2010

Preliminary Course

FINAL EXAMINATION

Mathematics

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using a black or blue pen
- Approved calculators may be used.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total Marks (84)

- Attempt Questions 1 – 7
- All questions are of equal value.

Question 1: (12 marks) *Use a separate sheet of paper*

Marks

(a) Express the decimal $0.\dot{2}0\dot{3}$ as a fraction in simplest form. 1

(b) Calculate $\frac{3}{2.78}+\frac{1}{\sqrt{3}}$ correct to 2 decimal places. 1

(c) If $\sqrt{27}+\sqrt{12}=\sqrt{A}$, find A 2

(d) Expand and simplify $(5x-2y)^2$ 1

(e) Simplify $\frac{x^2-9}{x^2+x-12}$ 2

(f) Factorise fully: $4x^3-32$ 2

(g) Solve simultaneously 2

$y=3x-7$

and $2x+y+2=0$

(h) Solve for x: 1

$$\frac{3x}{2}+3=x-4$$

Question 2: (12 marks) *Use a separate sheet of paper*

Marks

(a) Evaluate $\sqrt[3]{6.91\times10^{-5}}$ correct to three significant figures. **2**

(b) Factorize $2x^2+x-28$. **2**

(c) Simplify $\frac{2x+3}{3}-\frac{x+2}{4}$. **2**

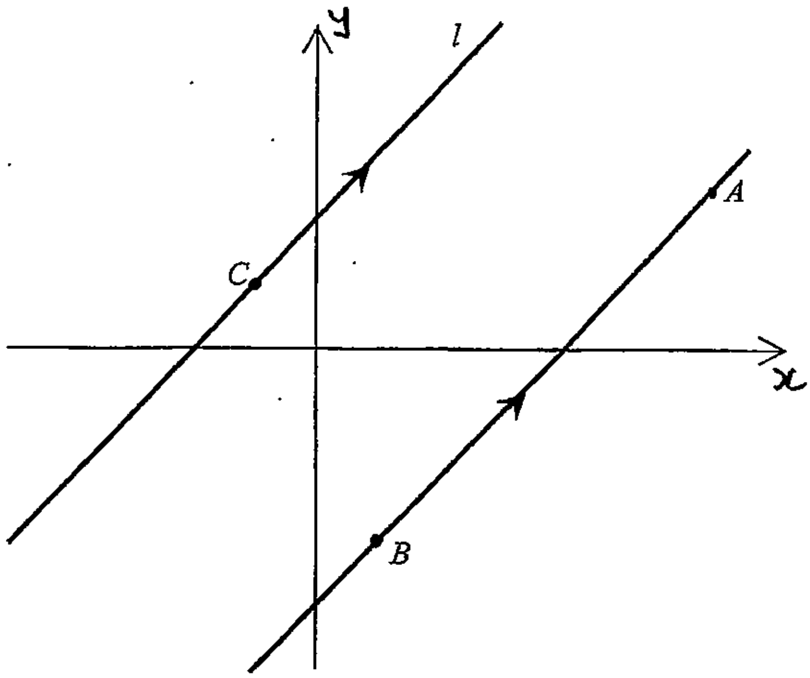
(d) Express $\left(2\sqrt{3}+1\right)\left(2-\sqrt{3}\right)$ in the form $a\sqrt{3}+b$. **1**

(e) A pair of jeans were discounted by 15% to a selling price of \$63.75. Find the original marked price of the jeans before the discount was applied. **2**

(f) Solve the following inequality. $|3x-5|\geq 2$. **3**

Question 3: (12 marks) *Use a separate sheet of paper*

(a)



NOT TO SCALE

The line l passes through $C(-1,\ 2)$ and has equation $y = 2x + 4$.
The point B has coordinates $(1,\ -6)$ and the line AB is parallel to line l.

