YEAR 11 PRELIMINARY EXAMINATION 2002

MATHEMATICS

Time Allowed – 85 minutes

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

The answers to all questions are to be returned in separate bundles clearly labelled Question 1, Question 2, etc. Each bundle must show your Candidate's Number.

Question 1: (START A NEW PAGE)

(a) Write down the exact value of: **marks**

(i) $\sin 210°$ **1**

(ii) $\tan \frac{\pi}{3}$ **1**

(iii) $\sec 315°$ **1**

(b) Solve the equation $4 \sin^2 A = 3$ for $0° \leq A \leq 360°$ **2**

(c) Find the exact value of x if:

$3 \log_{10} 3 - 2 \log_{10} x = 4$ **3**

(d) Solve for x: $|2x+2| = |x+3|$ **2**

Question 2: (START A NEW PAGE)

(a) Differentiate with respect to x:

(i) $y = \dfrac{x+1}{x-1}$ **2**

(ii) $y = xe^{3x^2+2}$ **2**

(iii) $y = [\ln(2x)]^3$ **2**

(b) Evaluate the following limits.

(i) $\displaystyle\lim_{x \to \infty} \frac{2x^3 + 5x}{4x^3 + x^2 - 3}$ **2**

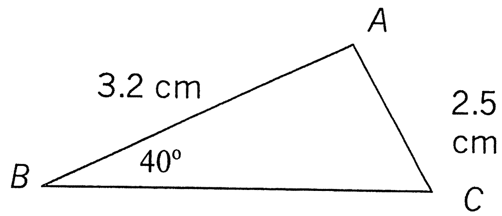
(ii) $\displaystyle\lim_{x \to 5} \frac{\sqrt{x} - \sqrt{5}}{x - 5}$ **2**

Question 3: (START A NEW PAGE)

(a) (i) Find the equation of the normal to the curve $y = 2x + \frac{1}{x}$ at the point $(\frac{1}{2}, 3)$. **3**

(ii) At what point does the normal meet the curve again? **2**

(b) In ΔABC, $AB = 3.2$ cm, $AC = 2.5$ cm and $\angle ABC = 40^\circ$.



Find the value of $\angle ACB$. **3**

(c) Show that $f(x)$ is an even function where:

$$f(x) = \ln(1 + e^x) - \frac{x}{2}$$ **2**

Question 4: (START A NEW PAGE)

(a) Prove the identity: $\dfrac{\cot A}{1 + \cot A} = \dfrac{1}{1 + \tan A}$ **2**

(b) (i) Differentiate the function $y = \dfrac{10x}{e^x}$ **1**

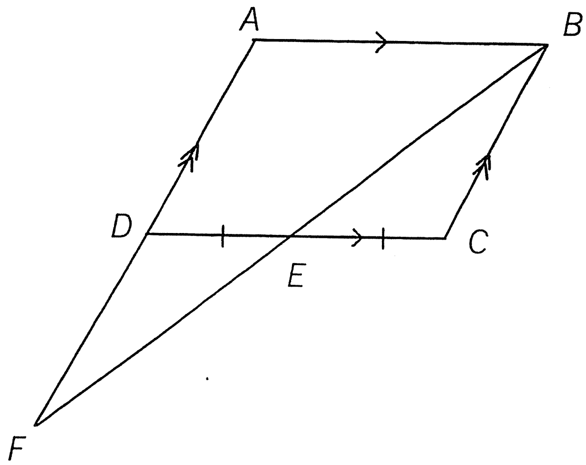
(ii) Find any stationary points and describe their nature **3**

(ii) Find the point of inflexion **2**

(iv) Sketch the curve of $y = \dfrac{10x}{e^x}$ **2**

Question 5: (START A NEW PAGE)

(a) $ABCD$ is a parallelogram. E is the midpoint of DC. AD produced meets BE produced at F.



(i) Copy the diagram and prove that $\Delta DEF \equiv \Delta CEB$. **2**

(ii) Prove that $BDFC$ is a parallelogram. **2**

(b) A hiker leaves his camp C, and walks for 25 km on a bearing of 125° T, to point A. He then changes his course, and walks in a direction of 190° T, for 30 km to point B.

