Hornsby Girls' High

Mathematics Preliminary Examination – 2004

QUESTION 1 (11 marks) (START A NEW PAGE) **Marks**

(a) Factorise completely

(i) $x^2 - 9$ **1**

(ii) $27 - 8y^3$ **1**

(b) Evaluate correct to 2 significant figures

$$\sqrt[4]{\frac{86.54 \times 3.16}{2.6^3}}$$ **2**

(c) Given that triangle ABC is a rightangled triangle in which $\angle ABC = 90°$, $AB = \left(2\sqrt{3} - 1\right)$ cm and $BC = \left(\sqrt{3} + 2\right)$ cm. Find the length of AC, leaving your answer in simplest surd form. **3**

(d) Convert $0.\dot{2}1\dot{2}$ into a fraction in its simplest form. **2**

(e) At a nursery, all plants increased in price by 35%. One plant now costs \$82.50. What was the old price for this plant? **2**

QUESTION 2 (9 marks) (START A NEW PAGE)

(a) Solve for x: $\frac{2x+3}{3} - \frac{3x-1}{4} = -3$ **2**

(b) Simplify: $\frac{x^2 - 2x}{y^2 - 4y + 4} \div \frac{x}{y-2}$ **3**

(c) The quadratic equation $3x^2 - 2x - 5 = 0$ has roots α and β. Without finding the roots evaluate:

(i) $\alpha\beta^2 + \beta\alpha^2$ **2**

(ii) $(\alpha - 3)(\beta - 3)$ **2**

QUESTION 3 **(10 marks)** **(START A NEW PAGE)** **Marks**

(a) For the parabola $x^2 + 6x - 12y + 3 = 0$

(i) Find the coordinates of the vertex. **2**

(ii) Calculate the focal length. **1**

(iii) Find the coordinates of the focus. **1**

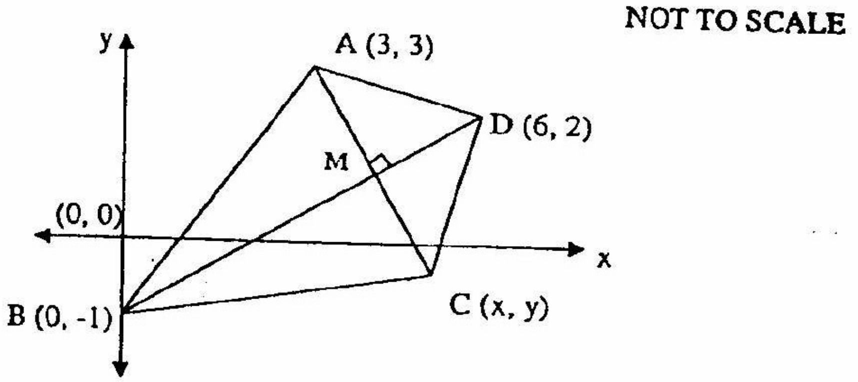
(iv) Find the equation of the directrix **1**

(b) Find the value(s) of k for which $x^2 + (k-1)x + 4 = 0$ has real roots. **2**

(c) Given $\dfrac{8}{x^2+5x+4} \equiv \dfrac{a}{(x+4)} + \dfrac{bx+c}{(x+1)}$ find the values of a, b and c. **3**

QUESTION 4 **(10 marks)** **(START A NEW PAGE)** **Marks**

(a) In the quadrilateral ABCD, the points A, B and D are (3, 3), (0, -1) and (6, 2) respectively. The line BD bisects the line AC at right angles at the point M.



(i) Find the distance BD. **1**

(ii) Show that the gradient of BD is $\frac{1}{2}$. **1**

(iii) Show that the equation of line BD is $x - 2y - 2 = 0$. **1**

Question 4 (CONT) **Marks**

(iv) Show that the equation of AC is $2x + y - 9 = 0$. **1**

(v) Find the coordinates of M. **2**

(vi) Hence, find the coordinates of C **1**

(b) A function $f(x)$ is defined as: $f(x) = \begin{cases} 2 - x & \text{for } x < -3 \\ 5 & \text{for } -3 \leq x < 0 \\ x^2 - 1 & \text{for } x \geq 0 \end{cases}$

