YEAR 11 PRELIMINARY EXAMINATION 1996

MATHEMATICS

Time Allowed – 150 minutes

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

The answers to the eight questions are to be returned in separate bundles clearly labelled Question 1, Question 2 etc. Each bundle must show your Candidate's Number.

QUESTION 1 (START A NEW PAGE)

(1) Factor $3x^2 + 11x - 4$.

(2) Simplify $\sqrt{80} + \sqrt{20}$.

(3) Find x if $2^{x+y} = 8$ and $3x - y = 5$.

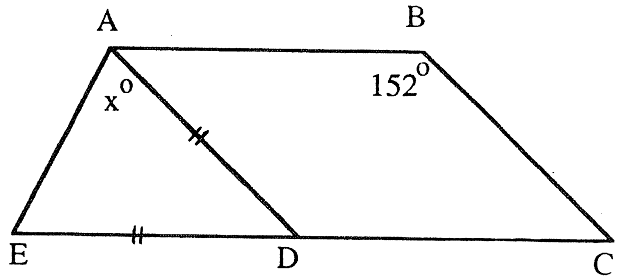
(4) Write down the centre and the radius of the circle with equation $x^2 + y^2 - 6y = 16$.

(5) Find the value of c if the line $y = 3x + c$ passes through the point $(-2,4)$.

(6) Find the value of k if the parabola $y = 3x^2 - 2x + k$ does not meet the x-axis.

QUESTION 2 (START A NEW PAGE)

(1) ABCD is a parallelogram. CD is extended to E so that AD = ED. Find the value of x giving reasons.



(2) Differentiate with respect to x:

(i) e^{4x},

(ii) $x^3 + 6\sqrt{x}$,

(iii) $\dfrac{x^2 - 2}{4x}$,

(iv) $\dfrac{\sin x}{3x + 2}$,

(v) $\log_e\left(\sqrt{x^2 + 6}\right)$.

QUESTION 3 (START A NEW PAGE)

(1) Find the co-ordinates of the point which divides the interval joining A(–4,2) and B(5,–7) in the ratio 3:1.

(2) Find the shortest distance from the point (2,5) to the line $y = \frac{3}{4}x + 1$.

(3) If all the letters of the word DELETED are arranged in a line, how many different arrangements can be made?

(4) Find the value of A and B if $4\sin\left(x - \frac{\pi}{3}\right)$ is expressed in the form $A\sin x + B\cos x$.

(5) If $x^2 + 2x + 3 = (x - 1)^2 + P(x - 1) + Q$ for all values of x, find the values of P and Q.

(6) Seven coloured discs (red, green, blue, purple, white, orange and yellow) are placed around a circle. In how many of these arrangements are the red, green and blue together?

QUESTION 4. (START A NEW PAGE)

(1) Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x^2 - 9}$.

(2) Find the equation of the tangent to the curve $y = \sqrt{4x + 1}$ at the point where $x = 2$.

(3) Assuming that the sequence {150, 144, 138, 132, ... } is an arithmetic sequence, find

(i) an expression for the general term T_n.

(ii) the first negative term in the sequence.

(iii) the greatest positive sum of the sequence.

QUESTION 5. (START A NEW PAGE)

(1) For the parabola $(x - 1)^2 = 8(y + 2)$ write down the co-ordinates of the vertex and focus and the equation of the directrix.

(2) Solve $\cos 2x = \frac{1}{2}$ for $0 \le x \le \pi$.

(3) Given that $y = \sin 2x$ for $0 \le x \le \frac{\pi}{4}$,

(i) find $\frac{dy}{dx}$,

(ii) deduce that $\frac{dy}{dx} = 2\sqrt{1 - y^2}$

(4) Given that the equation $x^3 + px + q = 0$ has roots $x = \alpha, \beta, \gamma$ write expressions for

(i) $\alpha + \beta + \gamma$,

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$,

(iii) $\alpha^2 + \beta^2 + \gamma^2$.

QUESTION 6. (START A NEW PAGE)

(1) Use induction to prove that:

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \quad \text{for all integers } n \ge 1.$$

(2) The points $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x^2 = 4y$.

