YEAR 11 PRELIMINARY EXAMINATION 1998

MATHEMATICS

Time Allowed – 150 minutes

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

The answers to the seven questions are to be returned in separate bundles clearly labelled Section A, Section B, etc. Each bundle must show your Candidate's Number.

SECTION A (Start a new page)

1. Fully factor:
   (a) $x^2 - 8x$
   (b) $6x^3 + 18x^2 - 24x$
   (c) $x^2 - y^2 - 2x - 2y$

2. If $x = \ln 2$ and $y = \ln 3$, express $\ln\sqrt{6}$ in terms of x and y.

3. Differentiate with respect to x: $y = 2\pi + x^2$.

4. Find k if $x^3 - kx + 5$ is divisible by $(x - 2)$.

5. Evaluate $1 - \tan^2 x$ if $x = 30°$.

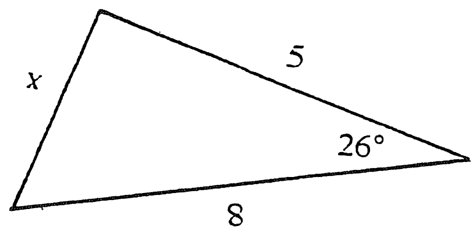
6. Find m if $\sqrt{20} + \sqrt{80} = m\sqrt{5}$.

7. Find the n th term of the geometric sequence $\{ 1, 2x^2, 4x^4, \ldots\ldots \}$.

SECTION B (Start a new page)

1. Solve $9^{2x+1} = 27$.

2. Find the length of the side x, correct to 2 decimal places.



3. The points A(−1,3), B(5,−1) and C(1,6) lie on a plane. Find:
   (a) the equation of the line AB.
   (b) the distance from C to the line AB.
   (c) the area of triangle ABC .

4. Find the acute angle between the lines $y = x$ and $2x - y = 1$.

SECTION C (Start a new page)

1. If $a = \frac{1}{2}$ and $b = \frac{1}{3}$, evaluate $a^{-1} + 2b^{-1}$.

2. Evaluate $\lim_{x \to 0} \left(\frac{2 \sin x \ \cos x}{5x} \right)$.

3. Differentiate with respect to x :

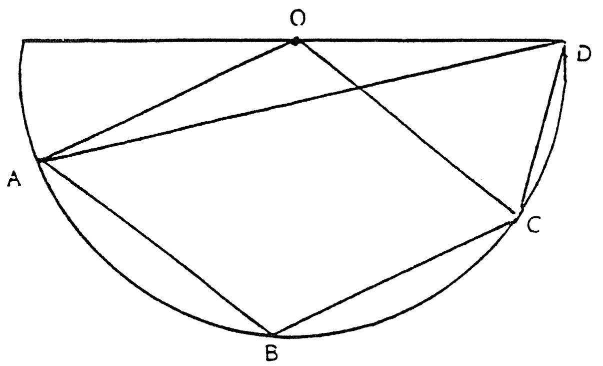
   (a) $y = \tan 4x$

   (b) $y = \frac{2x}{\sqrt{x}}$

   (c) $y = (2x + 1)^3 (2x - 1)^3$.

4. OABC is a parallelogram inscribed in a semi-circle where OD is a radius. Find $\angle$ ADC, giving all reasons.

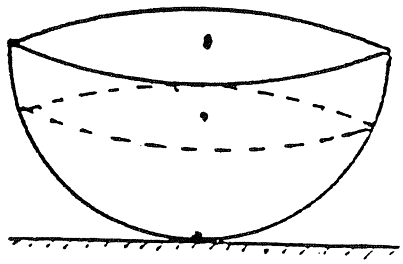
COPY DIAGRAM INTO YOUR ANSWER SHEET



5. Find the co–ordinates of the focus and vertex of the parabola with equation $2y = x^2 - 2x + 9$.

SECTION D (Start a new page)

1. A hemispherical bowl is 18 cm. in diameter and contains water to a depth of 7 cm. when level. How far will the bowl roll in order for the water to commence pouring. (Answer to 2 dec. places)



2. The letters of the word FACETIOUS are placed in a straight line. How many arrangements are possible if:

   (a) the word ACE is always present in the arrangement.
   (b) the letters of the word FACT are always together.
   (c) the vowels are never together.