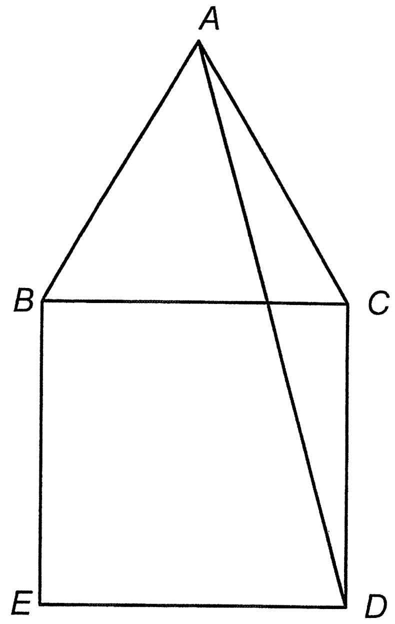
(i) Draw a labelled diagram showing the above information. **1**

(ii) Find the shortest distance between camp C and point B, to 2 dp. **3**

(iii) What is the bearing, to the nearest degree, of camp C from point B? **2**

Question 6: (START A NEW PAGE)

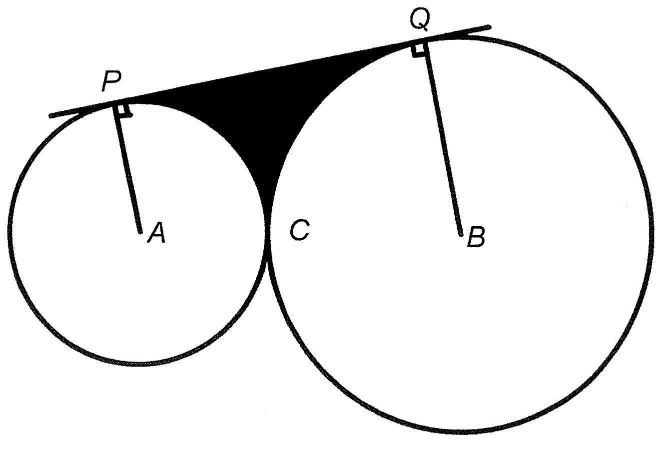
(a) ABC is an equilateral triangle. $BCDE$ is a square.



(i) Copy the diagram and find the size of $\angle ADC$. Give reasons for your answer. **2**

(ii) Show that $\angle BAD = 3\times\angle CAD$ **2**

(b) Two circles, centres A and B, with radii of 2 cm and 3 cm respectively, touch externally at C, so that A, B and C are collinear. PQ is a common tangent.



(i) What is the exact length of PQ? **1**

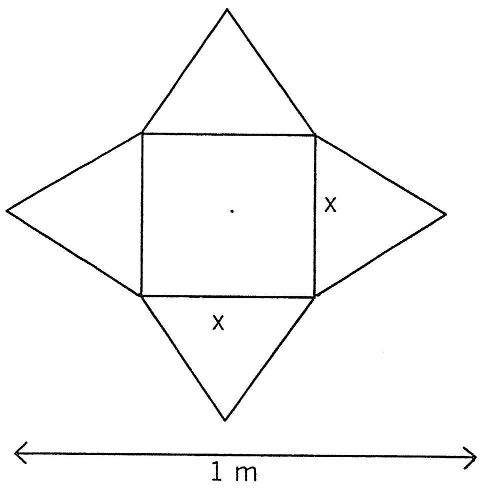
(ii) Find the size of $\angle ABQ$. Give your answer in radians to three decimal places. **1**

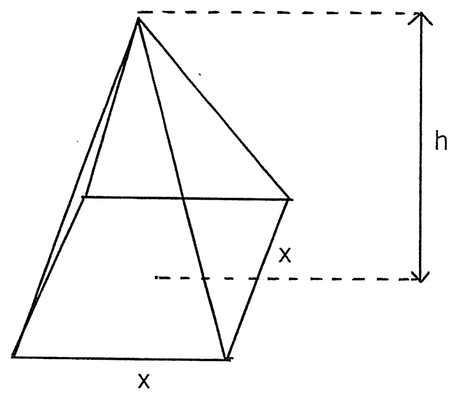
(iii) Calculate the shaded area of the region bounded by tangent PQ, arc PC and arc CQ. Give your answer to one decimal place. **4**

Question 7: (START A NEW PAGE)

(a) Prove that the equation $3x^2 - 2(n - 1)x + n^2 + 1 = 0$ has no real roots for any real value of n. **3**

(b) The net obtained by cutting along the sloping edges of a right square pyramid is such that the distance between the vertices of two opposite triangles is 1m.





$$V = \frac{1}{3}x^2h$$

(i) If the side of the square base is x, show that the volume of the pyramid V, is given by:

$$V = \frac{x^2}{6}\sqrt{1 - 2x}$$ **3**

(ii) Find the length of the side of the square base which gives a maximum volume for the pyramid. **4**

THIS IS THE END OF THE PAPER

Question 1.