		Marks
(i)	Copy the above diagram, writing the coordinates of B and C onto this diagram.	1
(ii)	Find the length of the interval BC.	1
(iii)	Find the midpoint of BC.	1
(iv)	Write down the slope of the line l and find the angle l makes with the positive x–axis. (answer to the nearest degree)	2
(v)	Show that AB has equation $y = 2x - 8$.	1
(vi)	If P is a point which lies on AB and on the line $y = 2$, find the coordinates of P.	1
(vii)	Show that the perpendicular distance of P from the line l is $\frac{12\sqrt{5}}{5}$ units	2
(viii)	Find the size of $\angle ABC$ to the nearest minute.	1
(ix)	Find the exact area of triangle PBC	2

Question 4: (12 marks) *Use a separate sheet of paper*

Marks

(a) Find in simplest form $\frac{f(x)-f(c)}{x-c}$ if $f(x) = 5x + 7$ 2

(b) Determine whether the function $f(x) = \frac{1}{x^2-4}$ is odd, even or neither odd nor even. WORKING MUST BE SHOWN. 1

(c) Consider $f(x) = \begin{cases} -3 & \text{if } x \le -1 \\ 2x-3 & \text{if } x > -1 \end{cases}$. Evaluate $f(-1) + f(1)$. 1

(d) Find the range of the function 1

$$y = \sqrt{4 - x^2}$$

(e) **Sketch** graphs of the following functions and state the **domain** of each.

(i) $y = \frac{3}{2x-1}$. 2

(ii) $y = |2 - 3x|$. 2

(f) Show the region of the number plane where the following hold simultaneously: 3

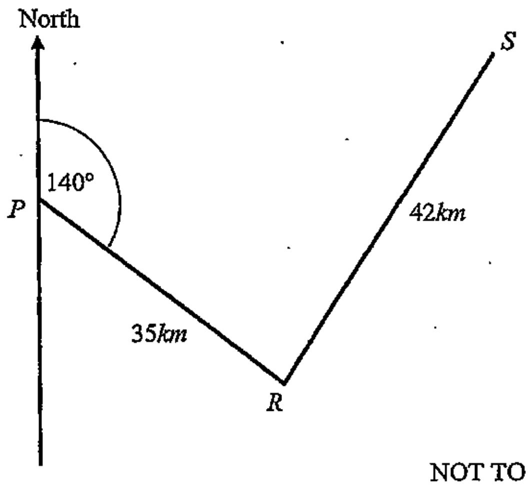
$$(x-1)^2 + y^2 \le 9$$
$$y < x + 1$$

Question 5: (12 marks) *Use a separate sheet of paper* | **Marks**

(a) Find the exact value of $\cot 330°$ | 1

(b) Solve $2\sin^2 x = 1$ where $-180° \le x \le 180°$ | 2

(c)



NOT TO SCALE

A tourist drives 35*km* from the town of Pine Vale (*P*) on a bearing of 140° T to the town of Radiatagrove (*R*).
He then drives 42*km* on a bearing of 38° T to the town of Spruceville (*S*).

Copy this **diagram**

(i) Show that $\angle PRS = 78°$. | 1

(ii) Show that the distance from Spruceville to Pine Vale (*SP*) is 49*km*, correct to the nearest kilometre. | 1

(iii) Show the size of $\angle SPR = 57°$ to the nearest degree. | 1

(iv) Hence, or otherwise, find the bearing of Pine Vale from Spruceville. Show all necessary working.. | 2

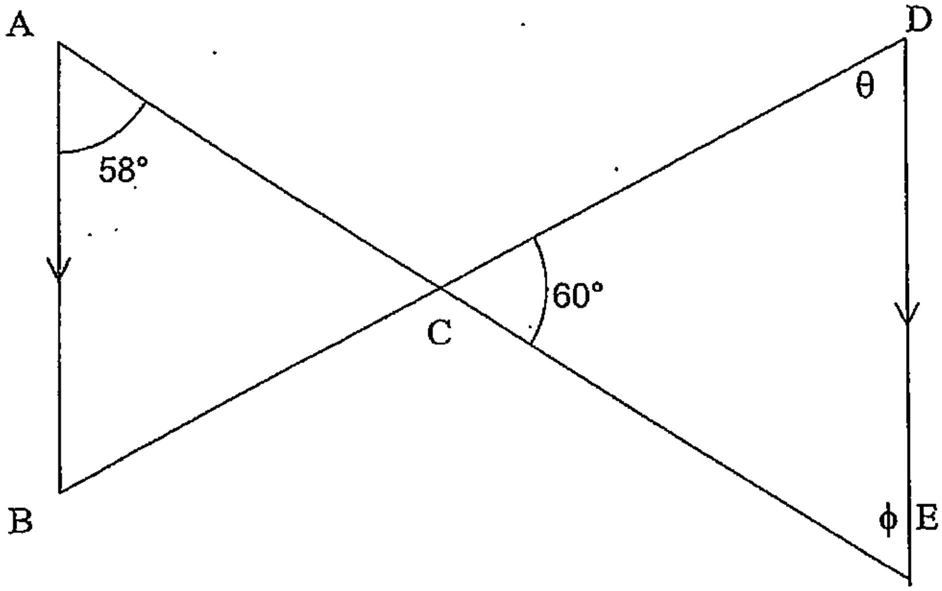
(d) If $\sin\theta = -\frac{4}{11}$ and $\tan\theta > 0$ find the exact value of $\cos\theta$. | 2