Calculate the value of

(α) $f(-10)$ **1**

(β) $f(-3)$ **1**

(γ) $f(a^2)$. **1**

QUESTION 5 **(11 marks)** **(START A NEW PAGE)**

(a) (i) Sketch $f(x) = 4 + \dfrac{1}{1-x}$ showing any asymptotes and intercepts with the axes. **2**

(ii) State the range of the above function. **1**

(b) Shade the region represented by

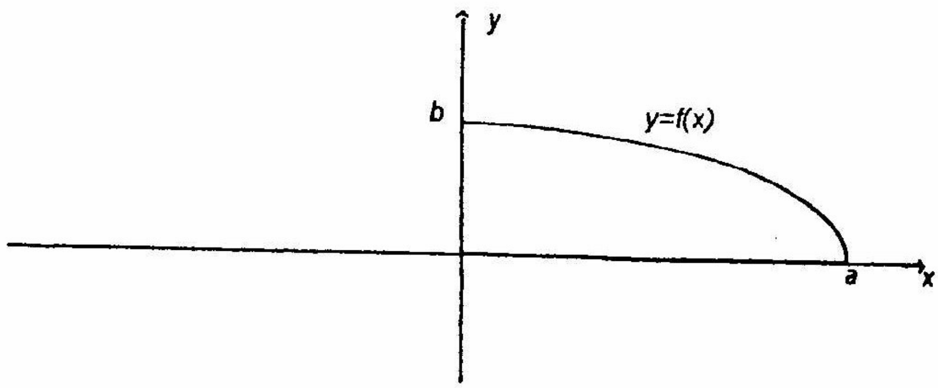
$4x + 3y - 12 < 0$ and $y \geq 3^x + 1$

(Note: Do not find the point of intersection of the two curves) **3**

Question 5 (continued)

Marks

(c)



The diagram above shows part of the graph of the function $y = f(x)$. You are told that it is an odd function. 1

Copy the diagram and complete the graph of the function

(d) The point P(x,y) moves so that its distance PA from the point A(1,5) is always twice its distance PB from the point B(4,−1).

(i) Show that the locus is a circle with equation $x^2 + y^2 - 10x + 6y + 14 = 0$ 2

(ii) Find the centre and radius of this circle

2

QUESTION 6 **(7 marks)** **(START A NEW PAGE)**

(a) If $\cos\alpha = \frac{2}{7}$ and $\sin\alpha < 0$, find the exact value of $\tan\alpha$ in the domain $0 \le \alpha \le 360^0$ 2

(b) Angela finds the angle of elevation of the top of a building to be 35^0.
After walking on horizontal ground for 60m towards the building she now finds the angle of elevation to be 70^0.

(i) Draw a neat sketch showing this information. 1

(ii) Find the height of the building to the nearest metre. 2

(c) Evaluate $\cos 135° + \operatorname{cosec} 60°$, in exact form.

2

QUESTION 7 (10 marks) **(START A NEW PAGE)** **Marks**

(a) Two surveyors, Phil and Chris place their surveying equipment at point X then set out in different directions. Phil walks due West for 200 metres while Chris walks 150 metres on a bearing of 130°.

(i) Draw a diagram showing the above information. **1**

(ii) Calculate the distance between Phil and Chris (to the nearest metre) **2**

(iii) Find the bearing of point X from Chris. **1**

(iv) Calculate the bearing of Phil from Chris (to the nearest degree) **2**

(b) (i) Show that: $\tan\theta + \cot\theta = \dfrac{1}{\sin\theta\cos\theta}$ **1**

(ii) Hence or otherwise, solve: **3**

$$\frac{1+\cot\theta}{\operatorname{cosec}\theta} - \frac{\sec\theta}{\tan\theta+\cot\theta} = -1, \quad 0° \le \theta \le 360°$$

QUESTION 8 (9 marks) **(START A NEW PAGE)**

(a) Find the derivatives of:

(i) $\sqrt{x^3} + \dfrac{1}{2x}$ **2**

(ii) $\left(3x^2 - 4x\right)^5$ **2**

(iii) $\dfrac{x+3}{x^2-4}$ **2**

(b) A tangent is drawn to the parabola $y = x^2 - 4x$ at the point P. The tangent has a gradient of 6.