(i) If the chord PQ is a focal chord prove that $pq = -1$.

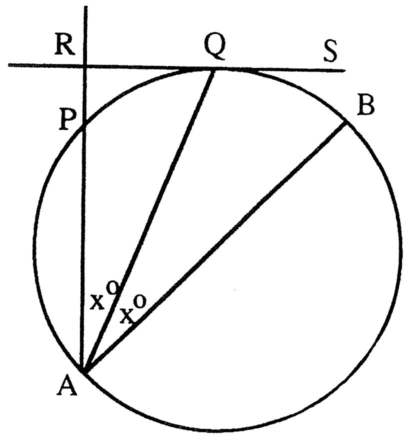
(ii) Show that the equation of the tangent at P is $px - y - p^2 = 0$.

(iii) Find the co-ordinates of R, the point where the tangent at P meets the axis of the parabola.

(iv) Find the locus of M, the midpoint of the interval QR.

QUESTION 7. (START A NEW PAGE)

(1) AB is the diameter of a circle. P and Q are points on the circle such that QA bisects $P\hat{A}B$. The tangent at Q meets the chord AP (extended) at R. Prove that $A\hat{R}Q = 90^o$.

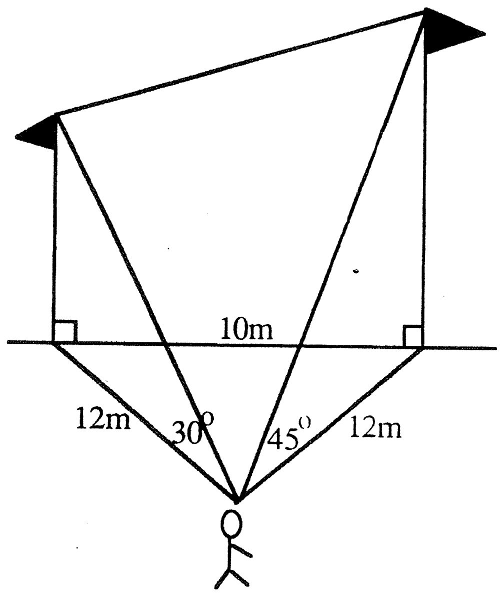


(2) John is standing 12 metres from the bases of two vertical flagpoles which are 10 metres apart. The angles of elevation of their tops, measured by John, are 30^o and 45^o. A taut wire joins the tops of two flagpoles.

(i) Find an expression for the exact difference in the heights of the two flagpoles.

(ii) Find the distance between the tops of the flagpoles. (Give your answer to the nearest metre)

(iii) Find the angle subtended by the wire at John's observation point. (Give your answer to the nearest degree)



QUESTION 8. **(START A NEW PAGE)**

(1) Prove that $\frac{\cos 3x}{\sin x} + \frac{\sin 3x}{\cos x} = 2\cot 2x$

(2) If $a + b \geq 0$, prove that $a^3 + b^3 \geq a^2b + ab^2$.

(3) P is a point on the circumference of a circle centre O and radius r. With centre P a circular arc QRS is drawn. (see diagram).

Given that $Q\hat{O}P = \theta$.

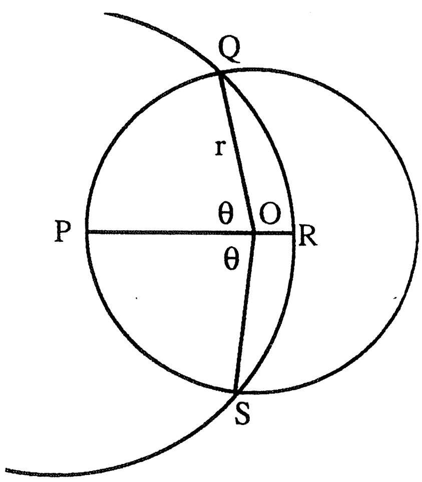
(i) Prove that the length of the chord QP is given by

$PQ = r\sqrt{2(1 - \cos\theta)}$.

(ii) Find the area of the sector with centre P and formed by the arc QRS.

