YEAR 11 PRELIMINARY EXAMINATION 1997

MATHEMATICS

Time Allowed – 150 minutes

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

The answers to the seven questions are to be returned in separate bundles clearly labelled Question 1, Question 2 etc. Each bundle must show your Candidate's Number.

QUESTION 1: **(START A NEW PAGE)**

(a) Express 48° in radians giving your answer correct to 3 decimal places.

(b) If $x = -5$, simplify $|3 - |4 + 2x||$.

(c) Solve $(2p + 3)^2 = 7$.

(d) Expand and simplify $(3\sqrt{2} - 4)^2$.

(e) Simplify $\dfrac{2m+1}{3} - \dfrac{m-1}{2}$.

(f) Solve $t^{\frac{2}{3}} = 16$.

(g) Write down the centre and radius of the circle $x^2 + 2x + y^2 - 6y - 15 = 0$.

QUESTION 2: **(START A NEW PAGE):**

(a) Differentiate with respect to x:

(i) $y = \sin 3x$,

(ii) $y = 4x^2 - x\sqrt{x}$,

(iii) $y = (5x + 3)^6$,

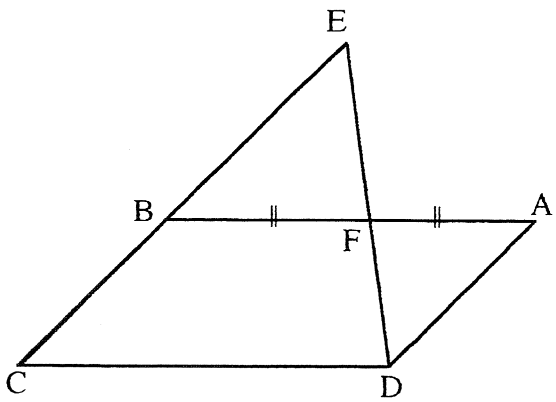
(iv) $y = \log(6x^2 - 2x + 1)$,

(v) $y = \dfrac{3x - 2}{x^2 + 4}$.

(b) ABCD is a parallelogram and F is the midpoint of AB. DF and CB are extended to meet at E. (see diagram)

(i) Prove that $\Delta BEF \equiv \Delta ADF$.

(ii) Prove that B is the midpoint of CE.



QUESTION 3: **(START A NEW PAGE)**

(a) If all the letters of the word CANADIAN are arranged in a line, how many different arrangements can be formed?

(b) Find the value of k if the polynomial $P(x) = x^3 + kx + 6$ is divisible by $2x + 3$.

(c) Solve $2\cos^2 x - 3\cos x - 2 = 0$ for $0^\circ \leq x \leq 360^\circ$.

(d) (i) Write down an expression for tan2A involving tanA .

(ii) Hence find the exact value of $\tan 22\frac{1}{2}^\circ$.

(e) The points A(1,3) , B(7,6) and C(11,8) are collinear. Find the ratio in which B divides the interval joining the points A and C.

QUESTION 4: **(START A NEW PAGE)**

(a) Solve $3 + x - 2x^2 \geq 0$.

(b) Given the parabola $y = \frac{1}{2}x^2 - x + 3$ find the co-ordinates of its vertex and focus.

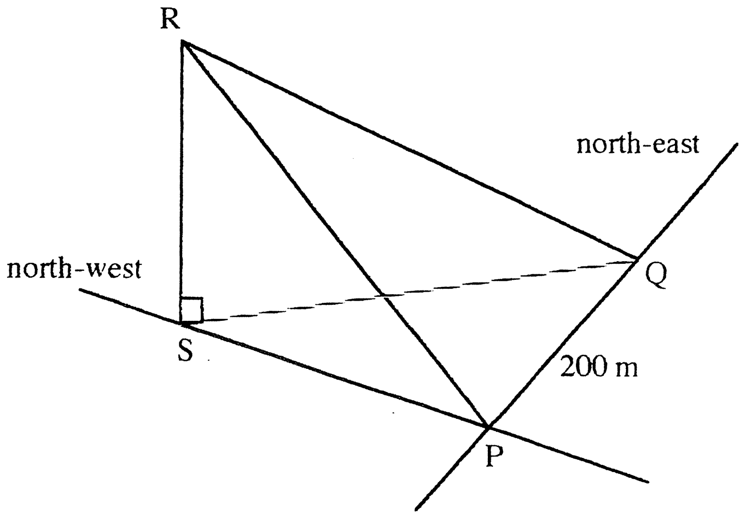
(c) Find the value of k for which the quadratic expression $(k+6)x^2 - 8x + k$ is positive definite.

