JAMES RUSE AGRICULTURAL HIGH SCHOOL

YEAR 11 PRELIMINARY EXAM

1999

MATHEMATICS

The exam is divided into two sections; A and B

2 Unit students :

The 2 Unit students have 90 minutes to complete section A. After 90 minutes the 2 Unit students must leave the hall (not any earlier).

3 Unit students :

The 3 Unit students have 150 minutes to complete both sections A and B.

INSTRUCTIONS:

1. Start each new question on a separate sheet of paper
2. Show all necessary working out and formulae
3. Approved calculators may be used
4. Your student number must be on every page
5. Marks may be deducted for untidy work
6. Every question in both sections is worth 15 marks.
7. The answers to the questions are to be returned in separate bundles.

SECTION A

QUESTION ONE

a) Factorise $8 - 2x - 3x^2$

b) Solve: i) $|x - 2| = 6$

ii) $2^{2p} \cdot 3^p = 144$

c) i) Express the following equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$

$$x^2 + y^2 - 4x + 6y + 3 = 0$$

ii) Write down the coordinates of the circle's centre and the length of its radius.

d) Simplify $(3 + \sqrt{2})^2 - (3 - \sqrt{2})^2$

e) Find the area of the sector, which subtends an angle of 112^0 at the centre of a circle with radius 5cm.

QUESTION TWO

a) Differentiate:
i) $y = e^{6-2x}$
ii) $y = x.\tan x$
iii) $y = \ln(5x + 1)$

b) Write down the domain and range for the following functions:
i) $y = 3^x$
ii) $y = \sqrt{x+3}$

c) A particle is traveling such that its displacement in metres at any time t seconds is given by $x = \sin(\frac{\pi}{2}t)$. Given that the particle starts from rest at the origin find:

i) the direction the particle first moves in
ii) when it next comes to rest
iii) how far the particle travels in the first three seconds.

QUESTION THREE

a) Find the equation of the tangent to the curve $y = x^2 - 2x - 4$ at the point where $x = 1$.

b) For the parabola $y = x^2 - 2x - 4$, write down the:

i) Coordinates of the vertex

ii) Coordinates of the focus

iii) equation of the directrix

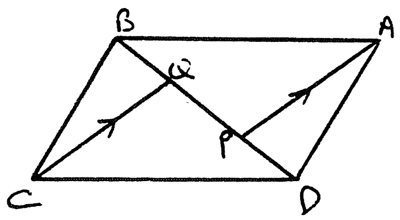
c) A surveyor travels from A, on a bearing of 130^0, to a point B 8km from A. She then travels on a bearing of 030^0 to a point C, due East of A. Find the distance AC in kilometres correct to two decimal places.

d) S_n denotes the sum of the first n terms of a Geometric series. If $s_n = \dfrac{a^n - b^n}{b^{n-1}}$, find the first term and the common ratio of the series in terms of a and b.

QUESTION FOUR

a) Find the set of values of k for which the roots of $2x^2 + kx + 3k - 10 = 0$ are real.

b) ABCD is a parallelogram
AP//CQ
Prove that Δ ADP $\equiv$ Δ CBQ.



c) A variable cylinder of radius r units and height h units is inscribed in a fixed cone, radius a units and height b units.

i) Prove that $\dfrac{b}{a} = \dfrac{h}{a-r}$

ii) Express the volume of the cylinder as a function of r.

iii) Prove that the maximum volume of the cylinder is $\dfrac{4}{9}$ of the cone's volume.

END OF SECTION A

SECTION B

QUESTION FIVE

a). When $P(x) = x^3 + 2x^2 - kx + 5$ is divided by $(x + 3)$ the remainder is 1, find the value of k.

b) Sketch the graph of $y = x(x - 1)^2$ without the use of calculus.

c). Prove that $\dfrac{1}{\tan A + \cot B} + \dfrac{1}{\cot A + \tan B} = \dfrac{\sin(A+B)}{\cos(A-B)}$

d) Find the exact value of $\tan 75^0$

e) Find $\lim\limits_{x \to -3} \dfrac{2x^3 + 54}{x+3}$

QUESTION SIX

a) If $f(x) = x^4 + bx^2 + cx + d$, find the values of b, c and d such that f(x) is an even function, $f(0) = 1$ and $f(1) = 0$.