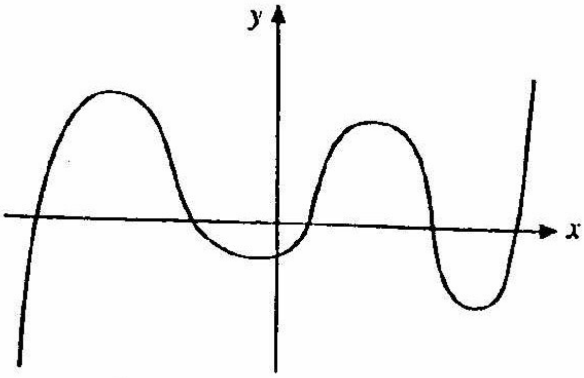
(i) Find the coordinates of P. **2**

(ii) Find the equation of the tangent at P. **1**

QUESTION 9 (16 marks) (START A NEW PAGE) **Marks**

(a) Find the values of x for which the curve $y = 2x^3 - 5x^2$ is decreasing **1**

(b) Copy the graph below and sketch the derivative function on the same graph **1**



(c) Differentiate $f(x) = 2x^2 - 7x$ from first principles **3**

(d) Sketch the graph of $y = x(2-x)^3$ by finding the stationary points and determining their nature, the points of inflection and x-intercepts. On your sketch clearly show all important features. **5**

(e) (i) The sum of the radii of two circles is 100 cm. If one of the circles has a radius of x cm, show that the sum of the areas of the two circles is given by $A = 2\pi(x^2 - 100x + 5000)$ **2**

(ii) Find the value of x for which A is the least. **3**

(f) Find the primitive function of x^{-3} **1**

END OF PAPER

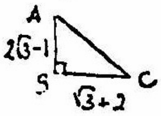
HGHS 2004 2U Yearly

1a) i) $(x-3)(x+3)$

ii) $3^3-(2y)^3 = (3-2y)(9+6y+4y^2)$

b) $1\cdot9860\ldots = 2\cdot0$

c)



$AC^2 = (2\sqrt{3}-1)^2 + (\sqrt{3}+2)^2$
$= 4\times3 - 4\sqrt{3}+1+3+4\sqrt{3}+4$
$= 20$
$\therefore AC = \sqrt{20} \quad (\sqrt{4\times5})$
$= 2\sqrt{5}$

d) let $x = 0\cdot212212212\ldots$
$1000x = 212\cdot212212\ldots$
$\therefore 999x = 212$
$x = 212/999$

e) 135% is $\$82\cdot50$
1% is $82\cdot50/135$
$\therefore 100\%$ is $82\cdot50/135 \times 100$
$= 61\cdot111\ldots$
$\therefore$ old price was $\$61\cdot11$

2a) $12^{4}x\frac{(2x+3)}{3} - 12^{3}\frac{(3x-1)}{4} = 12\times-3$
$8x+12-9x+3 = -36$
$-x+15 = -36$
$-x = -51$
$\therefore x = 51$

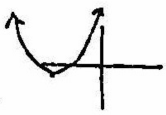
b) $\frac{x(x-2)}{(y-2)(y-2)} \times \frac{y-2}{x}$
$= \frac{x-2}{y-2}$

c) $\alpha+\beta = \frac{-b}{a} = \frac{2}{3} \qquad \alpha\beta = \frac{c}{a} = \frac{-5}{3}$

i) $\alpha\beta(\beta+\alpha) = \frac{-5}{3}\times\frac{2}{3}$
$= \frac{-10}{9}$

ii) $\alpha\beta - 3\alpha - 3\beta + 9 = \alpha\beta - 3(\alpha+\beta)+9$
$= \frac{-5}{3} - 3\times\frac{2}{3} + 9$
$= 5\frac{1}{3}$

3a) $x^2+6x+\left(\frac{6}{2}\right)^2 = 12y-3+9$
$(x+3)^2 = 12y+6$
$(x+3)^2 = 12\left(y+\frac{1}{2}\right)$



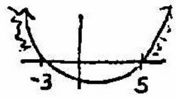
i) vertex : $\left(-3,-\frac{1}{2}\right)$

ii) $4a = 12 \quad \therefore a = 3$

iii) focus : $\left(-3, 2\frac{1}{2}\right)$

iv) directrix : $y = -3\frac{1}{2}$

b) real roots $\Delta \geq 0$
$\Delta = (k-1)^2 - 4\times1\times4$
$= k^2-2k+1-16$
$= k^2-2k-15$
$= (k-5)(k+3)$



so $(k-5)(k+3) \geq 0$
$\therefore k \geq 5, \; k \leq -3$

c) $\frac{8}{(x+4)(x+1)} \equiv \frac{a(x+1)+(bx+c)(x+4)}{(x+4)(x+1)}$

$\therefore 8 \equiv ax+a+bx^2+4bx+cx+4c$
$\equiv bx^2+(a+4b+c)x+a+4c$

$\therefore b=0$, $a+4b+c=0$, so $a+4c=8$
$\therefore a+c=0 \qquad -c+4c=8$
$a=-c \qquad \therefore 3c=8$
$\therefore c = 8/3$

