Hornsby Girls' High – 2006

Mathematics Preliminary Examination

<u>Question 1</u> (14 marks)

a) Evaluate the following. Express your answer in Scientific Notation correct to 3 significant figures. **2**

1

$$\frac{-0.1}{\sqrt{(0.1)^2+8}}$$

b) State the exact value of $\cos 210°$ **2**

c) Show that the roots of the quadratic equation $3x^2-8x+10=0$ are not real. **2**

d) Find $\lim\limits_{x\to 3}\dfrac{x^2-9}{2x-6}$ **2**

e) Simplify: $\tan\theta \times \sin(90°-\theta)$ **2**

f) A ship sails due south for 10km and due west for 3km. Calculate its bearing from its starting point. **2**

g) Solve simultaneously: **2**

$$2x+5y=16$$
$$10x-3y=-4$$

<u>Question 2</u> **(16 marks)** **START A NEW PAGE**

a) Differentiate the following:

i) $y=x^4-3x^3+12x$ **1**

ii) $y=(3x+5)(8-2x)$ **2**

iii) $y=\dfrac{x}{x^2+4}$ **2**

b) For the parabola $(x-4)^2 = 12(y+3)$ find:

i) the vertex 1
ii) the focus 1
iii) the equation of its directrix 1

c) For the quadratic $2x^2 - 6x + 1 = 0$ find:

i) $\alpha + \beta$ 1
ii) $\alpha\beta$ 1
iii) $\alpha^2 + \beta^2$ 2

d) Solve for θ: $0 \le \theta \le 360°$ 2
$2 \text{ Sin } \theta - \sqrt{3} = 0$

e) Solve for x: $4^x - 9(2^x) + 8 = 0$ 2

Question 3 (15 marks) START A NEW PAGE

a) If $f''(x) = 6x - 8$, $f'(2) = 0$ & $f(-1) = 3$. 2
Find an expression for $f(x)$.

b) Find the equation of a parabola with focus (5,-3) 2
and directrix $x = 9$.

c) For the equation $y = \frac{1}{3}x^3 - x^2 - 8x - 6$ find:

i) the y-intercept 1
ii) the stationary points and determine their nature 2
iii) any inflection points 2

Sketch the curve in the domain $-4 \le x \le 6$. 3
State the minimum value of the function in the above domain. 1

d) Find the equation of the normal to the curve
$y = \frac{1}{x^2}$ at the point where $x = 2$. Express your 2
answer in general form.

Question 4 (14 marks) START A NEW PAGE

a) What is the range of $y = 2\sqrt{9 - x^2}$? 2

b) For what values of a is the line $y = ax$ a tangent to the hyperbola $y = \dfrac{3}{x - 2}$? 2

c) Differentiate: $y = \left(\sqrt{x} + 1\right)^{100}$ 2

d) Consider the function $f(x) = 2x^2 + x$

i) Find $f'(x)$ using the First Principles Method 2

ii) Show that $a\left(1 + f'(a)\right) = 2f(a)$. 2

e) Show that the line $5x - 12y + 26 = 0$ is a tangent to the circle $x^2 + y^2 = 4$. 2

f) Find the domain over which the curve $f(x) = x^3 - 7x^2 + 1$ is concave downwards. 2

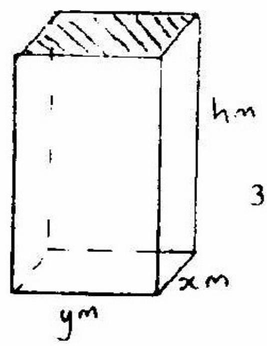
Question 5 (13 marks) START A NEW PAGE

a) What is the domain of $y = \sqrt{x^2 - 3x - 4}$? 2

b) Sketch a curve which is monotonic increasing and concave up for all values of x. 2

c) Prove: $\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = Cos^2\theta - Sin^2\theta$ 3

d) A box with a rectangular base and height h metres is open at the top and has a volume of 32m^3. The dimensions of the base of the box are x & y.



i) Show that the surface area (A) of the box is

$$A = xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$$

3

ii) Considering that for the surface area above y is constant and x is the variable, show that the minimum surface area of the box is $A = \left(16\sqrt{y} + \frac{64}{y}\right)m^2$.