b) A meeting room contains a round table surrounded by 10 chairs. These chairs are indistinguishable and equally spaced around the table. Find the number of different arrangements possible if:

i) there are no restrictions
ii) 3 particular people must sit together
iii) Person A must not sit next to person B.

c) Solve $2\sin\theta - \cos\theta = 1$ for $0 \le \theta \le \pi$ using the t- results.

d) The lengths of the sides of a scalene triangle are in an arithmetic sequence. If the largest angle in such a triangle is 120^0, show that the smallest angle is $21^0 47'$.

1999 Preliminary Exam

SECTION A

Q1 15

a) $8-2x-3x^2$

$\times\ -24$
$+\ -2$

$= 8-6x+4x-3x^2$

$= 2(4-3x)+x(4-3x)$

$=(2+x)(4-3x)$

b) i) $|x-2|=6$

$x-2=6$ or $-x+2=6$

$x=8$ or $x=-4$

ii) $2^{2p}.3^{p}=144$

$2^{2p}.3^{p}=2^4.3^2$

$\therefore p=2$

c) i) $x^2-4x+y^2+6y=-3$

$(x-2)^2+(y+3)^2=-3+4+9$

$(x-2)^2+(y+3)^2=10$

ii) Centre is $(2,-3)$

radius is $\sqrt{10}$ units

d) $(3+\sqrt{2}+3-\sqrt{2})(3+\sqrt{2}-3+\sqrt{2})$

$= 6\times 2\sqrt{2}$

$=12\sqrt{2}$

e) $112^\circ = \frac{28\pi}{45}$

$A = 25\times\frac{28\pi}{45} = 48.9\,cm^2$ $\quad \frac{140\pi}{9}\,cm^2$

Q2. 15

a) i) $y=e^{6-2x}$

$\frac{dy}{dx}=-2e^{6-2x}$

ii) $y=x.\tan x$

$\frac{dy}{dx}=x\sec^2 x+\tan x$

iii) $y=\ln(5x+1)$

$\frac{dy}{dx}=\frac{5}{5x+1}$

b) i) $y=3^x$

Domain is all real x

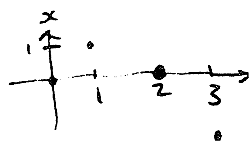
Range is $y>0$

ii) $y=\sqrt{x+3}$

Domain is $x\geqslant -3$

Range is $y\geqslant 0$

c) i) $x=\sin(\frac{\pi}{2}t)$



$v=\frac{\pi}{2}\cos(\pi/2\, t)$

when $t=0, v=\pi/2$

$\therefore$ first moves to the right

ii) at rest when $v=0$,

ie $\cos(\pi/2\, t)=0$

$\pi/2\, t=\pi/2, \frac{3\pi}{2}$

$t=1, 3$

$\therefore$ next comes to rest when $t=1$.

iii) 3 units.

Q3 (15)

a) $y = x^2 - 2x - 4$

$\frac{dy}{dx} = 2x - 2$

when $x = 1$, $m = 0$

$y = -5$

$\therefore$ eqn. of tangent is $y = -5$

b) i) $y + 4 = x^2 - 2x$

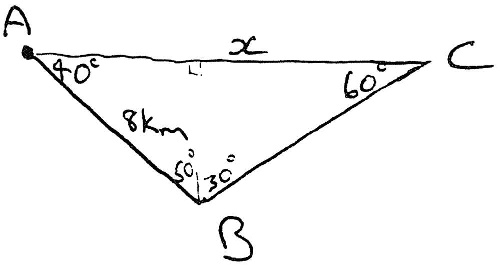
$y + 5 = (x-1)^2$

Vertex is $(1, -5)$

ii) focus $(1, -4\frac{3}{4})$

iii) Directrix $y = -5\frac{1}{4}$

c)