(iii) If the arc QRS divides the circle with centre O into two equals parts show that

$\sin\theta + (\pi - \theta)\cos\theta = \frac{\pi}{2}$.



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Question 1

(1) $(3x-1)(x+4)$

(2) $\sqrt{5\times16}+\sqrt{5\times4} = 4\sqrt{5}+2\sqrt{5}$
$= 6\sqrt{5}$

(3) $x+y=3$
$3x-y=5$

$4x = 8$
$x = 2$

(4) $x^2+y^2-6y+9=16+9$
$x^2+(y-3)^2=25$
$C(0,3)$ $r=5$

(5) $4=-6+c$
$c=10$

(6) $\Delta<0 \Rightarrow 4-12k<0$
$-12k<-4$
$k>\frac{1}{3}$

MEMO

(1) 1 for each factor

(2) 1 for $4\sqrt{5}$, 1 for $2\sqrt{5}$

(3) 1 for $x+y=3$
1 for $x=2$

(4) 1 for centre
1 for radius

(5) 1 for subs
1 for $c=10$

(6) 1 for Δ
1 for solve $\Delta<0$
$\Rightarrow k=\frac{1}{3}$

Question 2

(1) $A\hat{D}C = 152^\circ$ (opposite angles of a parm)

$A\hat{E}D = x^\circ$ (equal angles opposite equal sides)

$2x = 152^\circ$ (exterior angle of triangle equals sum of opp. interior angles) 3

$x = 76$

2) (i) $4e^{4x}$ 1

(ii) $3x^2 + 3/\sqrt{x}$ 2

(iii) $1/4 + \dfrac{1}{2x^2}$ 2

(iv) $\dfrac{(3x+2)\cos x - 3\sin x}{(3x+2)^2}$ 2

(v) $\frac{1}{2} \cdot \dfrac{2x}{x^2+6} = \dfrac{x}{x^2+6}$ 2

Question 3

(1) $A(-4,2)$ $\quad B(5,-7)$

$3:1$

$x = \frac{-4+15}{4}$ $\quad y = \frac{2-21}{4}$ **2**

pt. is $\left(\frac{11}{4}, \frac{-19}{4}\right)$

(2) $d = \frac{|3(2)-4(5)+4|}{\sqrt{3^2+(-4)^2}}$

$= \frac{|6-20+4|}{5}$ **2**

$= 2$

(3) $N^o = \frac{7!}{3!\,2!}$

$= \frac{7\times6\times5\times4}{2\times1}$ **2**

$= 420$

(4) $4\sin x\cos\frac{\pi}{3} - 4\cos x\sin\frac{\pi}{3} = 2\sin x - \sqrt{3}\cos x$ **2**

(5) $x=1$ $\quad 6 = B$

$x=0$ $\quad 3 = 1-A+B$

$A = B-2$

$= 4$

$\therefore A=4,\ B=6$ **2**

(6) N^o ways $= 3!\times4! = 144$ **2**

(1) $\lim_{x \to 3} \frac{(x-3)(x+2)}{(x-3)(x+3)} = \lim_{x \to 3} \frac{x+2}{x+3}$ 2