(e) Prove that $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$ | 2

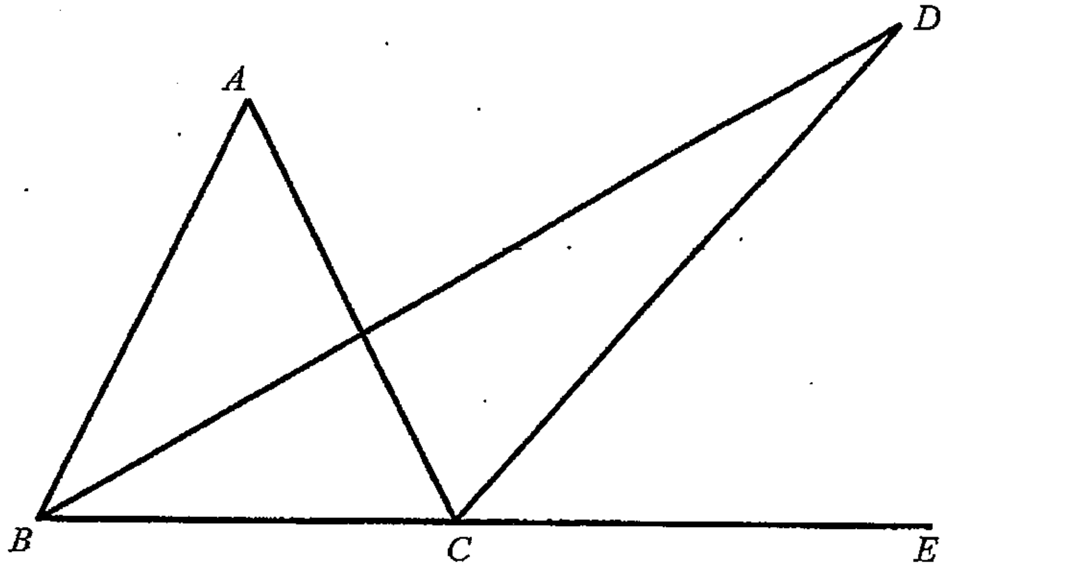
Question 6: (12 Marks) *Use a separate sheet of paper*

Marks

(a) Find θ and ϕ 1



(b)



NOT TO SCALE

ABC is an isosceles triangle in which $AB = AC$ and $\angle BAC = 64°$.
BC is produced to E. BD bisects $\angle ABC$ and CD bisects $\angle ACE$.

Copy or trace the diagram and mark on it all the given information.

(i) Find the size of $\angle ABC$ giving reasons. 1

(ii) Find the size of $\angle BDC$ giving reasons. 2

Question 6 *(continued)* **Marks**

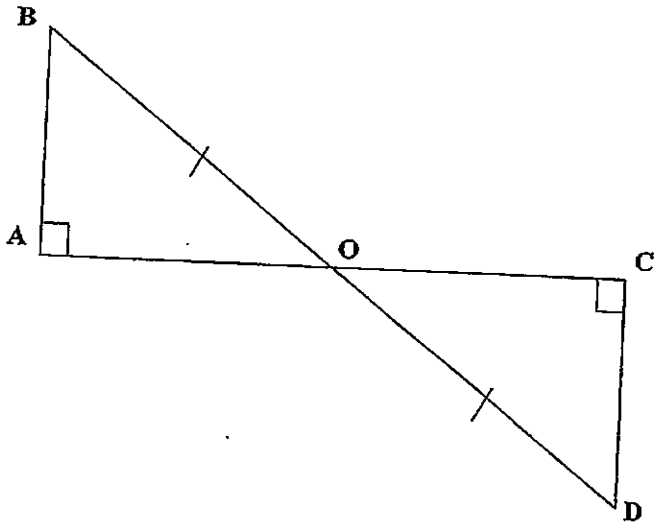
(c) The sum of the interior angles of a regular polygon is 3960°.