$\frac{x}{\sin 80°} = \frac{8}{\sin 60°}$

$x = \frac{8 \times \sin 80°}{\sin 60°}$

$= 9.097264341$

$\doteqdot 9.10$ km

d) $S_n = \frac{a^n - b^n}{b^{n-1}}$

$S_1 = a - b$ $\therefore$ first term is $(a-b)$

$S_2 = \frac{a^2 - b^2}{b}$

$S_2 - S_1 = T_2$

$\therefore T_2 = \frac{a^2 - b^2}{b} - a - b$

$= \frac{a^2 - ab}{b}$

ratio is $\frac{a(a-b)}{b} \times \frac{1}{a-b} = \frac{a}{b}$

Q4 (15)

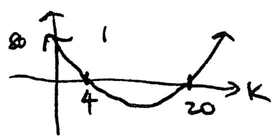
a) $2x^2 + Kx + 3K - 10 = 0$

$b^2 - 4ac \geq 0$

$K^2 - 8(3K - 10) \geq 0$

$K^2 - 24K + 80 \geq 0$

$(K - 20)(K - 4) \geq 0$



$K \leq 4$, $K \geq 20$

b)

$\angle APQ = \angle CQP$ (alternate angles; $\wedge$ $CQ \parallel$)

$\therefore 180° - \angle APQ = 180° - \angle CQP$

$\therefore \angle APD = \angle BQC$ (adjacent angles on a straight line)

$\therefore$ In $\triangle$s ADP, CBQ

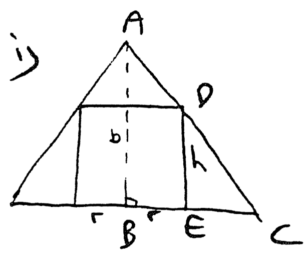
$\therefore \angle APD = \angle BQC$ (proven above)

$AD = BC$ (opp. sides of parallelogram ABC)

$\angle ADP = \angle CBQ$ (alternate angles; $CB \parallel DA$)

$\therefore \triangle ADP \equiv \triangle CBQ$ (A.A.S.)

c) i)



$\triangle ABC \parallel\parallel \triangle DEC$ (equiangular)

$\frac{a}{a-r} = \frac{b}{h}$ (ratio of corresponding sides in similar triangles)

$\therefore \frac{b}{a} = \frac{h}{a-r}$

ii) $V = \pi r^2 h$

$= \pi r^2 \cdot \frac{b}{a} \cdot (a - r)$

$= \frac{\pi b}{a}(ar^2 - r^3)$

iii) Volume of cone $= \frac{1}{3}\pi a^2 b$

$$\frac{dV}{dr} = \frac{\pi b}{a}(2ar - 3r^2)$$

$\frac{dV}{dr} = 0$ when $2ar = 3r^2$

ie $r = 0$, or $r = \frac{2a}{3}$

when $r=0$, there's no cylinder $\therefore$ only test when $r = \frac{2a}{3}$ using the first derivative as the fn is cts.

r	$< \frac{2a}{3}$ ($\frac{a}{2}$)	$= \frac{2a}{3}$	$> \frac{2a}{3}$ (a)
$\frac{dV}{dr}$	$\frac{\pi ab}{4}$	0	$-\pi ba$
slope	/	—	\

$\therefore$ rel. max. at $r = \frac{2a}{3}$, since there's only 1 stat. pt in the domain $0 < r < a$, the rel. max. is the absolute maximum.

$\therefore$ max. Volume of cylinder is:

$$V = \pi \frac{b}{a}\left(a\frac{4a^2}{9} - \frac{8a^3}{27}\right)$$
$$= b\pi\left(\frac{12a^2 - 8a^2}{27}\right)$$
$$= \frac{4a^2 b\pi}{27}$$

which is $4/9$ of the cone's volume

Q.E.D.