(a) (i) $\sin 210 = \sin(180+30)$

$= -\sin 30$

$= -\frac{1}{2}$ (1)

(ii) $\tan \pi/3 = \sqrt{3}$ (1)

(iii) $\sec 315 = \sec(360-45)$

$= \sec 45$

$= \frac{1}{\cos 45}$

$= \sqrt{2}$ (1)

(b)

$4\sin^2 A = 3$

$\sin A = \pm \sqrt{3}/2$ (1)

$A = \underbrace{60^\circ, 120^\circ}_{\frac{1}{2}}, \underbrace{240^\circ, 300^\circ}_{\frac{1}{2}}$ (1)

(c) $3\log_{10} 3 - 2\log_{10} x = 4.$

$\therefore \log_{10} 3^3 - \log_{10} x^2 = 4.$

$\therefore \log_{10}(3^3/x^2) = 4$ (1)

$9/x^2 = 10^4$ (1)

$\therefore x^2 = \frac{9}{10^4}$

$\therefore x = \frac{3}{100}$ as $x > 0$ (1)

(d) $|2x+2| = |x+3|$

for $x \geq -1$

$2x+2 = x+3$

$x = 1$

(1)

for $-3 \leq x \leq -1$

$2x+2 = -(x+3)$

$2x+2 = -x-3$

$3x = -5$

$x = -5/3$

$x = -1\frac{2}{3}$

(1)

Question 2

(a)(i) $y = \dfrac{x+1}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)(1) - (x+1)1}{(x-1)^2} \quad ①$$

$$= \frac{-2}{(x-1)^2} \quad ①$$

(ii) $y = xe^{3x^2+2}$ ①

$$\frac{dy}{dx} = x(6x)e^{3x^2+2} + (1)e^{3x^2+2} \quad ①$$

$$= (6x^2+1)e^{3x^2+2}$$

(iii) $y = [\ln(2x)]^3$

$$\frac{dy}{dx} = 3[\ln(2x)]^2 \times \frac{2}{2x} \quad ① \; ①$$

$$= \frac{3}{x}[\ln(2x)]^2$$

(b)

(i) $\lim_{x\to\infty} \dfrac{2x^3+5x}{4x^3+x^2-3}$

$$= \lim_{x\to\infty} \frac{\frac{2x^3}{x^3} + \frac{5x}{x^3}}{\frac{4x^3}{x^3} + \frac{x^2}{x^3} - \frac{3}{x^3}}$$

$$= \lim_{x\to\infty} \frac{2 + 5/x^2}{4 + \frac{1}{x} - \frac{3}{x^3}} \quad ①$$

$$= \frac{2+0}{4+0-0}$$

$$\frac{1}{2} \quad ①$$

(ii) $\lim_{x\to 5} \dfrac{\sqrt{x}-\sqrt{5}}{x-5}$

$$= \lim_{x\to 5} \frac{\sqrt{x}-\sqrt{5}}{(\sqrt{x}-\sqrt{5})(\sqrt{x}+\sqrt{5})} \quad ①$$

$$= \lim_{x\to 5} \frac{1}{\sqrt{x}+\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}+\sqrt{5}}$$

$$= \frac{1}{2\sqrt{5}} \quad ①$$

Question 3

(a)(i) $y = 2x + \frac{1}{x}$

$\frac{dy}{dx} = 2 - \frac{1}{x^2}$ (1)

at $x = \frac{1}{2}$.

$\frac{dy}{dx} = 2 - 4$

$= -2$ ($\frac{1}{2}$)

grad of tangent at $x = \frac{1}{2}$ $m_T = -2$

$\therefore$ grad of normal $= \frac{1}{2}$. ($\frac{1}{2}$)

Equation of normal

$y - 3 = \frac{1}{2}(x - \frac{1}{2})$ (1)

$y - 3 = \frac{1}{2}x - \frac{1}{4}$

$4y - 12 = 2x - 1$

$\therefore 2x - 4y + 11 = 0$

or $y = \frac{1}{2}x + 2\frac{3}{4}$

(ii) Solve Simultaneously.