$\therefore a = -8/3 \; (-2\frac{2}{3}), \; b=0, \; c = 8/3 \; (2\frac{2}{3})$

4a) i) $BD = \sqrt{(6-0)^2+(2--1)^2}$
$= \sqrt{36+9}$
$= \sqrt{45}$
$= 3\sqrt{5}$

ii) $m_{BD} = \frac{2--1}{6-0}$

$= \frac{1}{2}$

iii) $y--1 = \frac{1}{2}(x-0)$

$y+1 = \frac{1}{2}x$

$2y+2 = x$

$\therefore \quad 0 = x-2y-2$

iv) $m_{AC} = -2$

$\therefore \quad y-3 = -2(x-3)$

$y-3 = -2x+6$

$2x+y-9 = 0.$

v) $x-2y-2=0 \quad \cdots ①$

$2x+y-9=0 \quad \cdots ②$

①×2 $\quad 2x-4y-4=0 \quad \cdots ③$

②-③ $\quad 5y-5=0$

$5y=5$

$y=1$

sub in① $\quad x-2-2=0$

$\therefore \quad x=4$

So $\quad$ M is $(4,1)$

vi) M is midpoint

$\therefore \quad 4 = \frac{x+3}{2}$, $\quad 1 = \frac{y+3}{2}$

$8 = x+3 \quad\quad 2 = y+3$

$\therefore x=5 \quad\quad \therefore y=-1$

so C has coords $(5,-1)$

b)(α) $f(-10) = 2--10$

$= 12$

(β) $f(-3) = 5$

(γ) $f(a^2) = (a^2)^2-1$

$= a^4-1$

5a) $f(x) = 4 + \frac{1}{1-x}$

asymptotes : $x=1$

$y-4 = \frac{1}{1-x}$

$1-x = \frac{1}{y-4}$

$\therefore x = 1-\frac{1}{y-4}$

$\therefore$ asymptote : $y=4$

y-int: $(x=0)$

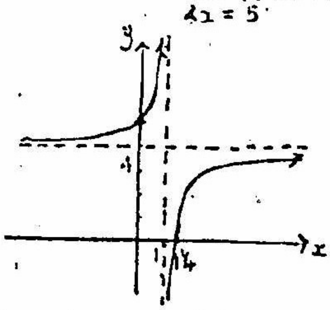
$\therefore y=5$

x-int: $(y=0)$: $0 = 4+\frac{1}{1-x}$

$0 = 4(1-x)+1$

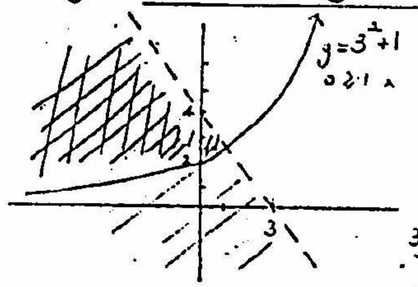
$0 = 4-4x+1$

$4x = 5$



ii) range: all real y, $y \neq 4$

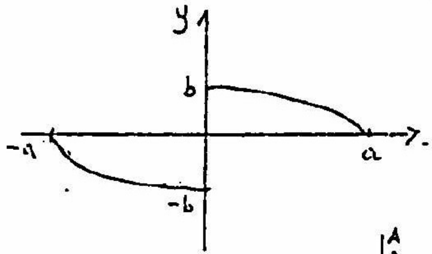
b)



