YEAR 11
PRELIMINARY EXAMINATION 2004

MATHEMATICS

Time Allowed – 85 minutes
(Plus 5 minutes Reading Time)

All questions may be attempted

All questions are of equal value

Department of Education approved calculators are permitted

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate's Number.

Question 1	**Marks**
a) Differentiate with respect to x, then simplify	
i. $f(x) = 3^{5x}$	1
ii. $f(x) = \dfrac{e^x}{\sin 2x}$	2
b) The equation of a parabola is $(x-3)^2 = -12(y-1)$. Find the:	
i. Coordinates of its vertex	1
ii. Equation of its directrix	2
c) i. Write down the discriminant of $x^2 - 2kx + 6k$.	1
ii. For what real values of k is $x^2 - 2kx + 6k$ always positive?	2
d) Given that $\tan\theta = T$ and θ is in the first quadrant, express in terms of T the value of $\sin(180° + \theta)$.	2
e) Solve $\sin x - \sqrt{3}\cos x = 0$ for $0° \le x \le 360°$	2
f) Solve for all real x: $\lvert x+3 \rvert = 2x-1$	2

Question 2 (start a new page)

Marks

a) Evaluate $\log_{27} 9$ **2**

b) Express as a single log: $2\log a + \frac{1}{2}\log b - \log c$ **2**

c) Solve for k: $9^k (3^3)^4 = 1$ **2**

d) If α and β are the roots of the equation $x^2 - 2x - 3 = 0$,

find the value of i. $\alpha + \beta$ **1**

ii. $\frac{1}{\alpha} + \frac{1}{\beta}$ **2**

e) Find the value of $\lim_{x \to 4} \left(\frac{x^2 - x - 12}{x - 4} \right)$ **2**

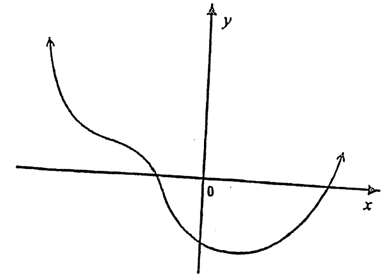
f) i. Find the stationary points of $y = x(\ln x)$ and determine their nature. **3**

ii. Show that y'' of $y = x(\ln x)$ is always positive ? **1**

Question 3 (start a new page)

Marks

a) This is the graph of $y = f(x)$.



Sketch the graph of its derivative, $y = f'(x)$ on the given set of axes. **2**

(see back page)

b) Given that in ΔABC, $XY // BC$, and $RY // SC$, **2**

prove that $AX{:}XB = AR{:}RS$

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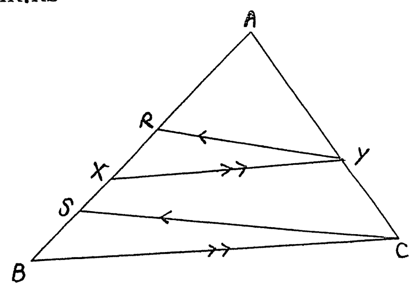

c) i. Write the domain of the function. **1**

$f(x) = \sqrt{3+x} - \sqrt{5-x}$

ii. Prove that this is a monotonic increasing function between its extreme points. **2**

iii. Find the range of this function. **1**

d) **Marks**

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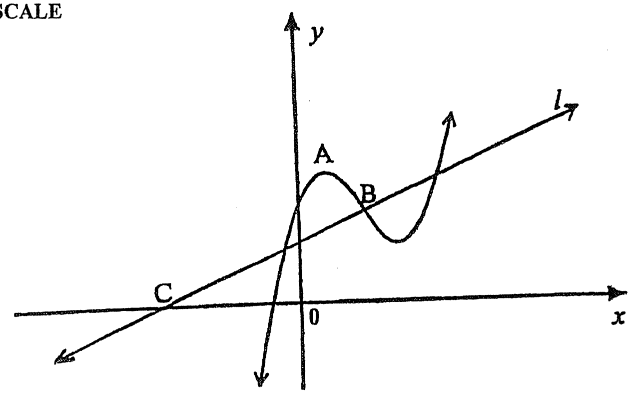


The diagram shows a sketch of the curve $y = x^3 - 6x^2 + 9x + 4$.

The curve has a local maximum point at A and a point of inflexion at B.