(i) How many sides has the polygon? 1

(ii) Find the size of each interior angle. 1

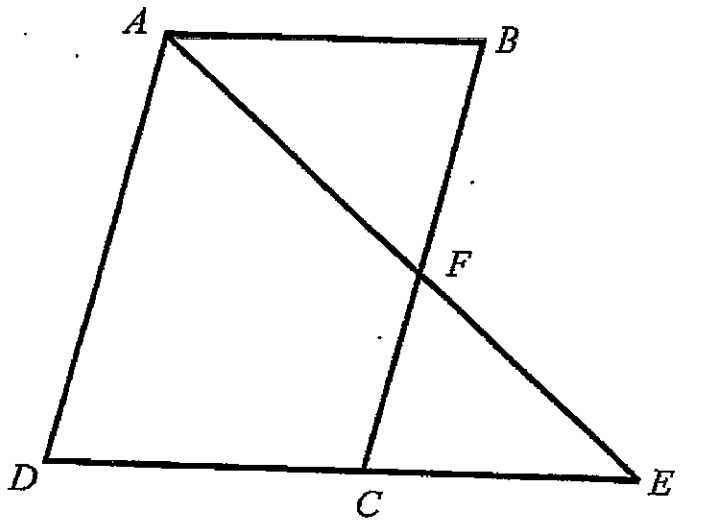
(iii) Hence or otherwise find the size of each exterior angle. 1

(d) 2



Prove that ΔAOB is congruent to ΔCOD

(e)



$ABCD$ is a parallelogram. DC is produced to E. AE cuts BC at F. $AD = 16cm$, $CE = 9cm$ and $BF = 10cm$.

(i) Prove that ΔABF is similar to ΔECF 2

(ii) Find AB 1

Question 7: (12 marks) *Use a separate sheet of paper*

Marks

(a) Express $9x^2 + 2x - 5$ in the form **2**

$ax(x + 1) + b(x+1)+C$

(b) For what values of k will the expression $kx^2 - 4x + k$ always be positive? **2**

(c) Given the equation $3x^2 + 7x - 4 = 0$ has roots α *and* β, without finding α *or* β evaluate $\alpha^2 + \beta^2$ **2**

(d) Find the value of k for which the equation $x^2 - (k + 4)x + (k - 3) = 0$ has **2**

(i) one root equal to -2

(ii) roots which are reciprocals of one another.

(e) Find all real numbers x which satisfy the equation $x^4 = 8(x^2 + 6)$. **2**

(f) Find, as a relationship between a, b *and* c, the condition for the quadratic equation in x **2**

$$(a^2 - b^2)x^2 + 2b(a - c)x + (b^2 - c^2) = 0$$

to have equal roots. Simplify your answer as far as possible.

END OF EXAM

GHS 2010 PRELIM. FINAL MATHEMATICS SOLUTIONS

QUESTION 1

a) $x = 0.\dot{2}0\dot{3}$

$1000x = 203.\dot{2}0\dot{3}$

$\therefore 999x = 203$ ①

$x = \frac{203}{999}$

b) $\frac{3}{2.78} + \frac{1}{\sqrt{3}} = 1.65648696\ldots$ (calc) ①

$= 1.66$ (2d.p)

c) $\sqrt{27} + \sqrt{12} = 3\sqrt{3} + 2\sqrt{3}$

$= 5\sqrt{3}$

$= \sqrt{75}$ ②

$\therefore A = 75$

d) $(5x-2y)^2 = 25x^2 - 20xy + 4y^2$ ①

e) $\frac{x^2-9}{x^2+x-12} = \frac{(x-3)(x+3)}{(x+4)(x-3)}$ ②

$= \frac{x+3}{x+4}$

f) $4x^3 - 32 = 4(x^3-8)$ ②

$= 4(x-2)(x^2+2x+4)$

g) $y = 3x - 7 \quad -(1)$

$2x + y + 2 = 0 \quad -(2)$

Sub ① in ②:

$2x + (3x-7) + 2 = 0$

$5x - 5 = 0$

$5x = 5$

$x = 1$ ②

Sub in ①: $y = -4$

$\therefore x = 1,\ y = -4$

h) $\frac{3x}{2} + 3 = x - 4$

$\frac{x}{2} = -7$ ①

$x = -14$

QUESTION 2

a) $\sqrt[3]{6.91 \times 10^{-5}} = 0.04103564$ (calc) ②

$= 0.0410$ (3s.f)

b) $2x^2 + x - 28 = \frac{(2x+8)(2x-7)}{2}$ ②

$= (x+4)(2x-7)$

c) $\frac{2x+3}{3} - \frac{x+2}{4}$

$= \frac{4(2x+3) - 3(x+2)}{12}$

$= \frac{8x+12-3x-6}{12}$ ②

$= \frac{5x+6}{12}$

d) $(2\sqrt{3}+1)(2-\sqrt{3})$

$= 4\sqrt{3} - 6 + 2 - \sqrt{3}$ ①

$= 3\sqrt{3} - 4$

e) $85\% = \$63.75$

$1\% = \$63.75 \div 85 = \$0.75$

$100\% = \$0.75 \times 100$

$= \$75$ ②

$\therefore$ Cost $= \$75$

f) $|3x-5| \geqslant 2$

Consider $|3x-5| = 2$

$3x-5=2$ or $3x-5=-2$

$3x = 7$ $\quad$ $3x = 3$

$x = \frac{7}{3}$ $\quad$ $x = 1$ ③

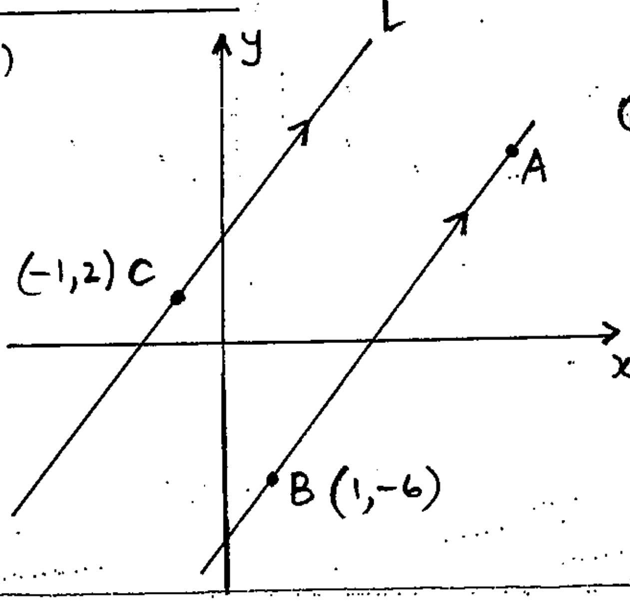
Test ✓ × ✓

1 $\quad$ $7/3$

$\therefore x \leq 1, x \geqslant \frac{7}{3}$

QUESTION 3

(I) ①



(II) $BC = \sqrt{(1+1)^2+(-6-2)^2}$

$= \sqrt{4+64}$ ①

$= \sqrt{68}$

$= 2\sqrt{17}$ units

(III) $\text{Midpt}_{BC} = \left(\frac{1-1}{2}, \frac{-6+2}{2}\right)$ ①

$= (0, -2)$

(IV) since $L \Rightarrow y = 2x+4$

$\therefore m = 2$

$\therefore$ slope is 2 ②

$\therefore \tan\theta = 2$

$\theta = 63.4349\ldots$

$\therefore \theta = 63^\circ$ (nearest degree)

(V) using $y - y_1 = m(x - x_1)$

and since $AB \parallel L$, $m = 2$

$\therefore y + 6 = 2(x-1)$ ①

$y + 6 = 2x - 2$

$y = 2x - 8$

as req.

(VI) $(y=2)$ $\quad 2 = 2x - 8$

$10 = 2x$ ①

$5 = x$

$\therefore P = (5, 2)$

(VII) $L \Rightarrow 2x - y + 4 = 0$

$\perp d = \dfrac{|2(5) - 1(2) + 4|}{\sqrt{2^2+(-1)^2}}$

$= \dfrac{12}{\sqrt{5}} = \dfrac{12\sqrt{5}}{5}$ units as req.