(d) The line $4x + 3y - 60 = 0$ and both co-ordinate axes are tangents to a circle . Find the radius of the circle.

QUESTION 5: **(START A NEW PAGE)**

(a) Given that the equation $x^3 + Px + Q = 0$ has roots $x = \alpha$, $x = \beta$ and $x = \gamma$ find an expression for $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$.

(b) A bushwalker on a horizontal straight road that runs north-east observes from a point P that a hill bears north-west and its peak R has an angle of elevation of 15°. On walking 200 metres farther along the road to a point Q, the angle of elevation of R is now 12°. (see diagram)

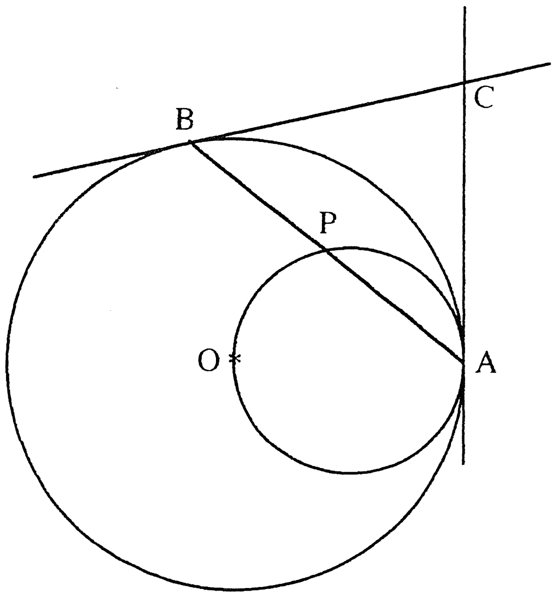


(i) If the height of the hill is h metres, show that $SP = h\cot 15^\circ$ and find a similar expression for SQ.

(ii) Show that $h = \dfrac{200}{\sqrt{\cot^2 12^\circ - \cot^2 15^\circ}}$

(iii) Find the height of the hill correct to the nearest metre.

(c) Two circles touch at A and the smaller circle passes through the centre O of the larger circle. AB is any chord of the larger circle, cutting the smaller circle at P. The tangents at A and B meet at C. (see diagram)



(i) Explain why OA is a diameter of the small circle.

(ii) Prove that OP bisects chord AB.

(iii) Prove that O, P and C are collinear.

QUESTION 6: **(START A NEW PAGE)**

(a) From a group of 4 adults and 7 children, how many different committees of 5 people can be formed if there is to be a majority of adults on the chosen committee?

(b) If $4p - 7$, $2p + 1$ and $p + 3$ are three successive terms of a geometric sequence find the value of p and hence find the value of the 6^{th} term.

(c) The point $P(2p,p^2)$ lies on the parabola $x^2 = 4y$. The normal at P meets the parabola again at point $Q(2q,q^2)$.

(i) By considering the slope of the tangent at P and the slope of interval PQ show that $p^2 + pq + 2 = 0$.

(ii) If M is the midpoint of interval PQ, show that M has co-ordinates $\left(-\frac{2}{p}, p^2 + 2 + \frac{2}{p^2}\right)$.

(iii) Find the equation of the locus of M as the position of P varies. (Do not find any restrictions on the locus).

QUESTION7: **(START A NEW PAGE)**

(a) (i) On the same set of axes sketch $x = \sin t$ and $x = \sin 2t$ for $0 \le t \le 2\pi$.

(ii) Two particles (A and B) move on a straight line and their position (x metres) from the origin O at time t hours is given by $x = \sin t$ and $x = \sin 2t$ respectively.

(α) Show that both particles start from the same position.

(β) Find when the particles next meet.

(γ) Write down an interval of time in which both particles are travelling towards the origin with positive velocity.

(b) A square based pyramid has total surface area $36\ m^2$.

(i) If the base of the pyramid has sides x metres, show that the volume of the pyramid is given by $V = x\sqrt{36 - 2x^2}$.

(ii) Show that $\frac{dV}{dx} = \frac{4(9 - x^2)}{\sqrt{36 - 2x^2}}$.

(iii) Hence find the dimensions of the base of the pyramid with greatest volume.

THIS IS THE END OF THE PAPER

JRAHS YR11 PRELIMINARY EXAM. 1997.

QUESTION 1

(a) $48^\circ = \frac{48 \times \pi}{180}$ rad.