The line l is a normal to the curve at point B and meets the x-axis at point C.

i. Find the coordinates of point A. **2**

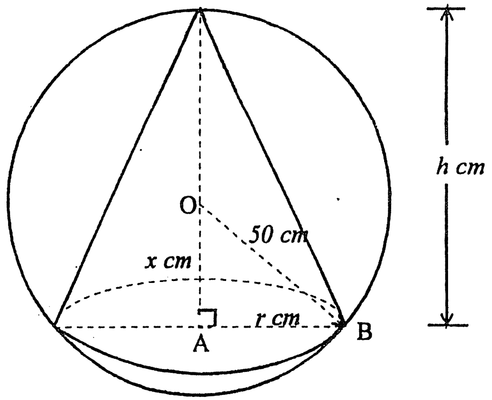
ii. Show that the coordinates of point B is (2,6). **2**

iii. Show that the equation of the line l is $x - 3y + 16 = 0$. **2**

iv. Find the coordinates of C. **1**

Question 4 (start a new page) **Marks**

a)



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The diagram shows a cone of base radius r cm and height h cm inscribed in a sphere of radius 50 cm. The centre of the sphere is O and $\angle OAB = 90^\circ$

Let $OA = x$ cm.

i. Show that $r = \sqrt{2500 - x^2}$. **1**

ii. Hence show that the volume, V cm^3, of the cone is given by: **1**

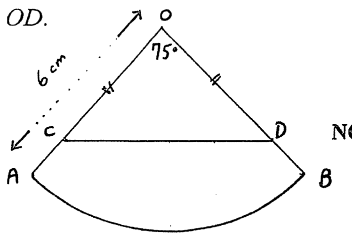
$$V = \frac{\pi}{3}(2500 - x^2)(50 + x)$$

iii. Find the radius of the largest cone which can be inscribed in the sphere. **5**

b) The diagram shows a sector AOB. The length of OA is 6 cm and **Marks**

$\angle AOB = 75°$. The points C and D lie on OA and OB respectively

and $OC = OD$.



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i. Convert 75° to radian measure, leaving the answer in terms of π. 1

ii. Find the exact area of the sector AOB. 1

iii. If the area of the triangle COD is one half the area of the sector AOB, find the length of OC correct to two decimal places. 2

c) The following diagram shows a circle with centre O and radius r *units*.

A radius divides a chord in the ratio of 2 : 1 and is bisected by the chord as shown in the diagram.

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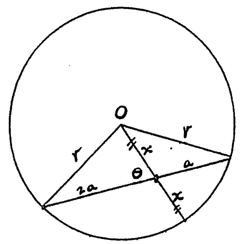

i) Show $r^2 = x^2 + 4a^2 - 4ax\cos\theta$ 1

ii) Show that the cosine of the angle θ between the chord and the radius is $\frac{1}{4}\sqrt{6}$. 3

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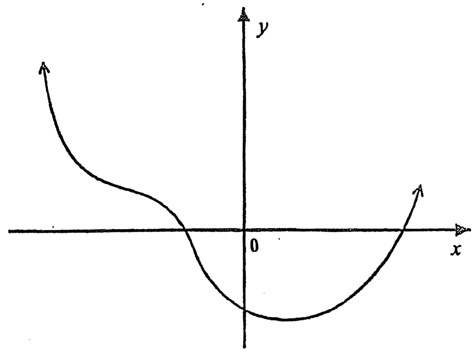
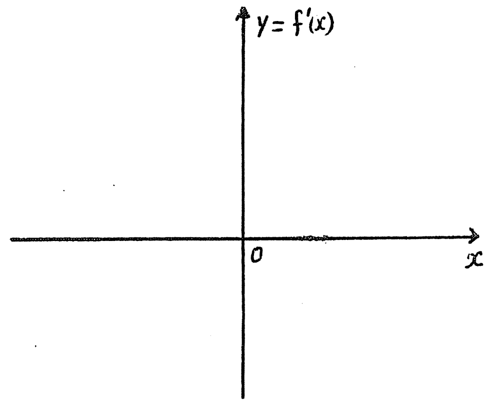
REMEMBER TO HAND IN THIS PAGE Student Number:__________

Question 3 (a)

This is the graph of $y = f(x)$.

Sketch the graph of its derivative, $y = f'(x)$ on the given set of axes.





