + 10 < 0$$

$$(k-5)(k-2) < 0$$



From the graph of $k^2 - 7k + 10$ this occurs when

$$2 < k < 5$$

Marking Guideline:

2	For correct response
1	For correct procedure with one error

c) Find the value of m if one root of the function $y = x^2 - 2mx + (m+3)$ is three times the other.

Solution

Let the roots be α and 3α then

$$\alpha + 3\alpha = \frac{-b}{a}$$

$$4\alpha = \frac{-(-2m)}{1}$$

$$\alpha = \frac{2m}{4}$$

$$= \frac{m}{2}$$

$$\alpha.3\alpha = \frac{c}{a}$$

$$3\alpha^2 = \frac{m+3}{1}$$

$$\alpha^2 = \frac{m+3}{3}$$

Substitute $\alpha = \frac{m}{2}$ into the equation for α^2

$$\left(\frac{m}{2}\right)^2 = \frac{m+3}{3}$$

$$\frac{m^2}{4} = \frac{m+3}{3}$$

$$3m^2 = 4m + 12$$

$$3m^2 - 4m - 12 = 0$$

$$m = \frac{-(-4) \pm \sqrt{16 - 4.3.(-12)}}{2.3}$$

$$= \frac{4 \pm \sqrt{160}}{6}$$

$$= \frac{4 \pm 4\sqrt{10}}{6}$$

$$= \frac{2 \pm 2\sqrt{10}}{3}$$

Marking Guideline:	
2	For correct response
1	For correct procedure with one error

Question 6. **(10 marks)** **Marks**

a) If $ax(x-b)+(x-c)\equiv 2x^2-x-1$, find the values of *a, b* and *c*.

Solution

$$ax(x-b)+(x-c)\equiv 2x^2-x-1$$
$$ax^2-abx+x-c\equiv 2x^2-x-1$$
$$ax^2+(1-ab)x-c\equiv 2x^2-x-1$$

Comparing coefficients

$c=1,\ a=2$

$$\begin{aligned}1-ab&=-1\\1-(2)b&=-1\\-2b&=-2\\b&=1\end{aligned}$$

Marking Guideline:

2 For correct response

1 For correct procedure with one error

b) A point *P*(*x,y*) moves so that its distance from the point *A*(6,0) is twice its distance from the point *B*(0,6) .

Find the equation of the locus and describe it geometrically.

Solution

$$\begin{aligned}PA&=2PB\\(PA)^2&=4(PB)^2\\(x-6)^2+(y-0)^2&=4\left[(x-0)^2+(y-6)^2\right]\\x^2-12x+36+y^2&=4x^2+4y^2-48y+144\\3x^2+12x+3y^2-48y+108&=0\\x^2+4x+y^2-16y+36&=0\\x^2+4x+\quad y^2-16y&=-36\\x^2+4x+4\quad y^2-16y+64&=-36+4+64\\(x+2)^2+(y-8)^2&=32\end{aligned}$$

Marking Guideline:

3 For correct response or

2 For correct locus without geometric description or

1 For demonstrating some understanding

This is the locus of a circle whose centre is $(-2,8)$ with a radius of $\sqrt{32}$ units.

c) A parabola has the equation $x = 12 + 4y - y^2$.

i) Find the coordinates of the vertex

Solution

The parabola has the form $(y-k)^2 = -4a(x-h)$

Rewriting

$$x = 12 + 4y - y^2$$

$$-x = y^2 - 4y - 12$$

$$-x + 12 + 4 = y^2 - 4y + 4$$

$$(y-2)^2 = -x + 16$$

$$(y-2)^2 = -1(x-16)$$

This implies the vertex is at $(16, 2)$

Marking Guideline:

2 For correct response

1 For correct procedure with one error

ii) Find the focal length

Solution

$$4a = 1$$

$$a = \frac{1}{4}$$

Marking Guideline:

1 For correct response

iii) Find the equation of the directrix

Solution

Since the vertex is at $(16, 2)$ and in the form $(y-k)^2 = -4a(x-h)$

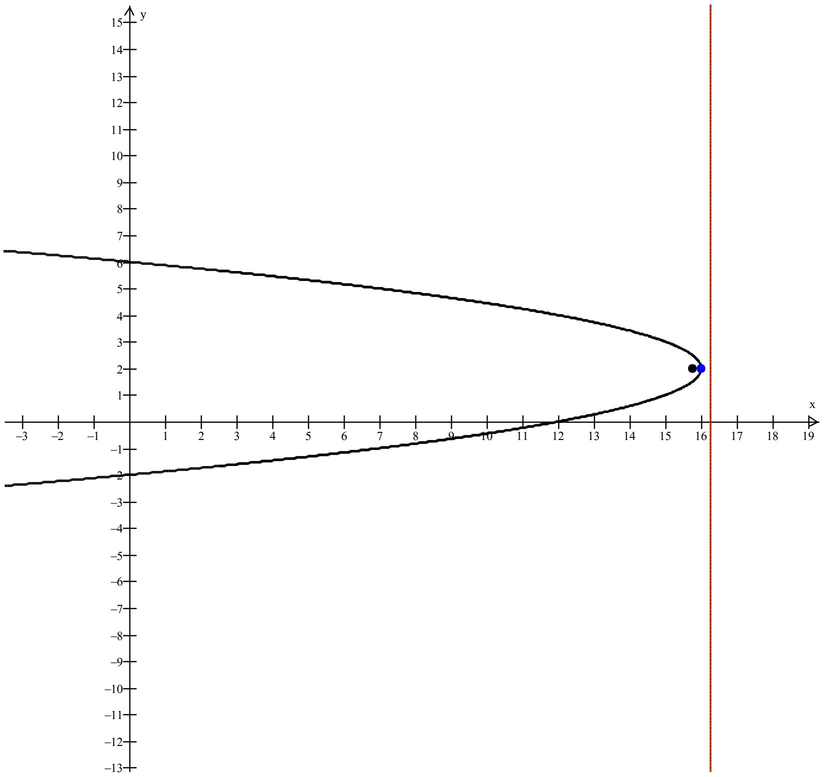
The equation of the directrix will be parallel to the y axis and $\frac{1}{4}$ units to the right of the vertex

Therefore the equation of the directrix is $x = 16\frac{1}{4}$

Marking Guideline:

1 For correct response

iv) Hence sketch $x = 12 + 4y - y^2$



Marking Guideline:

1 For correct response