$\frac{1}{2}x + 2\frac{3}{4} = 2x + \frac{1}{x}$.

$2x^2 + 11x = 8x^2 + 4$

$0 = 6x^2 - 11x + 4$ (1)

$0 = (2x - 1)(3x - 4)$

$\therefore x = \frac{1}{2}$ and $x = \frac{4}{3}$

$y = 3$ $\quad y = 2 \times \frac{4}{3} + \frac{3}{4}$

$= 3\frac{5}{12}$. (1)

$\therefore$ Normal meets curve again at $(1\frac{1}{3}, 3\frac{5}{12})$

3 (b) $\dfrac{\sin\angle ACB}{3\cdot2} = \dfrac{\sin 40}{2\cdot5}$

$\sin\angle ACB = \dfrac{3\cdot2\times\sin 40}{2\cdot5}$

$= 0.8228$ (1)

$\therefore \angle ACB = 55°22.$ (1)

or $180 - 55°22. = 124°38$ (1)

check obtuse angle: $124°38 + 40° < 180°$

(c) $f(-a) = \ln(1+e^{-a}) - \left(\frac{-a}{2}\right)$ ($\frac{1}{2}$)

$= \ln(1 + {}^1/e^{a}) + {}^a/_2.$

$= \ln\left(\dfrac{e^a+1}{e^a}\right) + {}^a/_2.$

$= \ln(e^a+1) - \ln e^a + {}^a/_2$

$= \ln(e^a+1) - a + {}^a/_2.$

$= \ln(1+e^a) - {}^a/_2.$ (1)

$= f(a)$

$\therefore f(-a) = f(a)$ ($\frac{1}{2}$)

$\therefore$ function is even.

Question 4

(a) $\dfrac{\cot A}{1+\cot A} = \dfrac{1}{1+\tan A}.$

LHS $= \dfrac{\frac{1}{\tan A}}{1+\frac{1}{\tan A}}$ (1)

$= \dfrac{1}{\tan A\left(1+\frac{1}{\tan A}\right)}$

$= \dfrac{1}{\tan A + 1}$ (1)

$=$ RHS.

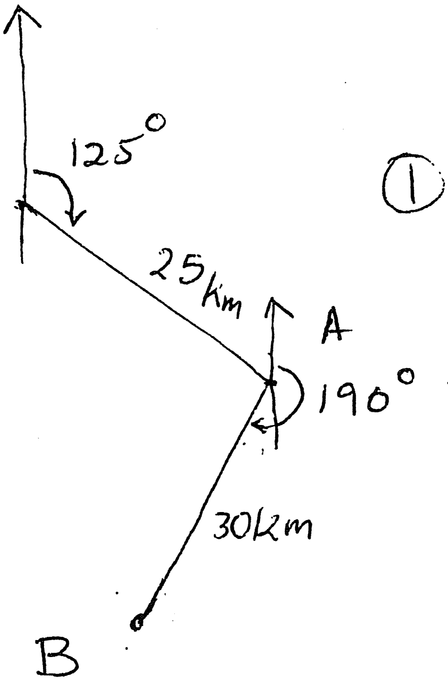
(b)

(i) $\angle CAB = 180^\circ - 55^\circ - 10^\circ$
$= 115^\circ$ (1)

$CB^2 = 25^2 + 30^2 - 2 \times 25 \times 30 \times \cos 115$ (1)
$= 625 + 900 - 1500 \cos 115$
$= 215892$

$CB = 46.46$ km.

Shortest dist $= 46.46$ km (1)



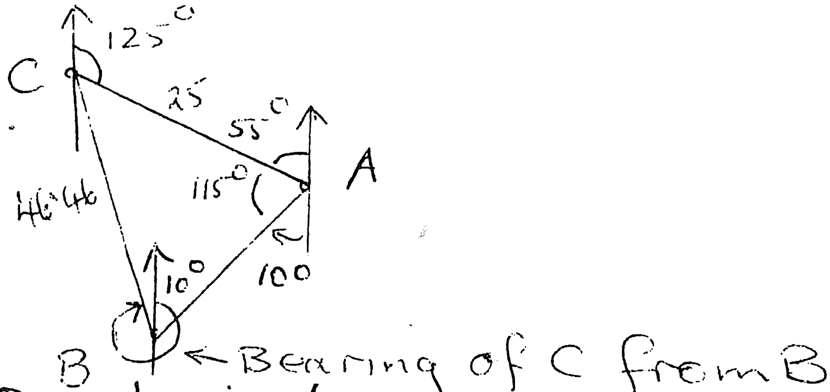
(1)