Question 1

a) i) $f'(x) = (3^{5x} \cdot \ln 3) \times 5$ 1

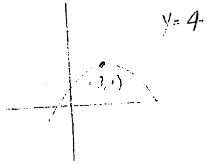
ii) $f'(x) = \dfrac{(\sin 2x)e^x - 2(\cos 2x)e^x}{(\sin 2x)^2}$ 1 $\frac{1}{2}$

$= \dfrac{e^x(\sin 2x - 2\cos 2x)}{(\sin 2x)^2}$ $\frac{1}{2}$

b) i) $(3, 1)$ 1

$a = 3$ 1

ii) $y = 4$ 1



c) i) $\Delta = (-2k)^2 - 4(6k)$

$= 4k^2 - 24k$ 1

ii) concave up

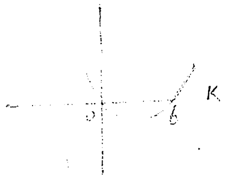
$\therefore \Delta < 0$

$4k^2 - 24k < 0$ 1

$k^2 - 6k < 0$

$k(k-6) < 0$

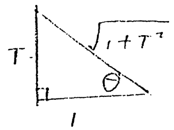
$0 < k < 6$ 1



d) $\sin(180° + \theta)$

$= -\sin\theta$ 1

$= -\dfrac{T}{\sqrt{1+T^2}}$ 1 #



e) $\sin x = \sqrt{3}\cos x$

$\tan x = \sqrt{3}$ 1

$x = 60°, 240°$

$\frac{1}{2}$ $\frac{1}{2}$

f) $|x+3| = 2x-1$

$x+3 = 2x-1$ or $x+3 = -(2x-1)$ 1

$4 = x$ $\quad 3x = -2$

$x = -\frac{2}{3}$

$\therefore x = 4$ 1

check $x = 4$

$|4+3| = |7| = 7$

$2x-1 = 2\times4-1 = 7$

$\therefore x = 4$

check $x = -\frac{2}{3}$

$\left|-\frac{2}{3}+3\right| = \left|\frac{7}{3}\right| = \frac{7}{3}$

$2(-\frac{2}{3})-1 = -\frac{7}{3} \neq \frac{7}{3}$

$\therefore x \neq -\frac{2}{3}$

Q 2

a) $\log_{27} 9 = \dfrac{\log 9}{\log 27} = \dfrac{\log 3^2}{\log 3^3} = \dfrac{2\cancel{\log 3}}{3\cancel{\log 3}} = \dfrac{2}{3}$ 1 1 #

b) $\log a^2 + \log\sqrt{b} - \log c = \log\left(\dfrac{a^2 \cdot \sqrt{b}}{c}\right)$ 1 1

c) $9^k(3^3)^4 = 1$

$3^{2k} \cdot 3^{12} = 3^0$ 1

$2k + 12 = 0$

$k = -6$ 1

d) i) $\alpha + \beta = -\frac{b}{a} = 2$ 1

ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha+\beta}{\alpha\beta} = \dfrac{2}{-3}$ 1 1 #

e) $\lim_{x\to 4} \dfrac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}} = 4+3 = 7$ 1 1 #

f) i) $y = x(\ln x)$

T.P: $y' = \dfrac{x}{x} + \ln x = 1 + \ln x = 0$ when $-1 = \ln x$ 1

ie $e^{-1} = x$

when $x = \frac{1}{e}$, $y = \frac{1}{e}\ln\frac{1}{e} = \frac{1}{e}\ln e^{-1} = \dfrac{-1}{e}$ $\therefore$ T.P $\left(\frac{1}{e}, -\frac{1}{e}\right)$ 1

$y'' = \frac{1}{x}$

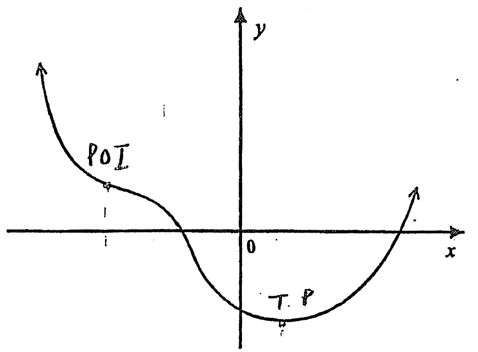
when $x = \frac{1}{e}$, $y'' = \dfrac{1}{\frac{1}{e}} = e > 0$