3. Solve the equations below for $0° \leq \theta \leq 180°$:

   (a) $\sin^2\theta = \sin \theta$

   (b) $\cos 4\theta = \cos 2\theta$

SECTION E (Start a new page)

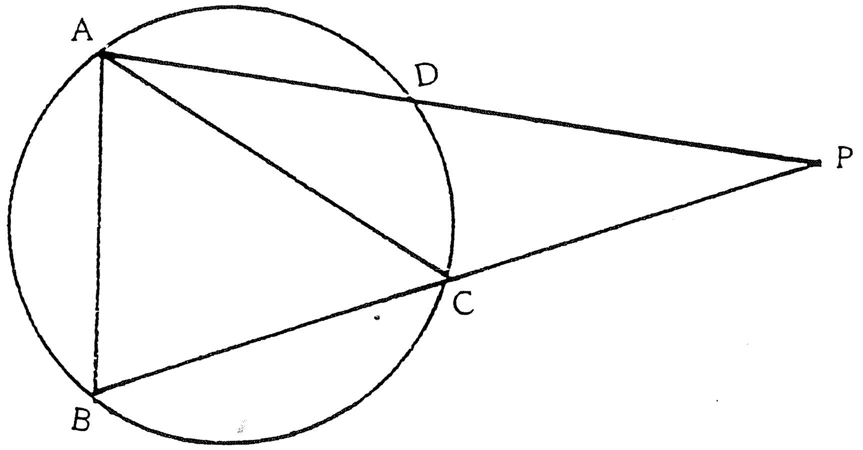
1. Solve for x : $2^{\log_4(x+3)} = 8$.

2. Prove that $\dfrac{2\cot\theta}{1+\cot^2\theta} = \sin 2\theta$.

3. Find n for which the sum of the first n terms of the arithmetic series { 15 + 13 + 11 + ········ } is 55.

4. Find the equation of the tangent to the curve $y = x^2 + x$ at the point where $x=1$.

5. In the circle ABCD given, chords AD and BC are extended to intersect at P, and $\angle APC = \angle BAC$. Prove that AB = BD. (COPY DIAGRAM INTO YOUR ANSWER SHEET.)



SECTION F (Start a new page)

1. A is the point (1,–3) and ℓ_1 is the line $x = 5$. Find the locus of P such that the distance from P to A is equal to the distance from P to the line ℓ_1 .

2. Find all solutions to $x^2 - x + \dfrac{6}{x^2 - x} = 7$.

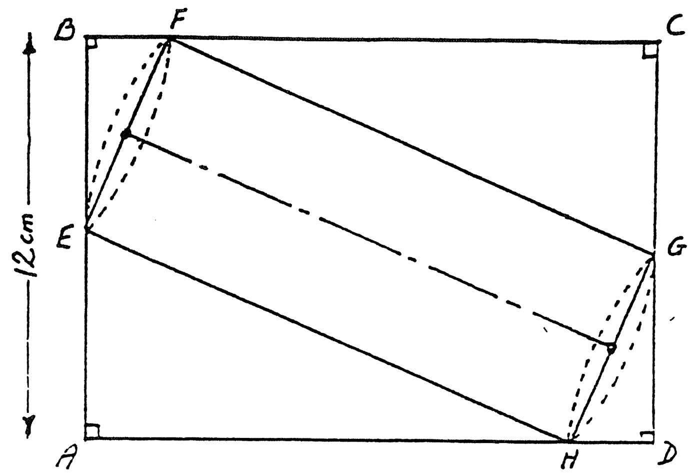
3. The n th term of the series { $3 + \frac{4}{3} + \frac{11}{18} +$ ············· } is given by $A\left(\frac{1}{3}\right)^n + B\left(\frac{1}{2}\right)^n$. Find:

   (a) the values of A and B.

   (b) the limiting sum of the series.

(continued next page)

4. A tilted right cylinder EFGH of radius 4 cm. and height 15 cm. neatly fits into a rectangular box which is 12 cm. high as shown in the cross–section given. Find $\angle$ AHE, the angle formed by the *side* EH of the cylinder and the bottom of the rectangular box. (COPY DIAGRAM INTO YOUR ANSWER SHEET)



SECTION G (Start a new page)

1. (a) Neatly sketch the curve $y = \dfrac{x+1}{x^2 - x}$, clearly showing all asymptotes and x,y intercepts.

(b) With the aid of the equation $x^2y - xy = x + 1$ or otherwise, find the RANGE of the function.

2. (a) Two points $P(2p,p^2)$ and $Q(2q,q^2)$ lie on the parabola $x^2 = 4y$.