3

END OF PAPER

SOLUTIONS - YEAR 11 YEARLY EXAM 2006

1) a) -3.53×10^{-2}

) $\cos 210 = -\cos 30$

$= -\frac{\sqrt{3}}{2}$

) $3x^2 - 8x + 10 = 0$

$\Delta = b^2 - 4ac$

$= 64 - 4(3)(10)$

$= -56 < 0$ ∴ unreal roots.

) $\lim_{x\to3} \frac{x^2-9}{2x-6}$ $x \neq 3$.

$= \lim_{x\to3} \frac{\cancel{(x-3)}(x+3)}{2\cancel{(x-3)}}$

$= 3$

) $\tan\theta \times \sin(90-\theta)$

$= \frac{\sin\theta}{\cancel{\cos\theta}} \times \cancel{\cos\theta}$

$= \sin\theta$

) $\tan\theta = \frac{3}{10}$

∴ $\theta = 16°42'$

∴ bearing is $196°42'$T

or S$16°42'$W

) $2x + 5y = 16$ — ① ×5

$10x - 3y = -4$ — ②

$10x + 25y = 80$ — ③

$-28y = -84$

∴ $y = 3$.

$2x + 15 = 16$

∴ $x = \frac{1}{2}$

∴ $x = \frac{1}{2}$, $y = 3$.

2 a) i) $y = x^4 - 3x^3 + 12x$

$y' = 4x^3 - 9x^2 + 12$

) $y = (3x+5)(8-2x)$

$y' = (8-2x)(3) + (3x+5)(-2)$

$y' = 24 - 6x - 6x - 10$

$y' = 14 - 12x$

Q2. (cont.)

iii) $y = \frac{x}{x^2+4}$ u v

$y' = \frac{(x^2+4)(1) - x(2x)}{(x^2+4)^2}$

∴ $y' = \frac{4 - x^2}{(x^2+4)^2}$

b) $(x-4)^2 = 12(y+3)$

$(x-4)^2 = 4(3)(y+3)$

i) ∴ Vertex $(4, -3)$

ii) focus is $(4, 0)$

iii) directrix : $y = -6$

c) $2x^2 - 6x + 1 = 0$

i) $\alpha + \beta = 3$

ii) $\alpha\beta = \frac{1}{2}$

iii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 3^2 - 2(\frac{1}{2})$

$= 8$

d) $2\sin\theta - \sqrt{3} = 0$ 180−60 | 60

$\sin\theta = \frac{\sqrt{3}}{2}$

∴ $\theta = 60°, 120°$

e) $4^x - 9(2^x) + 8 = 0$

let $m = 2^x$

∴ $m^2 - 9m + 8 = 0$

$(m-8)(m-1) = 0$

∴ $m = 8, m = 1$

but $m = 2^x$

∴ $2^x = 8$, $2^x = 1$

∴ $x = 3$, $x = 0$.

Q3. $f''(x) = 6x - 8$ $f'(2) = 0$ $f(-1) = 3$

$f'(x) = 3x^2 - 8x + c_1$

$0 = 3(2)^2 - 8(2) + c_1$

∴ $c_1 = 4$

∴ $f'(x) = 3x^2 - 8x + 4$

$f(x) = x^3 - 4x^2 + 4x + c_2$

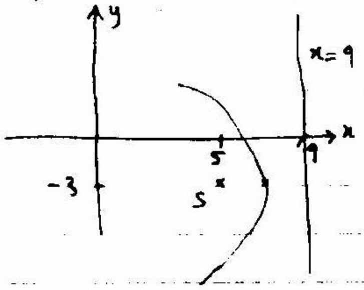
$3 = -1 - 4 - 4 + c_2$

∴ $c_2 = 12$

∴ $f(x) = x^3 - 4x^2 + 4x + 12$

Q3 (cont).