(END OF SECTION A)

Q5 (15 marks)

a) $P(x) = x^3 + 2x^2 - kx + 5$

$P(-3) = -27 + 18 + 3k + 5 = 1$

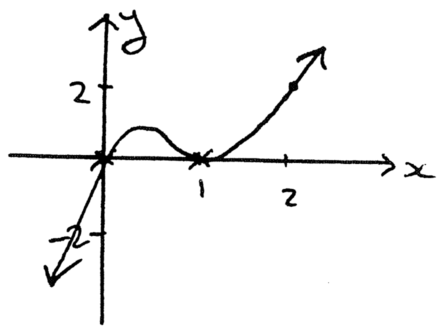
$-4 + 3k = 1$

$3k = 5$

$\therefore k = \frac{5}{3}$

b) $y = x(x-1)^2$

$(0,0)$ $(1,0)$



c) $\frac{1}{\tan A + \cot B} + \frac{1}{\cot A + \tan B} = \frac{\sin(A+B)}{\cos(A-B)}$

L.H.S. $= \frac{1}{\tan A + \cot B} + \frac{1}{\cot A + \tan B}$

$$= \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos B}{\sin B}} + \frac{1}{\frac{\cos A}{\sin A} + \frac{\sin B}{\cos B}}$$

$$= \frac{\sin B \cos A}{\sin A \sin B + \cos A \cos B} + \frac{\sin A \cos B}{\cos A \cos B + \sin A}$$

$$= \frac{\sin A \cos B + \sin B \cos A}{\sin A \sin B + \cos A \cos B}$$

$$= \frac{\sin(A+B)}{\cos(A-B)}$$

= R.H.S.

Q.E.D.

d) $\tan 75^\circ = \tan(30^\circ + 45^\circ) = \frac{\tan 30^\circ + \tan 45}{1 - \tan 30 . \tan 45}$

$$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{1 + \sqrt{3}}{\sqrt{3} - 1}$$

e) $\lim_{x\to -3} \frac{2x^3+54}{x+3}$

$= \lim_{x\to -3} \frac{2(x+3)(x^2-3x+9)}{x+3}$

$= \lim_{x\to -3} 2(x^2-3x+9)$

$= 2\times(9+9+9)$

$= 54$

Q6. 15 marks

a) $f(x) = x^4 + bx^2 + cx + d$

$f(0) = 1 \quad \therefore \underline{d=1}$

$f(1) = 0$

even $\therefore$ symmetrical about y-axis

$0 = 2 + b + c$

$\therefore b + c = -2 \quad \ldots \quad ①$

$f(-x) = x^4 + bx^2 - cx + d$

$f(-x) = f(x)$ (even)

$\therefore \underline{c=0}$

$\therefore \underline{b=-2}$ (from ①)

b) i) $9! = 362880$

ii) $1\times 3! \times 7! = 30\,240$

iii) $7\times 8! = 282\,240$

c) $2\sin\theta - \cos\theta = 1$

$2\left(\frac{2t}{1+t^2}\right) - \left(\frac{1-t^2}{1+t^2}\right) = 1$

$4t - 1 + t^2 = t^2 + 1$

$4t = 2$

$\therefore t = 1/2$

now $\tan\frac{\theta}{2} = \frac{1}{2}$

$\frac{\theta}{2} = 0.4636^c$

$\therefore \theta = 0.9273^c$

Test $\theta = \pi$,

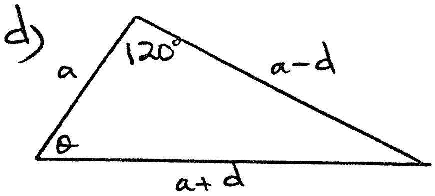
$2\sin\pi - \cos\pi = 1$

$0 - -1 = 1$

$1 = 1$ true

$\therefore$ solns. are $\theta = \pi$ and $0.927\dot{3}$

d)



$(a+d)^2 = (a-d)^2 + a^2 - 2a(a-d)\cos 120°$

$\cancel{a^2} + 2ad + \cancel{d^2} = \cancel{a^2} - 2ad + \cancel{d^2} + a^2 + (2a^2 - 2ad)\times\frac{1}{2}$

$0 = -4ad + a^2 + a^2 - ad$

$5ad = 2a^2 \qquad d \neq 0$

$a = \frac{5d}{2} \qquad a \neq 0$

$\frac{a-d}{\sin\theta} = \frac{a+d}{\sin 120°} \qquad \left(\sin 120° = \frac{\sqrt{3}}{2}\right)$

$\sin\theta = \frac{\sqrt{3}}{2}\cdot\frac{a-d}{a+d}$

$= \frac{\sqrt{3}}{2}\cdot\frac{2\frac{1}{2}d - d}{2\frac{1}{2}d + d}$

$= \frac{\sqrt{3}}{2}\cdot\frac{1\frac{1}{2}d}{3\frac{1}{2}d}$

$\sin\theta = 0.371153744$

$\theta = 21°47'$

QED.