$= 0.838$

(b) $|3-|4-10|| = |3-|-6||$

$= |3-6|$

$= |-3|$

$= 3.$

(c) $2p+3 = \pm\sqrt{7}$

$2p = -3 \pm \sqrt{7}$

$p = \frac{-3\pm\sqrt{7}}{2}$

(d) $9\times 2 - 2\times 3\sqrt{2}\times 4 + 16 = 18 - 24\sqrt{2} + 16$

$= 34 - 24\sqrt{2}$

(e) $\frac{2(2m+1) - 3(m-1)}{6} = \frac{4m+2-3m+3}{6}$

$= \frac{m+5}{6}$

(f) $(t^{2/3})^3 = 16^3$

$t^2 = 16^3$

$t = \pm\sqrt{16^3}$

$t = \pm 64$

(g) $(x^2+2x+1) + (y^2-6y+9) = 15+1+9$

$(x+1)^2 + (y-3)^2 = 25$

centre $(-1,3)$ & radius $= 5$

QUESTION 2.

(a) (i) $y' = 3\cos 3x$

(ii) $y' = 8x - \frac{3\sqrt{x}}{2x}$

(iii) $y' = 6(5x+3)^5 \cdot 5$
$= 30(5x+3)^5$

(iv) $y' = \frac{12x - 2}{6x^2 - 2x + 1}$

(v) $y' = \frac{(x^2+4)(3) - (3x-2)(2x)}{(x^2+4)^2}$
$= \frac{12 + 4x - 3x^2}{(x^2+4)^2}$

(b) (i) $EC \parallel AD$ (EC is extension of BC + opposite sides of parm ABCD are parallel)

In ΔBEF + ΔADF

$BF = FA$ (F is midpt. of AB)

$B\hat{F}E = D\hat{F}A$ (vertically opposite angles)

$B\hat{E}F = A\hat{D}F$ ($CE \parallel AD$, alternate angles)

$\therefore \Delta BEF \equiv \Delta ADF$ (AAS)

(ii) $BC = AD$ (opposite sides of parm ABCD)

$BE = AD$ (corresponding sides in congruent triangles)

$\therefore BC = BE$ (both equal AD)

$\therefore$ B is midpt. of CE ($BE = BC$)

QUESTION 3

(a) N° diff. arrangements $= \dfrac{8!}{3!2!}$

$= 3360$

(b) $P(-3/2) = 0$

$\therefore \left(-\tfrac{3}{2}\right)^3 + k\left(-\tfrac{3}{2}\right) + 6 = 0$

$\dfrac{21}{8} - \dfrac{3k}{2} = 0$

$k = 7/4.$

(c) $(2\cos x + 1)(\cos x - 2) = 0$

$\cos x = -\tfrac{1}{2}, 2.$

$x = 120°, 240° \quad (\cos x \neq 2)$

(d) (i) $\tan 2A° = \dfrac{2\tan A}{1 - \tan^2 A}$

(ii) let $A = 22\tfrac{1}{2}°$

$\tan 45° = \dfrac{2\tan 22\tfrac{1}{2}°}{1 - \tan^2 22\tfrac{1}{2}°}$

$1 = \dfrac{2x}{1-x^2}$ where $x = \tan 22\tfrac{1}{2}°$

$x^2 + 2x - 1 = 0$

$x = \dfrac{-2 \pm \sqrt{8}}{2}$

$= -1 \pm \sqrt{2}$

but $\tan 22\tfrac{1}{2}° > 0 \quad \therefore \tan 22\tfrac{1}{2}° = -1 + \sqrt{2}$

(e) let ratio be $k : l$

$A(1,3) \quad \times \quad C(11,8)$
$k : l$

from x: $7 = \dfrac{l + 11k}{k + l}$

$7k + 7l = l + 11k$

$6l = 4k$

$k : l = 6 : 4$

$\therefore$ ratio $= 3 : 2$

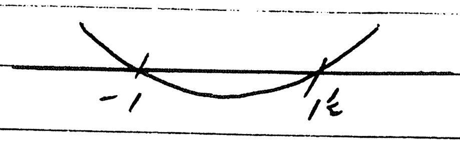
QUESTION 4.

(a) $3 + x - 2x^2 \geq 0$

$2x^2 - x - 3 \leq 0$

$(2x-3)(x+1) \leq 0$

$-1 \leq x \leq 1\tfrac{1}{2}$.



(b) $2y = x^2 - 2x + 6$

$x^2 - 2x = 2y - 6$

$x^2 - 2x + 1 = 2y - 5$

$(x-1)^2 = 4(\tfrac{1}{2})(y - 2\tfrac{1}{2})$

$\therefore$ vertex is $(1, 2\tfrac{1}{2})$ focal length $= \tfrac{1}{2}$

focus is $(1, 3)$

(c) for pos. def $a > 0$ & $\Delta < 0$

$\therefore K + 6 > 0$

$K > -6$ ——— (1)

& $\Delta = 64 - 4(K+6)(K)$

$= 64 - 24K - 4K^2$

$\therefore 64 - 24K - 4K^2 < 0$

$K^2 + 6K - 16 > 0$

$(K+8)(K-2) > 0$

$K < -8$ or $K > 2$ ——— (2)

from (1) and (2)

$K > 2$.