$= 5/6$

(2) $y' = \frac{1}{2}(4x+1)^{-1/2} \cdot 4$

$= \frac{2}{\sqrt{4x+1}}$

when $x = 2$, $y' = \frac{2}{\sqrt{9}}$

$= 2/3$ 4

$y = \sqrt{9}$

$= 3$

tangent is $y - 3 = \frac{2}{3}(x-2)$

$3y - 9 = 2x - 4$

$2x - 3y + 5 = 0$

(3) (i) $a = 150$, $d = -6$

$T_n = 150 - 6(n-1)$

$= 156 - 6n$ 2

(ii) $T_n < 0$ $156 - 6n < 0$

$-6n < -156$

$n > 26$ 2

$\therefore n = 27$

$T_{27} = -6$

(iii) $S_{26} = \frac{26}{2}\{150 + 0\}$ $= \frac{n}{2}(a+l)$ 2

$= 1950$

Question 2

(i) ~~vertex~~ $(1,-2)$ focal length $=2$

focus $(1,0)$

directrix $y=-4$

3

(2) $2x = \pi/3,\ 2\pi - \pi/3$

$x = \pi/6,\ 5\pi/6$

2

(3) (i) $\frac{dy}{dx} = 2\cos 2x$

1

(ii) $\cos^2 2x = 1 - \sin^2 2x$

$\cos 2x = \sqrt{1-\sin^2 2x}$ since $0 \le x \le \pi/4$

$\therefore +$ $\therefore \cos 2x \ge 0$

2

$\therefore \frac{dy}{dx} = 2\sqrt{1-\sin^2 2x}$

$= 2\sqrt{1-y^2}$

* explain why take $+\sqrt{}$

(4) (i) $\alpha+\beta+\gamma = 0$

1

(ii) $\alpha\beta+\alpha\gamma+\beta\gamma = p$

1

(iii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

1

$= 0 - 2p$

$= -2p$

2

Question 6

(i) $n=1$, $LHS = \frac{1}{1\times3}$

$= 3$

$RHS = \frac{1}{2+1}$

$= \frac{1}{3}$

$\therefore$ true

Assume true for integer $n=k$

ie $\frac{1}{1.3} + \frac{1}{3.5} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

To prove true for $n=k+1$

ie $\frac{1}{1.3} + \frac{1}{5.5} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

Now $LHS = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$ by assumption

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{k+1}{2k+3}$

$= RHS$

$\therefore$ if true for $n=k$ then true for $n=k+1$ & since true for $n=1$ then true for all $n \geq 1$.

5/

Qx 6 (a) (i) chord PQ $\left(\frac{p+q}{2}\right)x - y - pq = 0$

at $(0,1)$ $\quad 0 - 1 - pq = 0$

$pq = -1$ **2**

(ii) ~~slope~~ $= \frac{dy}{dx} = \frac{x}{2}$

at P, $y' = \frac{2p}{2}$

$= p$ **2**

tangent is $y - p^2 = p(x - 2p)$

$y - p^2 = px - 2p^2$

$px - y - p^2 = 0$

(iii) at axis $x = 0$ $\therefore$ $-y - p^2 = 0$ $\quad$ ($p \neq 0$ since PQ is focal chord.)

$y = -p^2$ **1**

$\therefore R$ is $(0, -p^2)$

(iv) $M\left[q, \frac{q^2 - p^2}{2}\right]$ $\quad Q(2q, q^2)\ \ R(0, -p^2)$

$\therefore x = q \quad y = \frac{1}{2}(q^2 - p^2)$ **2**

but $pq = -1$

$\therefore p = -\frac{1}{q}$

$\therefore y = \frac{1}{2}\left(q^2 - \left(\frac{1}{-q}\right)^2\right)$

$= \frac{1}{2}\left(q^2 - \frac{1}{q^2}\right)$

$y = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$

Note $y = \frac{q^4 - 1}{2q^2}$

$2q^2 y = q^4 - 1$

$q^4 - 2q^2 y - 1 = 0$

$\Delta = 4y^2 + 4 > 0$

$\therefore$ no restriction

Question 1

(i) $S\hat{Q}B = x^\circ$ (angle between tangent & chord equals angle in alt. segment)

$A\hat{Q}B = 90^\circ$ (angle at circumf. in semi ⊙)

$R\hat{Q}A = 180^\circ - (90+x)^\circ$ (adjacent angles on line RS)