(i) Derive the equation of the chord PQ.
(ii) The chord PQ passes through the focus S. Show that $pq = -1$.
(iii) Prove that the length of the chord PQ is $\left(p + \dfrac{1}{p}\right)^2$.
(iv) Find the minimum length of the chord PQ.

3. Given that $a^2 + b^2 < c^2$, show that the equation $a\cos\theta + b\sin\theta = c$ has no real roots.

END of PAPER

YEAR 11 PRELIM. 1998

SOLUTIONS

SECTION A

1 (a) $x^2 - 8x = x(x-8)$

(b) $6x^3 + 18x^2 - 24x$

$= 6x(x^2 + 3x - 4) = 6x(x+4)(x-1)$

(c) $x^2 - y^2 - 2x - 2y$

$= (x-y)(x+y) - 2(x+y) = (x+y)(x-y-2)$

2 $\ln\sqrt{6} = \frac{1}{2}(\ln 2 + \ln 3)$

$= \frac{1}{2}(x+y)$

3 $y = 2\pi + x^2$ $\qquad \frac{dy}{dx} = 2x$

4 $P(2) = 0 \quad \therefore (2)^3 - k(2) + 5 = 0$

$\therefore k = \frac{13}{2}$

5 $1 - \tan^2 x = 1 - \tan^2 30^\circ$

$= 1 - \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{2}{3}$

6 $\sqrt{20} + \sqrt{80} = m\sqrt{5}$

$\therefore 2\sqrt{5} + 4\sqrt{5} = 6\sqrt{5} \quad \therefore m = 6$

7 $1, 2x^2, 4x^4, \ldots \qquad T_n = (2x^2)^{n-1}$

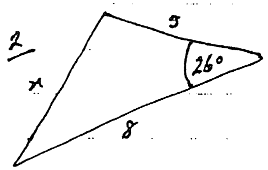
SECTION B

1 $9^{2x+1} = 27$

$\therefore 3^{4x+2} = 3^3$

$\Rightarrow 4x = 1$

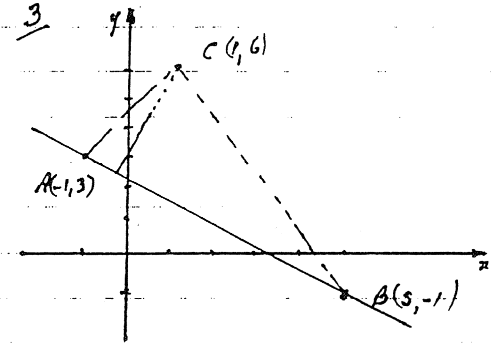
$x = \frac{1}{4}$

2 

$x^2 = 8^2 + 5^2 - 2\cdot8\cdot5\cdot\cos 26^\circ$

$= 64 + 25 - 80\cos 26^\circ$

$\therefore x = 4.13$ (2 dec. pl.)

3 

(a) $y - 3 = \frac{-4}{6}(x+1)$

$\therefore 3y - 9 = -2x - 2$

$\therefore 2x + 3y - 7 = 0$

(b) $d = \left|\frac{2\cdot1 + 3\cdot6 + -7}{\sqrt{2^2+3^2}}\right|$

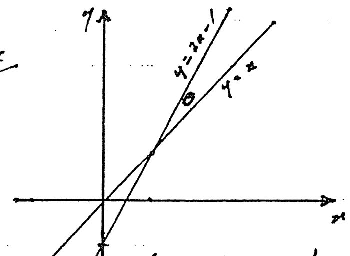
$= \frac{13}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}}$

$\therefore d = \sqrt{13}$

(c) $A = \frac{1}{2}\cdot\sqrt{36+16} \times \sqrt{13}$

$\therefore \quad = \frac{1}{2} \times 2\sqrt{13} \times \sqrt{13}$

$\therefore A = 13\ u^2$

4 

Let $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Let $m_1 = 1, \ m_2 = 2$

$\therefore \tan\theta = \left|\frac{1-2}{1+1.2}\right|$

$= \left|-\frac{1}{3}\right|$

$\therefore \theta \doteqdot 18^\circ 26'$

SECTION C

1. $a^{-1} + 2b^{-1}$
$= 2 + 2\cdot 3$
$= \underline{8}$

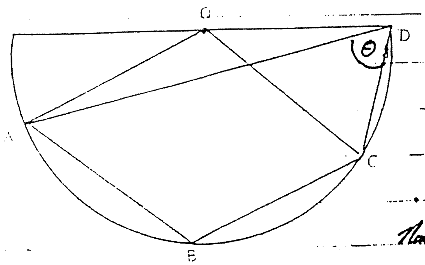
2. $\lim_{x\to 0}\left(\frac{\sin 2x}{5x}\right) = \lim_{x\to 0}\left(\frac{2}{5}\frac{\sin 2x}{2x}\right)$
$= \underline{\frac{2}{5}}$