(ii) $\dfrac{\sin \angle CBA}{25} = \dfrac{\sin 115^\circ}{46.46}$

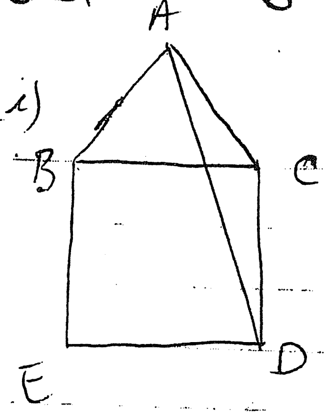
$\therefore \sin \angle CBA = 0.48768$

$\therefore \angle CBA = 29^\circ 11$ (1)

Bearing of C from B
$= 360 - (29^\circ 11 - 10^\circ)$
$= 341^\circ$ (1)



Question 6.



(a)(i)

$AC = BC$ (sides of equilateral triangle)
$CD = BC$ (sides of square).
$\therefore AC = CD$
$\therefore \angle CAD = \angle ADC$ (equal angles opposite equal sides) (1)

$\angle ACB = 60^\circ$ (angle in equilateral triangle)
$\angle BCD = 90^\circ$ (angle in square).

$\therefore \angle CAD + \angle ADC + 60 + 90 = 180^\circ$ (angle sum of triangle)
$\therefore \angle ADC = 15^\circ$

(1)

6(a)(ii) $\angle ADC = \angle CAD$ (from above)

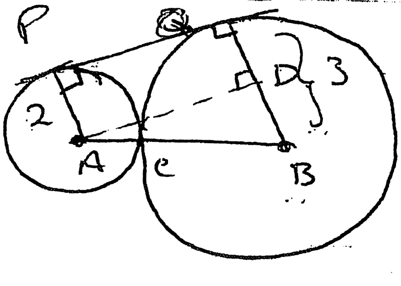
$\therefore \angle CAD = 15°$ (1)

$\angle BAC = 60°$ (angle in equilateral triangle)

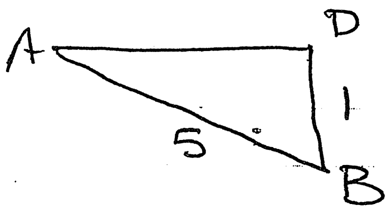
$$\angle BAD = \angle BAC - \angle CAD$$
$$= 60° - 15°$$
$$= 45°$$ (1)

$\therefore \angle BAD = 3 \times \angle CAD$

6(b)



Construct $AD \parallel PQ$



(i)
$$AD^2 = 5^2 - 1^2$$
$$= 25 - 1$$
$$= 24$$
$$AD = \sqrt{24}$$
$$= 2\sqrt{6}$$
$$AD = PQ$$
$\therefore PQ$ is $2\sqrt{6}$ cm. (1)

(ii) $\cos \angle ABD = \frac{1}{5}$

$\therefore \angle ABD = 1.369$

$\therefore \angle ABQ = 1.369$ radians (1)

(iii)
$$\angle PAB = \pi - 1.369$$
$$= 1.772 \text{ radians}$$ ($\frac{1}{2}$)

Area Sector PAC $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2}(2)^2(1.772)$

$= 3.544$ cm^2 (1)

Area Sector QBC $= \frac{1}{2}(3)^2(1.369)$

$= 6.161$ (1)

Area Trapezium PQBA $= \frac{1}{2}\, 2\sqrt{6}\,(2+3)$

$= 12.247$ (1)