(viii) ⊥d of C from AB $= \frac{12\sqrt{5}}{5}$

$\therefore \sin\theta = \dfrac{\frac{12\sqrt{5}}{5}}{2\sqrt{17}}$ (1)

$= \dfrac{6\sqrt{5}}{5\sqrt{17}}$

$\therefore \theta = 40°36'$

(ix) Area of $\triangle ABC = \frac{1}{2} AB \times \frac{12\sqrt{5}}{5}$

$d_{AB} = \sqrt{(5-1)^2+(2+6)^2}$

$= \sqrt{16+64}$ (2)

$= \sqrt{80}$

$= 4\sqrt{5}$

Area $= \frac{1}{2} \times 4\sqrt{5} \times \frac{12\sqrt{5}}{5}$

$= 24$ units^2

QUESTION 4

a) $\dfrac{f(x)-f(c)}{x-c} = \dfrac{5x+7-5c-7}{x-c}$

$= \dfrac{5x-5c}{x-c}$

$= \dfrac{5(x-c)}{x-c}$

$= 5$

b) $f(x) = \dfrac{1}{x^2-4}$

$f(-x) = \dfrac{1}{(-x)^2-4} = \dfrac{1}{x^2-4}$

$\therefore$ even since $f(x) = f(-x)$

c) $f(x) = \begin{cases} -3 & x \le -1 \\ 2x-3 & x > -1 \end{cases}$

$\therefore f(-1) = -3$ (1)

$f(1) = 2(1)-3 = -1$

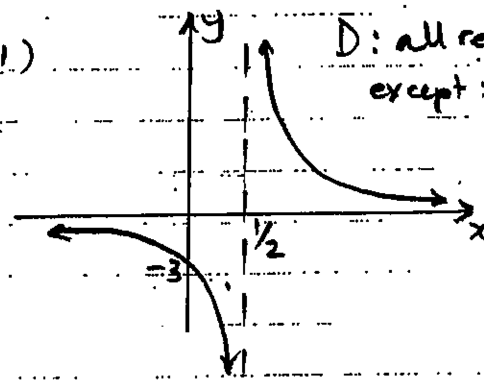
$\therefore f(-1)+f(1) = -3+(-1)$

$= -4$

d) Semi-circle (1)

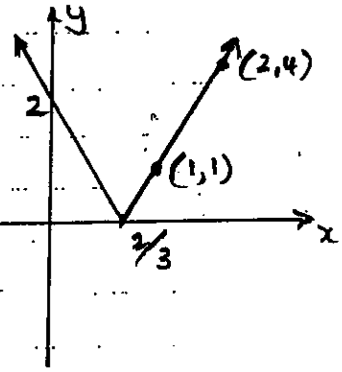
$\therefore$ Range $\Rightarrow 0 \le y \le 2$

e) (i)



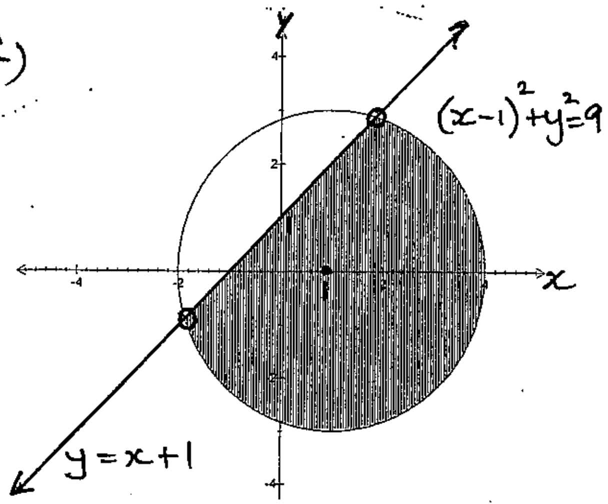
D: all real x except $x = \frac{1}{2}$

(ii)



D: all real x

f)