3. (a) $y = \tan 4x$
$\frac{dy}{dx} = \underline{4\sec^2 4x}$

(b) $y = \frac{2x}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}} = 2\sqrt{x}$
$\therefore \frac{dy}{dx} = \underline{\frac{1}{\sqrt{x}}}$

(c) $y = (2x+1)^3(2x-1)^3$
$= (4x^2-1)^3$
$\therefore \frac{dy}{dx} = 3(4x^2-1)^2 \times 8x$
$= \underline{24x(4x^2-1)^2}$

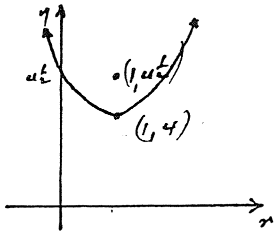
4.



Let $A\hat{D}C = \theta$
$A\hat{O}C = 2\theta$ ($A\hat{O}C$ at centre is twice $A\hat{D}C$ at circumference standing on same arc ABC)
$\therefore A\hat{B}C = 2\theta$ (Opposite angles of parm OABC)
Now $A\hat{B}C + A\hat{D}C = 180°$ (Opposite angles of cyclic quad. ABCD supplementary)
$\therefore 3\theta = 180°$ $\therefore \underline{\theta = 60°}$

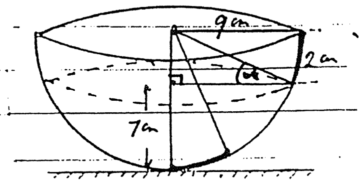
5. $2y = x^2 - 2x + 9$
$= (x-1)^2 + 8$
$\therefore (x-1)^2 = 2(y-4)$
i.e. $4a = 2$ $\therefore a = \frac{1}{2}$

Vertex at $(1, 4)$
and
Focus at $(1, 4\frac{1}{2})$



SECTION D

1.



$\sin\alpha = \frac{2}{9}$
$\therefore \alpha = 12°50'$
$= 0.2241^c$
$\therefore$ arc length for bowl to roll is $l = 9 \times 0.2241$
$\therefore \underline{l = 2.02 \text{ cm. (2 dec. pl.)}}$

2. (a) ACE ...... $\therefore 7!$ $= \underline{5040}$

(b) FACT ..... $\therefore 4!\,6!$ $= \underline{17280}$

(c) A × E × I × O × U $\therefore 5!\,4!$ $= \underline{2880}$

3. (a) $\sin^2\theta = \sin\theta$
$\therefore \sin^2\theta - \sin\theta = 0$
$\therefore \sin\theta(\sin\theta - 1) = 0$
i.e. $\sin\theta = 0$ $\sin\theta = 1$
$\therefore \underline{\theta = 0°, 180°, 90°}$

(b) $\cos 4\theta = \cos 2\theta$
Let $U = 2\theta$
$\therefore \cos 2U = \cos U$
$\therefore \cos^2 U - \sin^2 U = \cos U$
$\therefore 2\cos^2 2\theta - \cos 2\theta - 1 = 0$
$\therefore \cos 2\theta = \frac{1 \pm \sqrt{1+8}}{4}$
$\therefore \cos 2\theta = 1, -\frac{1}{2}$
$\therefore 2\theta = 120°, 240°, 0°, 360°$
$\therefore \underline{\theta = 60°, 120°, 0°, 180°}$

SECTION E

1. $2^{\log_4(x+3)} = 2^3$

$\therefore \log_4(x+3) = 3$

$\therefore x+3 = 4^3$

$\therefore \underline{x = 61}$

2. LHS $= \dfrac{2\cot\theta}{1+\cot^2\theta}$ RHS $= \sin 2\theta$

LHS $= \dfrac{2\cos\theta}{\sin\theta} \div \operatorname{cosec}^2\theta$

$= \dfrac{2\cos\theta}{\sin\theta} \times \sin^2\theta$

$= 2\cos\theta\sin\theta$

$= \sin 2\theta =$ RHS.

3.