) $S(5,-3)$, directrix is $x=9$



$\therefore$ vertex is $(7,-3)$

$a = 2$

$\therefore (y+3)^2 = -8(x-7)$

) $y = \frac{1}{3}x^3 - x^2 - 8x - 6$

) y-intercept is -6

) $y' = x^2 - 2x - 8 = 0$ for stat pts.

$(x-4)(x+2) = 0$

$\therefore x = 4,\ x = -2$

$y = -32\frac{2}{3},\ y = 3\frac{1}{3}$

$\therefore$ stat pts are $(4,-32\frac{2}{3})$ & $(-2,3\frac{1}{3})$

$y'' = 2x - 2$

at $x = 4$, $y'' = 6 > 0$ $\therefore$ minimum at $(4,-32\frac{2}{3})$

at $x = -2$ $y'' = -6 < 0$ $\therefore$ max at $(-2,3\frac{1}{3})$

i) For pts of inflexion $y'' = 0$

$\therefore 2x - 2 = 0$

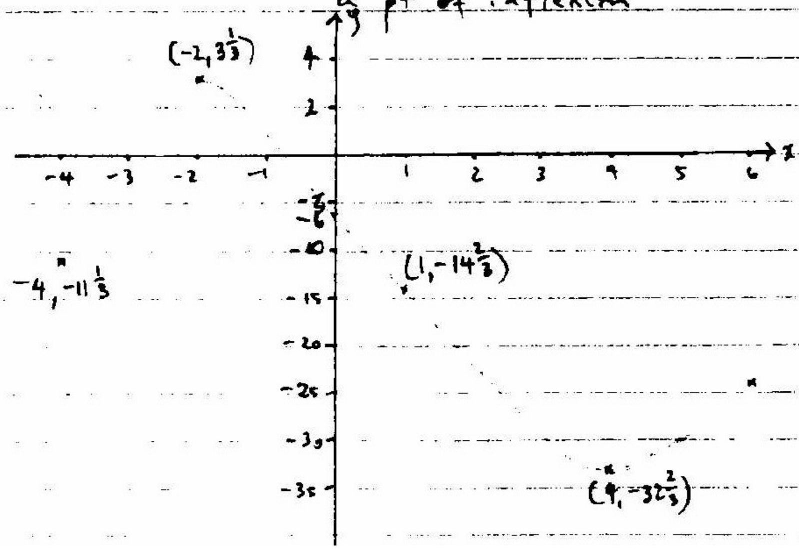
$\therefore x = 1$

$y = -14\frac{2}{3}$

x	0	1	2
y''	-2	0	$+2$

$\therefore$ change in concavity $\therefore (1,-14\frac{2}{3})$ is a pt of inflexion

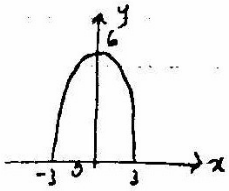
)



$\therefore$ the minimum value is $-32\frac{2}{3}$

Q4 $y = 2\sqrt{9 - x^2}$

i)



Range : all real y where $0 \le y \le 6$.

) $ax = \frac{3}{x-2}$

$ax(x-2) = 3$

$ax^2 - 2ax - 3 = 0$

For $y = ax$ to be a tangent to the hyperbola, there must only be one solution.

ie $\Delta = 0$.

$b^2 - 4ac = 0$

$4a^2 - 4(a)(-3) = 0$

$4a^2 + 12a = 0$

$4a(a+3) = 0$

$\therefore a = 0$ or $a = -3$.