(d) let radius of circle be r $\therefore$ centre is (r, r)

$\therefore \perp$ dist. from line to centre $= r$

$$r = \frac{|4(r) + 3(r) - 60|}{\sqrt{16+9}}$$

$$r = \frac{|7r - 60|}{5}$$

$$5r = |7r - 60|$$

$\therefore 7r - 60 = 5r$ or $-7r + 60 = 5r$

$r = 5$ or 30

QUESTION 5

(a) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$

$\alpha\beta+\alpha\gamma+\beta\gamma = P$ & $\alpha\beta\gamma = -Q$.

$\therefore \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{-P}{Q}$

(b) (i) $\frac{h}{SP} = \tan 15^\circ$

$SP = \frac{h}{\tan 15^\circ}$

$SP = h\cot 15^\circ$ & $SQ = h\cot 12^\circ$

(ii) $(h\cot 15^\circ)^2 + 200^2 = (h\cot 12^\circ)^2$

$200^2 = h^2(\cot^2 12^\circ - \cot^2 15^\circ)$

$\therefore h^2 = \frac{200^2}{\cot^2 12^\circ - \cot^2 15^\circ}$

$h = \frac{200}{\sqrt{\cot^2 12^\circ - \cot^2 15^\circ}}$

(iii) $h = 69.82$ m

$= 70$ m (to nearest m)

(c) (i) $O\hat{A}C = 90^\circ$ (radius of large circle is $\perp$ to tangent at A)

But OA is also a chord of the small circle & since it is $\perp$ to the tangent at A then OA is a diameter of the small circle

(ii) $O\hat{P}A = 90^\circ$ (angle at circumference in semicircle)

$\therefore$ OP bisect AB (radius (OP) perpendicular to a chord bisects the chord)

(iii) In ΔBCP & ΔACP

$BP = AP$ (OP bisect AB)

$BC = AC$ (tangents from an external point are equal)

$PC = PC$ (common)

$\Delta BPC \equiv APC$ (SSS)

$\therefore B\hat{P}C = A\hat{P}C$ (corresponding angles in congruent triangles)

$\therefore B\hat{P}C = 90°$ (sum of adjacent angles on line AB)

$O\hat{P}A + A\hat{P}C = 180°$

$\therefore O, P, C$ are collinear (sum of adjacent angles $= 180°$)

QUESTION 6

(a) 3 adults, 2 children or 4 adult, 1 child

$$N^\circ \text{ committees} = {}^4C_3 \cdot {}^7C_2 + {}^4C_4 \cdot {}^7C_1$$
$$= 91$$

(b)
$$\frac{p+3}{2p+1} = \frac{2p+1}{4p-7}$$
$$(p+3)(4p-7) = (2p+1)^2$$
$$\therefore p = 22$$
$$a = 81$$
$$r = \frac{2p+1}{4p-7}$$
$$= \frac{45}{81}$$
$$\therefore T_6 = 81\left(\tfrac{5}{9}\right)^5$$
$$= 4\tfrac{209}{729}$$

(c) (i) slope of tangent at $P = p$

slope of chord $PQ = \dfrac{p^2-q^2}{2p-2q}$

$$= \frac{p+q}{2}$$

since PQ is $\perp$ to tangent at P

$$(p)\left(\frac{p+q}{2}\right) = -1$$
$$p^2 + pq = -2$$
$$p^2 + pq + 2 = 0$$

(ii) M is $\left(\dfrac{2p+2q}{2}, \dfrac{p^2+q^2}{2}\right)$

but $q = \dfrac{-2-p^2}{p} = -\dfrac{(p^2+2)}{p}$

$$\therefore M \text{ is } \left[p - \frac{2+p^2}{p}, \frac{1}{2}\left(p^2 + \left(\frac{2+p^2}{p}\right)^2\right)\right]$$

M is $\left[-\frac{p}{2}\ ;\ p^2+2+\frac{2}{p^2}\right]$

(iii) $x = -2/p$

$p = -2/x$

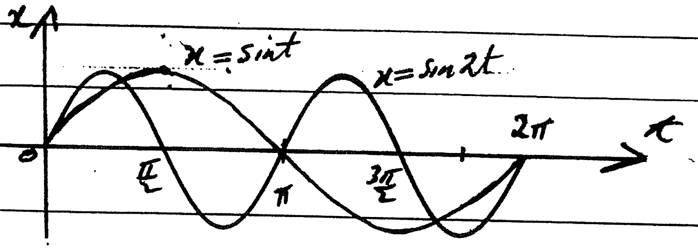
$\therefore y = \left(-2/x\right)^2 + 2 + \frac{2}{\left(-2/x\right)^2}$