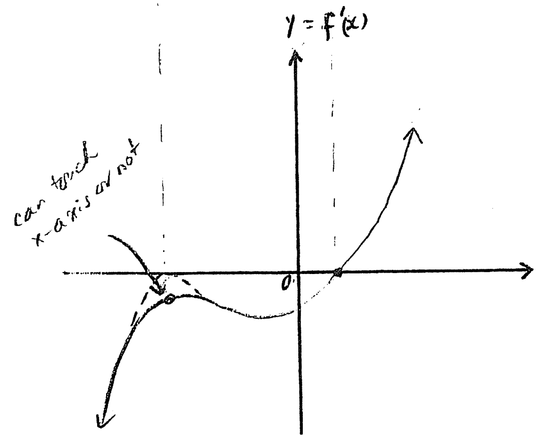
$\therefore \left(\frac{1}{e}, -\frac{1}{e}\right)$ is minimum 1

2f ii) $y'' = \frac{1}{x} > 0$ because $y = x(\ln x)$ is only defined for $x > 0$ 1

Question 3

a)





2

3 b) In $\triangle ABC$

$XY \parallel BC$ (given)

$AX : XB = AY : YC$ (line parallel to one side of triangle divides the other 2 sides into same ratio) 1

Similary in $\triangle ASC$

$RY \parallel SC$ (given)

$AR : RS = AY : YC$ $\frac{1}{2}$

$\therefore AX : XB = AR : RS$ $\frac{1}{2}$

c) Domain: $x \geq -3$ and $x \leq 5$

ie $-3 \leq x \leq 5$ 1

ii) $f'(x) = \frac{1}{2\sqrt{3+x}} + \frac{1}{2\sqrt{5-x}} > 0$ for $-3 < x < 5$

1 m for $f'(x)$

1 m for > 0

$\therefore$ montonic increasing between its extreme points

iii)

When $x = -3$ $f(x) = -\sqrt{8}$

$x = 5$ $f(x) = \sqrt{8}$

$\therefore$ Range: $-\sqrt{8} \leq f(x) \leq \sqrt{8}$ 1

d) i) $y = x^3 - 6x^2 + 9x + 4$

$y' = 3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$ 1

$(x-3)(x-1) = 0$ $\therefore x = 1$ or 3

$\therefore x_A = 1$ from graph A is closer to y-axis

$y_A = 1^3 - 6 \cdot 1^2 + 9 \cdot 1 + 4 = 1 - 6 + 9 + 4 = 8$ 1

$\therefore A(1, 8)$

(Q3a cont'd)

ii) $y'' = 6x - 12 = 0 \quad \therefore x = 2$

$x = 2,\ y = 2^3 - 6\times 4 + 18 + 4 = 8 - 24 + 22 = 6$

POI at $B\ (2, 6)$

iii) $y' = 3x^2 - 12x + 9$

At B, gradient $= y' = 3x^2 - 12x + 9 = 12 - 24 + 9 = -3$

gradient of normal at $B = \frac{1}{3}$ $\quad \frac{1}{2}$

$\therefore$ gradient of line $l = -\frac{1}{3}$ $\quad \frac{1}{2}$

equation of line l : $y - 6 = \frac{1}{3}(x-2)$

$y - 6 = \frac{x}{3} - \frac{2}{3}$

$3y - 18 = x - 2$

$x - 3y + 16 = 0$

iv) At $y = 0$, $x = -16$

$\therefore C = (-16, 0)$

Q4 a) i) Using Pythagoras' Theorem

$50^2 = r^2 + x^2$

$\therefore r^2 = 2500 - x^2$

$r = \sqrt{2500 - x^2} \quad (r > 0)$

$\frac{1}{2}$m off for not stating $r > 0$

ii) $V = \frac{1}{3}\pi r^2 h \qquad h = 50 + x$

$V = \frac{1}{3}\pi(2500 - x^2)(50 + x)$

at least 1 step between

iii) $V' = \frac{\pi}{3}\left[(-2x)(50+x) + (2500 - x^2)\right]$

$V' = \frac{\pi}{3}\left[-100x - 2x^2 + 2500 - x^2\right]$

$V' = \frac{\pi}{3}\left[-3x^2 - 100x + 2500\right] = 0$

$3x^2 + 100x - 2500 = 0$

$x = \dfrac{-100 \pm \sqrt{10000 - 4(3)(-2500)}}{6}$

$x = \dfrac{-100 \pm \sqrt{40000}}{6}$

$x = \frac{50}{3}$ or -50

But $x > 0 \quad \therefore x = \frac{50}{3}$ only

$r = \sqrt{2500 - \left(\frac{50}{3}\right)^2}$

$r = \frac{100\sqrt{2}}{3}$ cm or 47.14 cm

To determine max/min : Test $\frac{dV}{dx}$ as V, $\frac{dV}{dx}$ are continuous for $0 \le x \le 50$

x	16.6	$50/3$	16.7
V'	13.95	0	-6.98

$\therefore$ Relative max when $r = \frac{100\sqrt{2}}{3}$ cm.