) $y = (\sqrt{x} + 1)^{100}$

$y' = 100(\sqrt{x}+1)^{99} \times \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore y' = \frac{50(\sqrt{x}+1)^{99}}{\sqrt{x}}$

1) i) $f(x) = 2x^2 + x$

$f(x+h) = 2(x+h)^2 + (x+h)$

$\therefore f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 + (x+h) - 2x^2 - x}{h}$

$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h - \cancel{2x^2} - \cancel{x}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h + 1)}{\cancel{h}}$ ($2h \to 0$)

$\therefore f'(x) = 4x + 1$

) Show that $a(1 + f'(a)) = 2f(a)$

LHS $= a(1 + 4a + 1)$

$= 2a + 4a^2$

$= 2(2a^2 + a)$

$= 2f(a)$

$\therefore$ LHS = RHS.

e) Centre $(0,0)$ $r = 2$

$d = \left| \frac{5(0) - 12(0) + 26}{\sqrt{5^2 + 12^2}} \right|$

$d = \frac{26}{13} = 2.$

Since the perp. dist = the radius of the circle the line must be a tangent.

f) $f(x) = x^3 - 7x^2 + 1$

$f'(x) = 3x^2 - 14x$

$f''(x) = 6x - 14 < 0$ for concave down

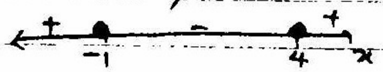
$6x < 14$

$\therefore x < \frac{7}{3}$

Q5 a) $x^2 - 3x - 4 \ge 0$

$(x-4)(x+1) \ge 0$

$\therefore x = 4,\ x = -1$

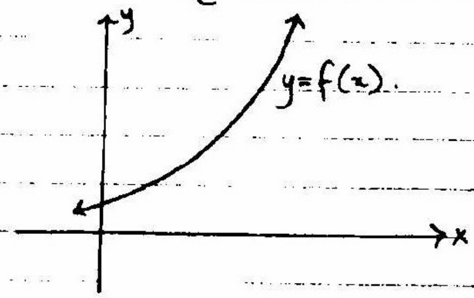


$\therefore x \le -1,\ x \ge 4$

2

$\therefore$ Domain is : $\{x \in \mathbb{R}$ where $x \le -1,\ x \ge 4\}$

b)



2

c) LHS $= \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$

$= \frac{1 - \tan^2\theta}{\sec^2\theta}$

$= \cos^2\theta - \frac{\sin^2\theta}{\cos^2\theta} \times \frac{\cos^2\theta}{1}$

$= \cos^2\theta - \sin^2\theta$

$=$ RHS

$\therefore$ LHS = RHS QED!

3

Q5 (cont.) i)

$SA = 2xh + 2yh + xy$

$V = xyh$

$V = 32$

$\therefore 32 = xyh$

$h = \frac{32}{xy}$

$SA = 2x\left(\frac{32}{xy}\right) + 2y\left(\frac{32}{xy}\right) + xy$

$SA = \frac{64}{y} + \frac{64}{x} + xy$

$\therefore SA = xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$ QED!

$= xy + 64\left(x^{-1} + \frac{1}{y}\right)$ (3)

ii) $SA' = y - 64x^{-2}$ (keeping y constant)

$= y - \frac{64}{x^2} = 0$ for stat. pts.

$\therefore \frac{64}{x^2} = y$

$x^2 = \frac{64}{y}$

$\therefore x = \frac{8}{\sqrt{y}}$ (since $x > 0$)

Test for a minimum.

$SA'' = \frac{128}{x^3}$

when $x = \frac{8}{\sqrt{y}}$, $SA'' > 0$ $\therefore$ minimum Surface Area.

OR

$x < \frac{8}{y^{1/2}}$ $A' < 0$

$x = \frac{8}{y^{1/2}}$ $A' = 0$

$x > \frac{8}{y^{1/2}}$ $A' > 0$

$\therefore x = \frac{8}{y^{1/2}}$ gives A_{min}

$\therefore SA = \frac{8}{\sqrt{y}} \cdot y + 64\left(\frac{\sqrt{y}}{8} + \frac{1}{y}\right)$

$= 8\sqrt{y} + 8\sqrt{y} + \frac{64}{y}$

(3)

$SA = \left(16\sqrt{y} + \frac{64}{y}\right) m^2$ is the minimum surface Area.