$y = 4/x^2 + 2 + x^2/2$

$y = \frac{8 + 4x^2 + x^4}{2x^2}$

QUESTION 7

(a) (i)



(ii) (α) $t=0 \quad x_A = \sin 0$
$= 0$

$x_B = \sin 0$
$= 0$

$\therefore$ both start at $x=0$.

(β) meet when $\sin t = \sin 2t$

$\sin t = 2\sin t\cos t$

$2\sin t\cos t - \sin t = 0$

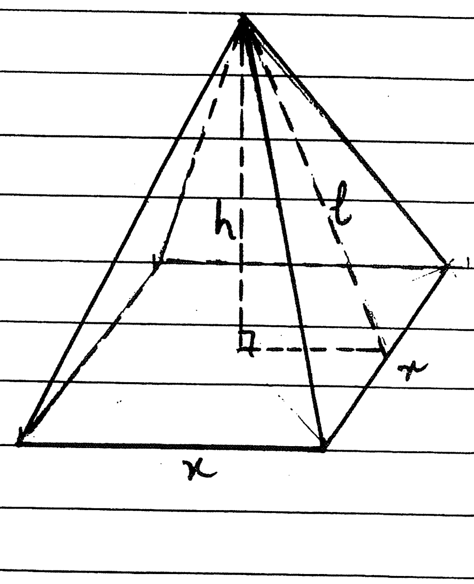
$\sin t(2\cos t - 1) = 0$

$\sin t = 0$ or $\cos t = \frac{1}{2}$

$t = 0, \pi, 2\pi, \quad t = \frac{\pi}{3}, \frac{5\pi}{3}$

$\therefore$ first time (after $t=0$) is $t = \frac{\pi}{3}$.

(γ) $\frac{7\pi}{8} < t < 2\pi$.



(b) (i) $V = \frac{1}{3}x^2h$ — ①

$36 = x^2 + 2xl$ — ②

$l^2 = h^2 + \left(\frac{x}{2}\right)^2$ — ③

from ② $l = \dfrac{36 - x^2}{2x}$

from ③ $h^2 = l^2 - \dfrac{x^2}{4}$

$$= \left(\frac{36 - x^2}{2x}\right)^2 - \frac{x^2}{4}$$

$$= \frac{1296 - 72x^2 + x^4}{4x^2} - \frac{x^2}{4}$$

$$h^2 = \frac{1296 - 72x^2}{4x^2}$$

$$h^2 = \frac{36(36-2x^2)}{4x^2}$$

$$h = \frac{6\sqrt{36-2x^2}}{2x}$$

from ① $V = \frac{1}{3}x^2 \cdot \frac{6\sqrt{36-2x^2}}{2x}$

$$V = x\sqrt{36-2x^2}$$

(ii) $\frac{dV}{dx} = (1)\sqrt{36-2x^2} + (x)\cdot\frac{1}{2}(36-2x^2)^{-\frac{1}{2}}\cdot(-4x)$

$$= \sqrt{36-2x^2} - \frac{2x^2}{\sqrt{36-2x^2}}$$

$$= \frac{36-2x^2-2x^2}{\sqrt{36-2x^2}}$$

$$= \frac{36-4x^2}{\sqrt{36-2x^2}}$$

$$\frac{dV}{dx} = \frac{4(9-x^2)}{\sqrt{36-2x^2}}$$

(iii) for stat. pt $\frac{dV}{dx} = 0$

$$\therefore \frac{4(9-x^2)}{\sqrt{36-2x^2}} = 0$$

$$9 - x^2 = 0$$

$$x = \pm 3$$

but $x > 0$ $\therefore x = 3$

Test nature of stat. pt

x	$3-\varepsilon$ (2)	3	$3+\varepsilon$ (4)
$\frac{dV}{dx}$	$\doteqdot 3.78$ > 0	0	$\doteqdot -2.2$ < 0

Since $V(x)$ is cts for $0 < x < \sqrt{18}$ & there is a change in sign of $V'(x)$ from >0 to <0 for x ~~near~~ $x = 3$ then there is a local max. tp at $x = 3$.

Since there is only one tp. for $0 < x < \sqrt{18}$ it is the absolute max tp.

$\therefore$ dimension of base is 3m × 3m