$a = 15$ $S_n = \frac{n}{2}[2a+(n-1)d] = 55$

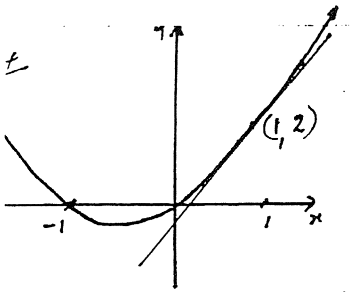
$d = -2$ $\therefore n[30+(n-1)\cdot -2] = 110$

$\therefore n(32-2n) = 110$

$\therefore n^2 - 16n + 55 = 0$

$\underline{n = 5, 11}$

4.



Let equation of tangent be:

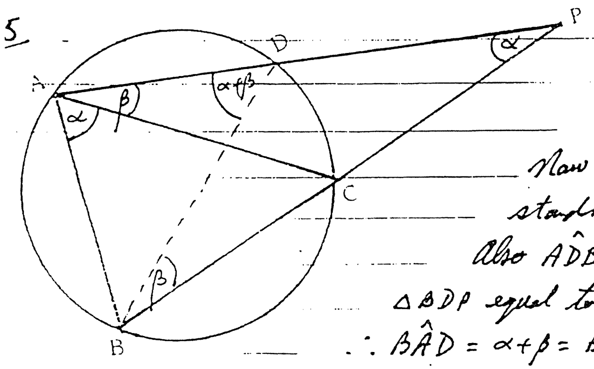
$y - y_1 = m(x - x_1)$ $m = \left.\frac{dy}{dx}\right)_{x=1}$

$\therefore y - 2 = 3(x-1)$ $= 2x+1)_{x=1}$

$y - 2 = 3x - 3$ $\therefore m = 3$

$\therefore \underline{3x - y - 1 = 0}$

5.



Join BD, and let $A\hat{P}C = B\hat{A}C = \alpha$ (given)

Let $C\hat{B}D = \beta$.

Now $D\hat{A}C = \beta$ (Angles at circumference standing on same arc CD)

Also $A\hat{D}B = \alpha+\beta$ (exterior angle of ΔBDP equal to sum of interior opposites)

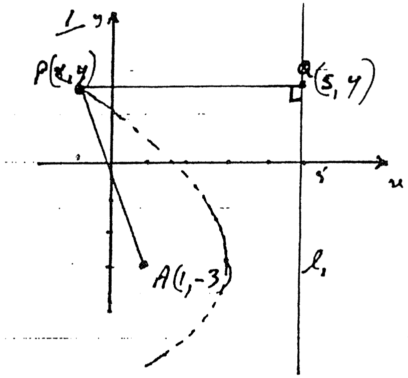
$\therefore B\hat{A}D = \alpha+\beta = B\hat{D}A$

$\therefore \Delta ABD$ is isosceles.

$\therefore AB = BD$ (equal sides opposite equal angles in isosceles ΔABD)

SECTION F

1.



Since $PA = PQ$

$\therefore (x-1)^2 + (y+3)^2 = (x-5)^2$

$\therefore x^2 - 2x + 1 + y^2 + 6y + 9 = x^2 - 10x + 25$

$\therefore (y+3)^2 = -8x + 24$

$= -8(x-3)$

$\therefore \underline{(y+3)^2 = -8(x-3)}$

2. $x^2 - x + \dfrac{6}{x^2-x} = 7$

Let $U = x^2 - x$

$\therefore U + \dfrac{6}{U} = 7$

$\therefore U^2 - 7U + 6 = 0$

$\therefore (U-6)(U-1) = 0$

$\therefore x^2 - x - 6 = 0$ or $x^2 - x - 1 = 0$

$x = \dfrac{1 \pm \sqrt{1+24}}{2}$ or $x = \dfrac{1 \pm \sqrt{1+4}}{2}$

$\therefore x = \dfrac{1 \pm 5}{2}$ $x = \dfrac{1 \pm \sqrt{5}}{2}$

$\therefore \underline{x = 3, -2, \dfrac{1 \pm \sqrt{5}}{2}}$

3. $n = 1$, $n = 2$

$\dfrac{A}{3} + \dfrac{B}{2} = 3$ —(1)

$\dfrac{A}{9} + \dfrac{B}{4} = \dfrac{4}{3}$ —(2)

$\therefore A = 3, B = 4$

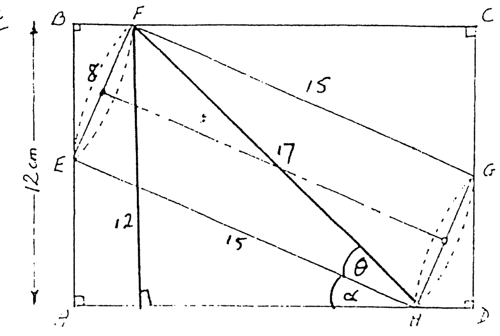
[Test for $n=3$: $3 \times \frac{1}{27} + 4 \times \frac{1}{8} = \frac{11}{18}$ ✓]

Now $S = 3\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots\right) + 4\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)$

$\therefore S_\infty = \left(3 \times \dfrac{\frac{1}{3}}{1-\frac{1}{3}}\right) + \left(4 \times \dfrac{\frac{1}{2}}{1-\frac{1}{2}}\right)$

$= \underline{5\tfrac{1}{2}}$

4.