QUESTION 5

a) $\cot 330^\circ = -\cot 30^\circ$

$= -\sqrt{3}$ ①

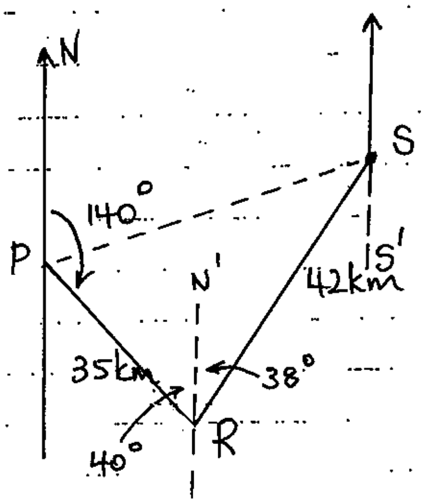
b) $2\sin^2 x = 1$

$\sin^2 x = \frac{1}{2}$ ②

$\sin x = \pm\frac{1}{\sqrt{2}}$

$\therefore x = -135^\circ, -45^\circ, 45^\circ, 135^\circ$

c)



(i) $\hat{PRN'} = 40^\circ$ (co-interior ∠'s in parallel lines are supplementary)

$\hat{N'RS} = 38^\circ$ (given in data)

$\therefore \angle PRS = 40 + 38 = 78^\circ$ ①

(ii) $SP^2 = 35^2 + 42^2 - 2 \times 35 \times 42 \times \cos 78^\circ$

$= 2377.739629\ldots$ (calc)

$\therefore SP = 48.76207\ldots$ (calc)

$\therefore SP = 49$ (nearest whole no.) ①

(iii) Let $\angle SPR = \theta$

$\therefore \frac{\sin\theta}{42} = \frac{\sin 78^\circ}{SP}$

$\sin\theta = \frac{42\sin 78^\circ}{SP}$ ①

$= 0.8425\ldots$ (calc)

$\therefore \theta = 57.4053\ldots$ (calc)

$\therefore \angle SPR = 57^\circ$ (nearest degree)

(iv) $\angle PSR = 180 - (78 + 57)$ (∠ sum of Δ = 180°)

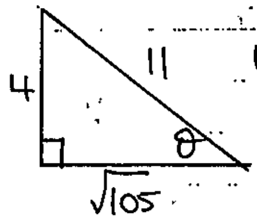
$= 45^\circ$

$\angle RSS' = 38^\circ$ (alternate ∠'s in parallel lines are equal) ②

$\therefore$ Bearing $= 180 + 38 + 45$

$= 253^\circ T$.

d)



Using this triangle $\cos\theta = \frac{\sqrt{105}}{11}$

since $\sin\theta < 0$ $\tan\theta > 0$

$\therefore$ in 3rd Quad and $\cos\theta < 0$

$\therefore \cos\theta = -\frac{\sqrt{105}}{11}$

e) $LHS = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

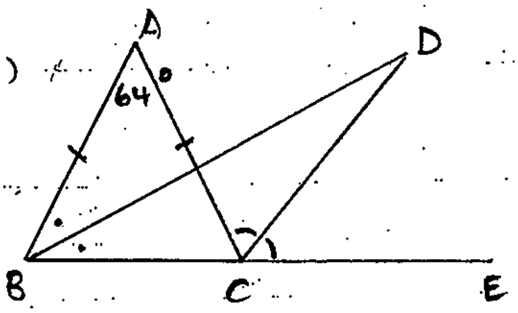
$= \frac{1}{\cos\theta\sin\theta}$

$= \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta} = \sec\theta\,\mathrm{cosec}\,\theta$

$= RHS$.

QUESTION 6 (a) $\phi = 58$, $\theta = 62$ (1)

(b)



(i) Since ΔABC isosceles

$\therefore \angle ABC = \angle ACB$ (1)

$\therefore \angle ABC = 58^\circ$ ($\angle$ sum of $\Delta = 180^\circ$)

(ii) $\angle DBC = 29^\circ$ ($\frac{1}{2} \times \angle ABC$)

$\angle BCD = 58^\circ + \frac{1}{2} \times \angle ACE$ (2)

$\angle ACE = 116^\circ$ (supplementary to $\angle ACB$)

$\therefore \angle BCD = 116^\circ$

and $\angle BDC = 35^\circ$ ($\angle$ sum of $\Delta = 180^\circ$)