Since there is only one turning point in the domain $0 \le x \le 50$ it is also the absolute max.

Alternatively assume $-50 \le x \le 50$

$V' = 0$ when $x = -50, \frac{50}{3}$

Test V'' for concavity as V, V', V'' are continuous for $-50 \le x \le 50$

$V'' = \frac{\pi}{3}(-6x - 100)$

$V''(-50) = \frac{200\pi}{3} > 0 \quad \therefore$ rel min at $x = -50 \qquad V(-50) = 0\ cm^3$

$V''\left(\frac{50}{3}\right) = -\frac{200\pi}{3} < 0 \quad \therefore$ rel max at $x = \frac{50}{3} \qquad V\left(\frac{50}{3}\right) = 155140.4\ cm^3$

Check end points $\quad V(50) = 0\ cm^3$

$\therefore x = \frac{50}{3}$ gives absolute max

or $r = \frac{100\sqrt{2}}{3} = 47.1$ cm gives absolute max

ii) Area sector $AOB = \frac{1}{2}r^2\theta = \frac{1}{2}\times 6^2 \times \frac{5\pi}{12} = \frac{15\pi}{2}\ cm^2$ 1
(or $23.56\ cm^2$)

iii) Let $OC = x$ cm
Area $\Delta COD = \frac{1}{2}x^2 \sin\frac{5\pi}{12}$

$\frac{15\pi}{2} = 2\left[\frac{x^2}{2}\sin\frac{5\pi}{12}\right]$ 1

$\frac{15\pi}{2} = x^2\sin\frac{5\pi}{12}$

(Q4b cont'd)

$x^2 = \frac{15\pi}{2} \div \left(\sin\frac{5\pi}{12}\right)$

$x^2 = 24.3931\ldots$

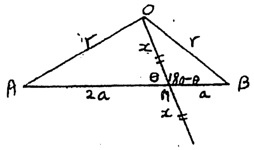
$x = 4.94$ cm

$OC = 4.94$ cm (2dp) 1
$(OC > 0)$

c) i) In ΔOAM

$r^2 = x^2 + (2a)^2 - 2(2a\,x\cos\theta)$

$r^2 = x^2 + 4a^2 - 4ax\cos\theta$ 1



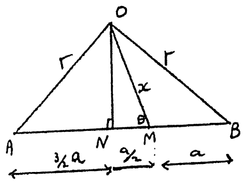
ii) Similarly in ΔOBM

$r^2 = x^2 + a^2 - 2ax\cos(180° - \theta) = x^2 + a^2 + 2ax\cos\theta$

$\therefore\ \cancel{x^2} + 4a^2 - 4ax\cos\theta = \cancel{x^2} + a^2 + 2ax\cos\theta$

$3a^2 = 6ax\cos\theta$

$\frac{a}{2x} = \cos\theta$ ... Eq. 1 1



Let N is the midpoint of AB
$\therefore ON \perp AB$
$AN = BN = \frac{3a}{2}$
$(ON)^2 = r^2 - \left(\frac{3a}{2}\right)^2 = x^2 - \frac{a^2}{4}$

$r = 2x$

$(2x)^2 - \frac{9a^2}{4} = x^2 - \frac{a^2}{4}$

$3x^2 = 2a^2$

$x = \sqrt{\frac{2}{3}}\,a$ $(x > 0)$ 1

Alternatively (3u)
Products of intercepts of intersecting chords are equal
$x(3x) = 2a^2$
$3x^2 = 2a^2$
$x = \sqrt{\frac{2}{3}}\,a$
$\frac{r}{2} = \sqrt{\frac{2}{3}}$

Since $\cos\theta = \frac{a}{2x}$ (from Eq. 1)

$\cos\theta = \frac{\cancel{a}}{2\sqrt{\frac{2}{3}}\cancel{a}} = \frac{\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{4}$ 1