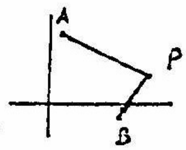
$3y < 12 -$

$y < 4-\frac{4}{3}$

c)



d) i) $PA = 2PB$



$\sqrt{(x-1)^2+(y-5)^2} = 2\sqrt{(x-4)^2+(y+1)^2}$

$x^2-2x+1+y^2-10y+25 = 4[x^2-8x+16+y^2+2y+1]$

$x^2-2x+y^2-10y+26 = 4x^2-32x+64+4y^2+8y+4$

$0 = 3x^2-30x+3y^2+18y+42$

$\therefore \quad 0 = x^2-10x+y^2+6y+14$

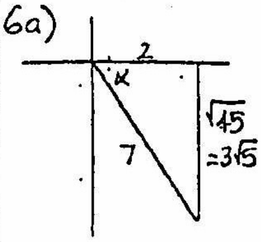
ii) $-14+25+9 = x^2-10x+\left(\frac{-10}{2}\right)^2+y^2+6y+\left(\frac{6}{2}\right)^2$

$20 = (x-5)^2+(y+3)^2$

$\therefore$ circle centre $(5,-3)$, radius $\sqrt{20}$

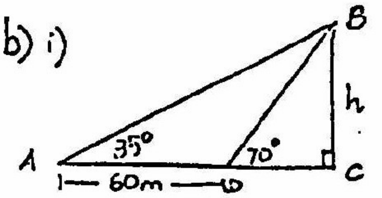
$= 2\sqrt{5}$

6a)



$\therefore \tan\alpha = -\frac{3\sqrt{5}}{2}$

b) i)



ii) $\angle ABD = 35°$ (ext $\angle$ of Δ theorem)

$\therefore \Delta ABD$ is isosceles

$\therefore BD = 60\,m$

so $\sin 70° = \frac{h}{60}$

$\therefore h = 60 \sin 70$

$= 56.38155...$

$= 56\,m$

c) $\cos 135 = -\cos(180-45)$, $\text{cosec } 60 = \frac{1}{\sin 60}$

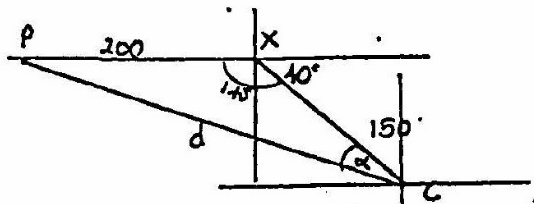
$= -\cos 45$ $\qquad = \frac{1}{\sqrt{3}/2}$

$= -\frac{1}{\sqrt{2}}$ $\qquad = 2/\sqrt{3}$

$\therefore \cos 135° + \text{cosec } 60° = \frac{-1}{\sqrt{2}} + \frac{2}{\sqrt{3}}$

$= \frac{-\sqrt{3}+2\sqrt{2}}{\sqrt{6}}$

7a)
i)



ii) $d^2 = 200^2 + 150^2 - 2 \times 200 \times 150 \times \cos 140$

$= 108462.66....$

$\therefore d = \sqrt{108462.66...}$

$= 329.336...$

$\approx 329\ m$

iii) $360 - 50 = 310°$ (or $270 + 40$)

iv) $\frac{\sin\alpha}{200} = \frac{\sin 140}{329}$

$\therefore \sin\alpha = \frac{200 \sin 140}{329}$

$\sin\alpha = 0.3907...$

$\therefore \alpha = 23$

$\therefore$ bearing of Phil from Chris:

$310 - 23 = 287°$

b) i) LHS $= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta}$

$= \frac{1}{\cos\theta \sin\theta}$

$=$ RHS

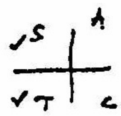
ii) LHS: $\frac{1 + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} - \frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta\cos\theta}}$

$= \left(\frac{\sin\theta + \cos\theta}{\sin\theta} \times \frac{\sin\theta}{1}\right) - \left(\frac{1}{\cos\theta} \times \frac{\sin\theta\cos\theta}{1}\right)$

$= \sin\theta + \cos\theta - \sin\theta$

$= \cos\theta$

$\therefore \cos\theta = -1$



$\therefore \theta = 180°$.