$= (90-x)^\circ$

$A\hat{R}Q = 180 - \{(90-x)+x\}$ (angle sum of ΔARQ

$= 90^\circ$

4

2) (i) diff $= x - y$

2

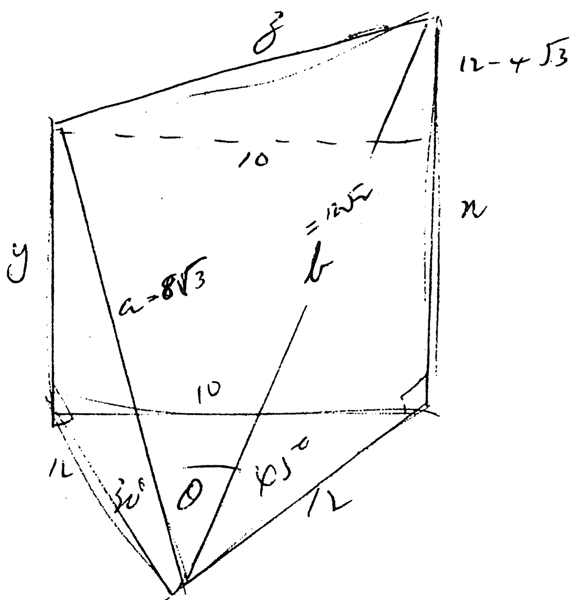
$x = 12$ (isosceles Δ.)

$y = 12 \tan 30^\circ$

$= 12 \cdot \frac{1}{\sqrt{3}}$

$= 4\sqrt{3}$

diff $= 12 - 4\sqrt{3}$, m



(ii) ~~dist~~ =

$z^2 = 10^2 + (12-4\sqrt{3})^2$

$z \doteqdot 11.2$

$= 11$ m (to nearest m.)

2

(iii) $\cos\theta = \dfrac{a^2 + b^2 - z^2}{2ab}$

$\frac{12}{a} = \cos 30$

$a = \frac{12}{\cos 30}$

$= \frac{12}{\frac{\sqrt{3}}{2}} = \frac{24}{\sqrt{3}} = 8\sqrt{3}$

$b^2 = 12^2 + 12^2$

$= 12\sqrt{2}$

$\cos\theta = \dfrac{(8\sqrt{3})^2 + (12\sqrt{2}) - z^2}{2.(8\sqrt{3})(12\sqrt{2})}$

$\theta = 4^\circ 36$

$\doteqdot 5^\circ$

4

(1) $\frac{\cos 3x}{\sin x} + \frac{\sin 3x}{\cos x} = \frac{\cos 3x \cos x + \sin 3x \sin x}{\cos x \sin x}$ 3

$= \frac{\cos(3x - x)}{\frac{1}{2}\sin 2x}$

$= \frac{\cos 2x}{\frac{1}{2}\sin 2x}$

$= 2\cot 2x$

(2) $a^3+b^3-a^2b-ab^2 = a^2(a-b) - b^2(a-b)$ 3

$=(a^2-b^2)(a-b)$

$= (a+b)(a-b)^2$

≥ 0 since $a+b \geq 0$ & $(a-b)^2$ is a square & $\therefore \geq 0$

$\therefore a^3+b^3 \geq a^2b + ab^2$

(3)(i) $PQ^2 = r^2+r^2-2r.r.\cos\theta$ 1

$= 2r^2(1-\cos\theta)$

$\therefore PQ = r\sqrt{2(1-\cos\theta)}$

(ii) $Q\hat{P}R = P\hat{Q}R = \frac{\pi-\theta}{2}$ 2

$\therefore QPS = \pi - \theta$

area $= \frac{1}{2}.PQ.PR.Q\hat{P}R$

$= \frac{1}{2}\cdot 2r^2(1-\cos\theta)(\pi-\theta)$

$= r^2(1-\cos\theta)(\pi-\theta)$

(iii) Area segments $= 2\times\frac{1}{2}r^2(\theta-\sin\theta)$

$= r^2(\theta - \sin\theta)$

Total area $= r^2(1-\cos\theta)(\pi-\theta) + r^2(\theta-\sin\theta)$

$\therefore$ if area $= \frac{1}{2}$ area of ⊙

$r^2(1-\cos\theta)(\pi-\theta)+r^2(\theta-\sin\theta) = \frac{1}{2}$

$(1-\cos\theta)(\pi-\theta)+\theta-\sin\theta = \pi/2$

$\pi-\theta-(\pi-\theta)\cos\theta+\theta-\sin\theta = \pi/2$

$-(\pi-\theta)\cos\theta + \sin\theta = -\pi/2$

$\sin\theta + (\pi-\theta)\cos\theta = \pi/2$

3