(c)(i) $(n-2) \times 180 = 3960$

$n - 2 = 22$ (1)

$n = 24$

$\therefore$ polygon has 24 sides

(ii) $3960 \div 24 = 165$

$\therefore$ Each angle is 165° (1)

(iii) Either $180 - 165 = 15^\circ$

or $360 \div 24 = 15^\circ$ (1)

$\therefore$ Each exterior $\angle = 15^\circ$

(d) $\angle BAO = \angle DCO = 90^\circ$ (given)

$\angle BOA = \angle DOC$ (vertically opposite $\angle$'s are equal)

$BO = DO$ (given) (2)

$\therefore \Delta AOB \equiv \Delta COD$ (AAS)

(e)(i) $\angle AFB = \angle CFE$ (vertically opposite $\angle$'s are equal)

$\angle BAF = \angle FEC$ (alternate $\angle$'s in parallel lines are equal)

(parallel lines since sides of parallelogram and sides produced) (2)

(ii) $\frac{AB}{CE} = \frac{BF}{CF}$ (corresponding sides are in proportion)

$\frac{AB}{9} = \frac{10}{6}$ (1)

$6AB = 90$

$AB = 15$.

QUESTION 7

a) $9x^2 + 2x - 5 \equiv ax(x+1) + b(x+1) + c$

RHS $= ax^2 + ax + bx + b + c$

$= ax^2 + (a+b)x + (b+c)$

$\therefore a = 9$, $a + b = 2$, $b + c = -5$

$9 + b = 2$, $-7 + c = -5$

$b = -7$, $c = 2$

$\therefore 9x^2+2x-5 = 9x(x+1)-7(x+1)+2$

(b) Quadratic is always positive when $a>0$ and $\Delta<0$

$\Delta = b^2-4ac$

$= 16-4k^2$

$\therefore 16-4k^2<0$

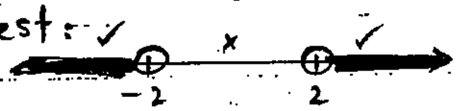
Consider $16-4k^2=0$

$16=4k^2$

$4=k^2$

$\pm 2 = k$

Test:



$\therefore k<-2,\ k>2$

Since $a>0$, $k>2$ are only values.

(c) $3x^2+7x-4=0$

$\alpha+\beta = -\frac{7}{4}$

$\alpha\beta = -\frac{4}{3}$

Since $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$

$= \left(-\frac{7}{4}\right)^2 - 2\left(-\frac{4}{3}\right)$

$= \frac{49}{16}+\frac{8}{3}$

$= \frac{275}{48} \quad \left(5\frac{35}{48}\right)$

(d) $x^2-(k+4)x+(k-3)=0$

$(x=-2)$ $\quad 4+(k+4)2+(k-3)=0$

$4+2k+8+k-3=0$

$3k=-9$

$\therefore k=-3$

(ii) let α, β be roots

and since $\beta = \frac{1}{\alpha}$

product of roots $\Rightarrow \alpha \times \frac{1}{\alpha} = \frac{c}{a}$

$1 = k-3$

$\therefore 4 = k$

(e) $x^4 = 8(x^2+6)$

$x^4 = 8x^2+48$

$x^4-8x^2-48=0$

$(x^2-12)(x^2+4)=0$

$\therefore x^2 = 12, -4$

$x = \pm\sqrt{12},$

($x^2=-4$ no real solutions)

$\therefore x = \pm 2\sqrt{3}$

(f) $(a^2-b^2)x^2+2b(a-c)x+(b^2-c^2)=$

For equal roots $\Delta=0$.

$\Delta = [2b(a-c)]^2-4(a^2-b^2)(b^2-c^2)$

$= 4b^2(a-c)^2-4(a^2-b^2)(b^2-c^2)$

$= 4b^2(a^2-2ac+c^2)-4(a^2b^2-a^2c^2-b^4+b^2c^2)$

$= 4a^2b^2-8ab^2c+4b^2c^2-4a^2b^2+4a^2c^2+4b^4-4b^2c^2$

$= 4b^4+4a^2c^2-8ab^2c$

$= 4(b^4-2ab^2c+a^2c^2)$

$= 4(b^2-ac)^2$

$\therefore$ For equal roots $b^2-ac=0$

$b^2=ac$