Q7 15 marks

a) $\frac{x-2}{x-4} \geq 2 \qquad x \neq 4$

$\frac{x-2-2(x-4)}{x-4} \geq 0$

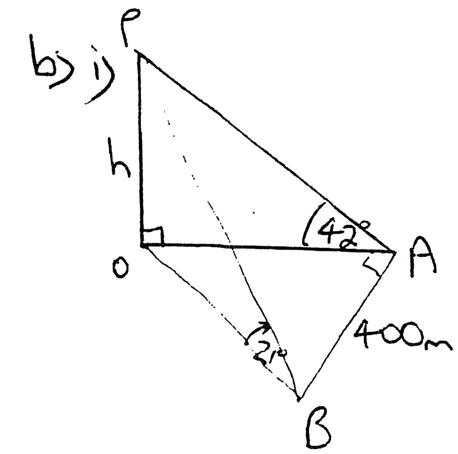
$\frac{-x+6}{x-4} \geq 0$

	$x<4$	$4<x<6$	$x>6$
	$-$	$+$	$-$
$6-x$	$+$	$+$	$-$
$x-4$	$-$	$+$	$+$

$\therefore \underline{4 < x \leq 6}$

Q7 Continued

b) i)



In ΔPOB,

$$\tan 21° = \frac{h}{OB}$$

$$\therefore OB = \frac{h}{\tan 21°}$$

$$= h\cot 21°$$

In ΔPOA,

$$\tan 42° = \frac{h}{OA}$$

$$OA = \frac{h}{\tan 42°}$$

$$OA = h\cot 42°$$

In ΔOAB

$$OB^2 = OA^2 + AB^2$$

$$h^2\cot^2 21° = h^2\cot^2 42° + 400^2$$

$$h^2(\cot^2 21° - \cot^2 42°) = 400^2$$

$$h^2 = \frac{400^2}{(\cot^2 21° - \cot^2 42°)}$$

$$h = \frac{400}{\sqrt{\cot^2 21° - \cot^2 42°}}$$

QED

ii) $$h = \frac{400}{\sqrt{6.786489 - 1.23346}}$$

$$= \frac{400}{\sqrt{5.553}}$$

$$= 169.744$$

$\doteqdot 170$m (nearest metre)

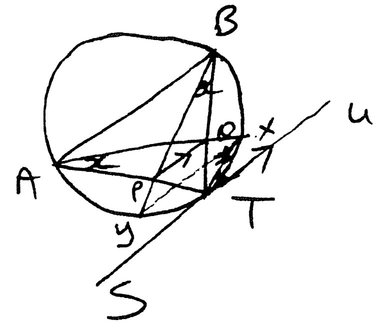
c) i) $Q\hat{T}U = B\hat{A}T$ (angle in the alternate segment)

$Q\hat{T}U = P\hat{Q}T$ (alternate angles; $PQ \parallel TU$)

$\therefore B\hat{A}T = P\hat{Q}T$ (both equal to $Q\hat{T}U$)

$\therefore ABQP$ is a cyclic quad. (exterior angle equals the interior opposite angle.)

ii)



let $P\hat{B}Q = x$

$\therefore P\hat{A}Q = x$ (angles in same segment, PABQ is cyclic quad).

$\therefore X\hat{T}U = x$ (angle between chord XT and ~~tangent~~ tangent equals angle in alternate segment)

$Y\hat{B}T = Y\hat{X}T = x$ (angles in same segment).

$Y\hat{X}T = X\hat{T}U$ (both x)

$\therefore YX \parallel STU$ (one pair of alternate angles is equal)