Area PCQ $= 12.247 - (3.544 + 6.161)$

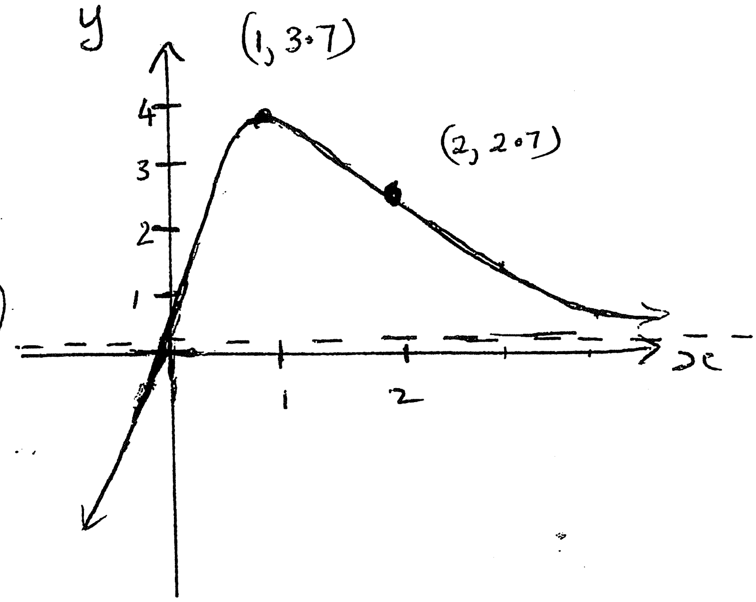
$= 2.542$ ($\frac{1}{2}$)

$\therefore$ Area of region $= 2.5$ cm^2.

(b)

(iv) graph

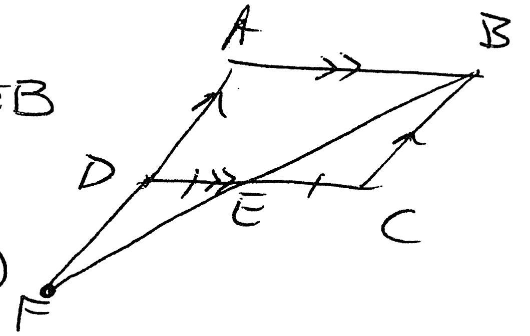
asces/scale $\frac{1}{2}$
graph $\frac{1}{2}$
asymptote $\frac{1}{2}$
plotting max pt $\frac{1}{2}$
& pt of inflexion



(2)

(a) Question 5

(i) In $\triangle DEF$ and $\triangle CEB$



$DE = EC$ (given)

$\angle DEF = \angle CEB$ (vertically opposite)

$\angle FDE = \angle BCE$
(AF ∥ BC alternate angles)

$\therefore \triangle DEF \equiv \triangle CEB$ (ASA)

(2)

(ii)

AD ∥ BC (opposite sides of parallelogram)
$\therefore$ DF ∥ BC (DF is extension of AD)
DF = BC (corresponding sides of congruent triangles)
$\therefore$ BDFC is a parallelogram
(one pair of equal parallel sides)

(2)

Alternative solution could use
Bisecting diagonals
BE = EC (given)
FE = EB (corresponding sides of congruent triangles.)

4(b) (i) $y = \frac{10x}{e^x} = 10xe$

$$\frac{dy}{dx} = -10xe^{-x} + 10e^{-x}$$ 1

$$= 10(1-x)e^{-x}$$

(ii) Stationary point occurs when $\frac{dy}{dx} = 0$.

$\therefore\ 10(1-x)e^{-x} = 0$

$\therefore\ x = 1$

$y = 10e^{-1} \approx 3 \cdot 7$ 1

Test Nature. $\frac{d^2y}{dx^2} = -10(1-x)e^{-x} + (-1)10e^{-x}$

$= 10(x-2)e^{-x}$ 1

when $x = 1$

$\frac{d^2y}{dx^2} = 10(1-2)e^{-1}$

$= 10e^{-1} < 0$

$\therefore\ \frac{d^2y}{dx^2} < 0$ when $x = 1$

$\therefore$ concave down

$\therefore$ local max at $(1, 10e^{-1})$ 1

1 Alternate Test with table

Function is continuous for $0 \le x \le 2$

$\therefore$ local max at $(1, 10e^{-1})$

x	0	1	2
$\frac{dy}{dx}$	10	0	-1.4

+ 0 −

(iii) Point of Inflexion occurs when $\frac{d^2y}{dx^2} = 0$ and has a change in sign.

$\frac{d^2y}{dx^2} = 10(x-2)e^{-x}$ (from above).