8a) i) let $y = x^{3/2} + \frac{1}{2}x^{-1}$

$y' = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-2}$

$= \frac{3\sqrt{x}}{2} - \frac{1}{2x^2}$

ii) $5(3x^2 - 4x)^4(6x-4)$

or $10(3x^2-4x)^4(3x-2)$

iii) let $y = \frac{x+3}{x^2-4}$, $u = x+3$ $\quad v = x^2-4$

$u' = 1$ $\quad v' =$

$y' = \frac{(x^2-4) \times 1 - (x+3)(2x)}{(x^2-4)^2}$

$= \frac{x^2 - 4 - 2x^2 - 6x}{(x^2-4)^2}$

$= \frac{-x^2 - 6x - 4}{(x^2-4)^2}$

b)i) $m = 6,\ y' = 2x - 4$

$\therefore 2x - 4 = 6$

$2x = 10$

$x = 5 \quad \therefore y = 5^2 - 4\times 5 = 5$

$\therefore$ P has co-ords $(5,5)$

ii) $y - 5 = 6(x-5)$

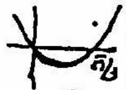
$y = 6x - 25$ or $6x - y - 25 = 0$

9a) decreasing $y' < 0$,

$y' = 6x^2 - 10x$

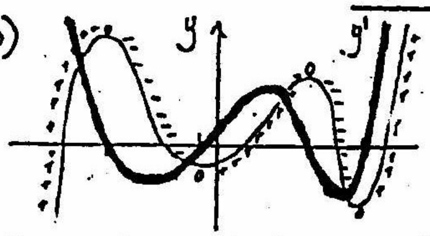
$= 2x(3x-5)$

so $2x(3x-5) < 0$



ie. $0 < x < 5/3$ (1)

b)



(1)

c) $f(x) = 2x^2 - 7x,\ f(x+h) = 2(x+h)^2 - 7(x+h)$

$= 2(x^2 + 2xh + h^2) - 7x - 7h$

$= 2x^2 + 4xh + 2h^2 - 7x - 7h$

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \dfrac{2x^2 + 4xh + 2h^2 - 7x - 7h - (2x^2 - 7x)}{h}$

$= \lim_{h\to 0} \dfrac{h(4x + 2h - 7)}{h}$

$= 4x - 7$ (3)

d) $y = x(2-x)^3$

$y' = x \times 3(2-x)^2 \times -1 + (2-x)^3$

$= -3x(2-x)^2 + (2-x)^3$

$= (2-x)^2(-3x + 2 - x)$

$= (2-x)^2(2-4x)$ (1)

$= 2(2-x)^2(1-2x)$

stat pts: $y' = 0$ ie. $x = 2,\ x = \frac{1}{2}$

$y'' = 2(2-x)^2 \times -2 + 2\times 2(2-x)\times -1 \times (1-2x)$

$= -4(2-x)^2 - 4(2-x)(1-2x)$

$= -4(2-x)(2-x+1-2x)$

$= -4(2-x)(3-3x)$

$= -12(2-x)(1-x)$

when $x = 2,\ y = 0$

$y'' = 0$

$\therefore$ poss. pt of infl.

when $x = \frac{1}{2},\ y = \frac{27}{16}$

$y'' = -9 < 0$

$\therefore \left(\frac{1}{2}, \frac{27}{16}\right)$ max t.p.

now poss. pts of inflection $y'' = 0$

ie. $-12(2-x)(1-x) = 0$

$\therefore x = 2,\quad x = 1$

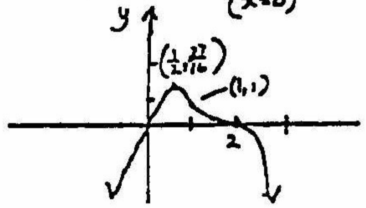
x	2^-	2	2^+
y''	+	0	−

$\therefore (2,0)$ is a horizontal pt. of infl.

x	1^-	1	1^+
y''	−	0	+

$\therefore (1,1)$ is a pt of inf.

x-int $(y=0)$: $x = 0, x = 2$ y-int $(x=0)$: $y = 0$.



(1)

e)i)

$A_1 = \pi x^2$ $A_2 = \pi(100-x)^2$

$\therefore A = \pi x^2 + \pi(10000 - 200x + x^2)$

$= \pi(x^2 + 10000 - 200x + x^2)$

$= \pi(2x^2 + 10000 - 200x)$

$= 2\pi(x^2 - 100x + 5000)$ (2)

ii) $A' = 2\pi(2x - 100)$

$= 4\pi(x - 50)$ (1)

min. when $A' = 0$

ie. $4\pi(x-50) = 0$

when $x = 50$ (1)

now $A'' = 4\pi > 0$ (1)

$\therefore x = 50$ cm is least value.

f) $\dfrac{x^{-2}}{-2} + c$ or $\dfrac{-1}{2x^2} + c$ (1)