$\sin\theta = \frac{8}{17}$ $\therefore \theta = 28°4'$

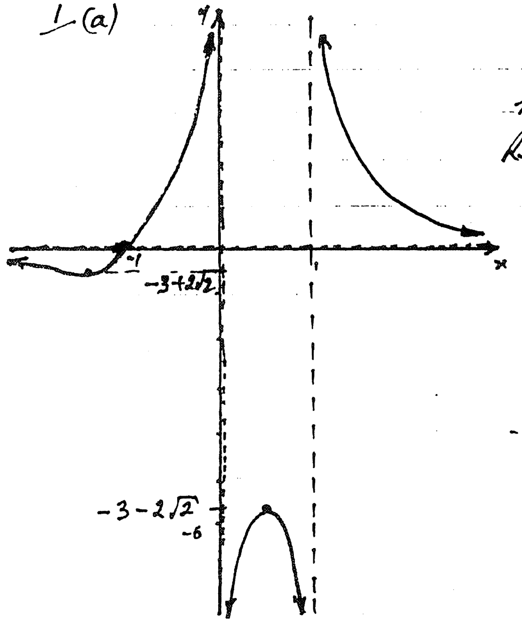
Now $\sin(\theta+\alpha) = \frac{12}{17}$

$\therefore \theta + \alpha = 44°54'$

$\therefore \underline{\alpha = 16°50'}$

SECTION G

1 (a)



To find RANGE, we solve

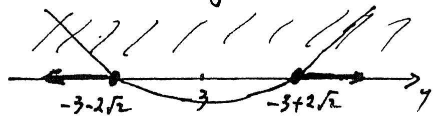
$x^2y - xy - x - 1 = 0$ for y:

Now $(y)x^2 - (y+1)x - 1 = 0$

Require $\Delta \geq 0$ ie; $(y+1)^2 + 4y \geq 0$

ie; $y^2 + 6y + 1 \geq 0$

ie; $(y+3)^2 \geq 8$



∴ Required RANGE is

$y \leq -3-2\sqrt{2}$, $y \geq -3+2\sqrt{2}$

2 (a)(i) $P(2p, p^2)$ $Q(2q, q^2)$

$y - p^2 = \frac{q^2-p^2}{2(q-p)}(x-2p)$

$= \frac{p+q}{2}(x-2p)$

$\therefore y - \left(\frac{p+q}{2}\right)x + pq = 0$

(ii) at $S(0,1)$ $pq + 1 = 0$

$\therefore pq = -1$

(iii) $PQ^2 = (q^2-p^2)^2 + (2q-2p)^2$

$= (q+p)^2(q-p)^2 + 4(q-p)^2$

$= (q-p)^2\left[(q+p)^2 + 4\right]$

$= \left(\frac{1}{p}+p\right)^2\left[\left(-\frac{1}{p}+p\right)^2 + 4\right]$

$= \left(\frac{1}{p}+p\right)^2\left(\frac{1}{p}+p\right)^2$

$\therefore PQ = \left(p + \frac{1}{p}\right)^2$ ※

(iv) We require $\frac{d\,PQ}{dp} = 0$

Now $\frac{d\,PQ}{dp} = 2\left(p+\frac{1}{p}\right)\times\left(1-\frac{1}{p^2}\right)$

$= \frac{2}{p^2}(p^4-1) = 0$

$\Rightarrow p = 1$ $\frac{d^2PQ}{dp^2} > 0$ ∴ MIN.

When $p = 1$, $\left(p + \frac{1}{p}\right) = 2$

∴ Minimum length of PQ is 4

3. Write $a\cos\theta + b\sin\theta = c$

as $\cos(\theta - \alpha) = \frac{c}{\sqrt{a^2+b^2}}$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

Since $a^2 + b^2 < c^2$

$\therefore \cos(\theta - \alpha) > 1$

∴ no real solution.