$\therefore\ 10(x-2)e^{-x} = 0$

$\therefore\ x = 2$

$y = 20e^{-2} \approx 2 \cdot 7$ 1

Test sign change

x	1	2	3
$\frac{d^2y}{dx^2}$	-3.6	0	0.50

$\therefore$ Point of inflexion occurs at $(2, 20e^{-2})$ 1

(a) $3x^2 - 2(n-1)x + n^2 + 1 = 0$

for no roots $\Delta < 0$

$$\Delta = b^2 - 4ac$$

$$\Delta = [-2(n-1)]^2 - 4(3)(n^2+1) \quad 1$$

$$= 4(n^2 - 2n + 1) - 12n^2 - 12$$

$$= -8n^2 - 8n - 8$$

$$= -8(n^2 + n + 1) \leftarrow \text{Alternatively show this is negative definite ie } \Delta < 0$$

$$= -8\left(n^2 + n + \tfrac{1}{4} - \tfrac{1}{4} + 1\right)$$

$$= -8\left[\left(n + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right] \quad 1$$

as $\left(n+\frac{1}{2}\right)^2 \geq 0$ (perfect square)

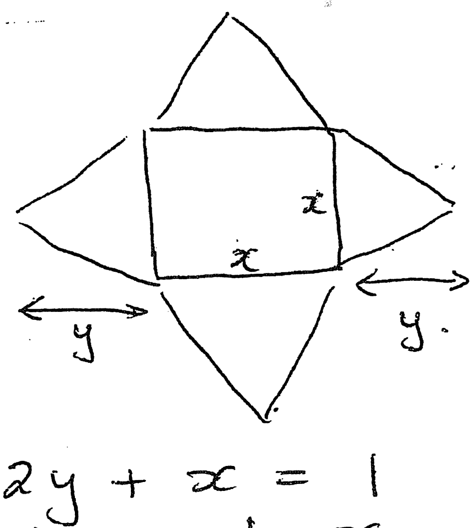
$$\therefore \left(n+\tfrac{1}{2}\right)^2 + \tfrac{3}{4} > 0$$

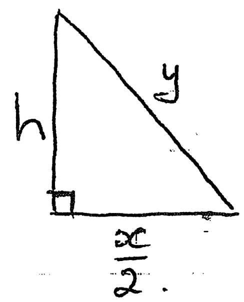
$$\therefore -8\left[\left(n+\tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right] < 0$$

$$\therefore \Delta < 0 \quad 1$$

$\therefore$ no ~~roots~~ real roots for any value of n

(b)





$$2y + x = 1$$

$$y = \frac{1-x}{2}$$

$$h^2 + \left(\frac{x}{2}\right)^2 = y^2$$

$$h^2 + \frac{x^2}{4} = \frac{(1-x)^2}{4}$$

$$h^2 = \frac{1 - 2x + x^2 - x^2}{4}$$

$$h^2 = \frac{1-2x}{4}$$

$$\therefore h = \sqrt{\frac{1-2x}{4}} \quad 2$$

$V = 3x\ h$

$= \frac{1}{3} x^2 \sqrt{\frac{1-2x}{4}}$ 1

$\therefore V = \frac{x^2\sqrt{1-2x}}{6}$

(ii)

$\frac{dV}{dx} = \frac{x^2}{6}\left(\frac{1}{2}\right)(1-2x)^{-\frac{1}{2}}(-2) + \frac{2x\sqrt{(1-2x)}}{6}$ 1

$= \frac{-x^2}{6\sqrt{1-2x}} + \frac{2x\sqrt{1-2x}}{6}$

$= \frac{x^2 + 2x(1-2x)}{6\sqrt{1-2x}}$

$= \frac{-x^2 + 2x - 4x^2}{6\sqrt{1-2x}}$

$= \frac{2x - 5x^2}{6\sqrt{1-2x}}$ 1

for max volume $\frac{dV}{dx} = 0$

$\therefore$ for max volume $\frac{2x-5x^2}{6\sqrt{1-2x}} = 0$

$\therefore x(2-5x) = 0$

$\therefore x = 0 \quad x = 0{\cdot}4$ 1

as $x > 0$ $\therefore$ Test $x = 0{\cdot}4$ for max.

function is continuous for $\cdot 3 \le x \le \cdot 45$

x	$0{\cdot}3$	$0{\cdot}4$	$0{\cdot}45$
$\frac{dv}{dx}$	$0{\cdot}04$	0	$-0{\cdot}06$

\+ 0 −

1

$\therefore$ local max at $x = 0{\cdot}4$

as there is only one stationary point for $x > 0$ then $x = 0{\cdot}4$ gives the absolute max volume.

$\therefore$ length of square base for max vol is $0{\cdot}4$m (= 40cm)