YR 11 PRELIMINARY MATHS (2U) 2008

Question 1 (15 marks)

a) Write down the exact value of: **2**

i. $\sin 150^o$
ii. $\text{cosec } 315^o$

b) Find the exact value of x if: **3**
$3 \log_{10} 4 \quad 2 \log_{10} x \quad 6$

c) Shade the region(s) in the Cartesian plane which represent the simultaneous solution of $y \le x + 4$ and $y \ge x^2 - 4$. **2**

d) Given that $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$ can be written in the form $p + q\sqrt{2}$, find the values of p and q. **2**

e) For what values of k is the expression $kx^2 - (3k - 1)x + k$ positive definite? **3**

f) Prove that: $\dfrac{\cos A}{1+\sin A} + \dfrac{1+\sin A}{\cos A} \quad 2\sec A$ **3**

Question 2 (15 marks) *Start a new page*

a) Differentiate with respect to x.

i $-5(2 - \pi x)^3$ **1**

ii $(e^x - e^{x})^2$ **1**

iii. $2x\log_e x$ **2**

iv. $\dfrac{2x+4}{x-1}$ **2**

b) A lighthouse stands on top of a 20 m high cliff. From a ship anchored off shore, the angle of elevation of the top of the lighthouse is 65° and the angle of elevation of the base of the lighthouse is 63°12′. Find, to the nearest metre, the height of the top of the lighthouse above sea level. **3**

c) The distance between the point (x , -2) and the line $4x - 3y + 2 \quad 0$ is 8 *units*. Find two possible values of x. **2**

d) i. Sketch the graphs of $y = \cos x$ and $y = \frac{3}{2}\tan x$ from $x = \frac{-\pi}{2}$ to $x = \frac{\pi}{2}$ on the same set of axes **1**

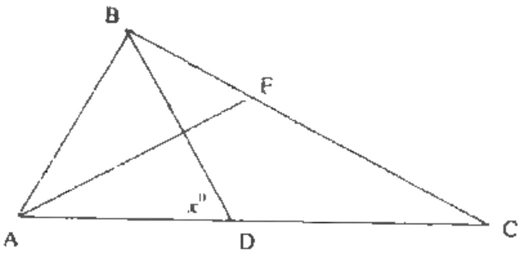
ii. By solving the equation $\cos x = \frac{3}{2}\tan x$, find the coordinates of the point of intersection of the two graphs that lie between $x = 0$ and $x = \frac{\pi}{2}$. **3**

Question 3 (15 marks) *Start a new page*

a) Write down the equation of the parabola with the vertex at the origin and directrix $x = 2$. **1**

b) One root of the equation $x^2 - (r + 3)x + 2(r + 1) = 0$ is twice the other root. Find the two possible values of r. **3**

c)



ΔABC is a right-angled triangle with $\angle ABC = 90°$. D is a point on AC such that $AB = BD = DC$. E lies on BC such that AE bisects $\angle BAD$. Let $\angle ADB = x°$. Copy the diagram showing this information.

i. Show that $\angle DBC = (2x - 90)°$. **1**

ii Hence find the value of x. **1**

iii. Show that ΔAEC is isosceles. **2**

d) The function $y = x^3 - 3x^2 - 9x + 1$ is defined in the domain $-4 \le x \le 5$.

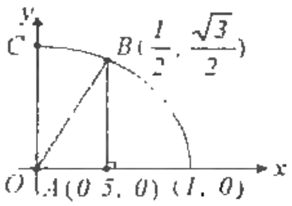
i. Find the co-ordinates of any turning points and determine their nature. **3**

ii. Find the coordinates of any points of inflexion. **2**

iii. Draw a neat sketch of the curve for the defined domain. **2**

Question 4 (15 marks) *Start a new page*

a) The diagram shows the first quadrant of the circle $x^2 + y^2 = 1$. The point A has coordinates $\left(\frac{1}{2}, 0\right)$ and $B\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

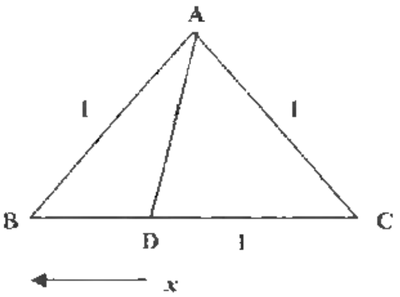


Not to scale

i. What is the exact value of $\angle COB$? **1**

ii Show that the exact value of the shaded area $OABC$ is $\frac{2\pi + 3\sqrt{3}}{24}$. **2**

b) In the diagram, ABC is an isosceles triangle where $\angle BAC = \frac{3\pi}{5}$ and $AB = AC = 1$. The point D is chosen on BC such that $CD = 1$. Let $BC = x$.



Not to scale

i Show that $\angle ADC = \frac{2\pi}{5}$. **2**

ii. Given that ΔDBA and ΔABC are similar, deduce that $x^2 - x - 1 = 0$. **1**

iii. By using the cosine rule in ΔABC, deduce that $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$. **3**

c) The point $P\left(n, \frac{n^2}{2}\right)$ lies on the parabola with equation $2y = x^2$.

$A(4,1)$ is a fixed point

i. Show that $PA = \left(\frac{n^4}{4} - 8n + 17\right)^{\frac{1}{2}}$ **2**

ii. As P moves, find the minimum distance of P from A. **3**

iii. Prove that for this position of P, AP is normal to the parabola. **1**

End of the paper

Question 1

a) i) $\sin 150 = \sin 30$

$= \frac{1}{2}$ #

ii) $\operatorname{cosec} 315 = -\operatorname{cosec} 45$

$= \frac{-1}{\sin 45}$

$= -\sqrt{2}$ #

b) $3\log_{10} 4 - 2\log_{10} x = 6$

$\log_{10} 4^3 - \log_{10} x^2 = 6$

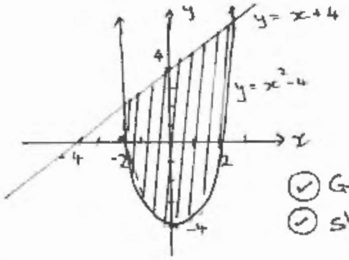
$\log_{10}\left(\frac{4^3}{x^2}\right) = 6$

$10^6 = \frac{4^3}{x^2}$

$x^2 = \frac{4^3}{10^6}$

$x = \frac{8}{10^3}$ #

c)



Graphs

shading

d) $\frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$

$= \frac{(\sqrt{2}+1)^2}{2-1}$

$= \frac{2+2\sqrt{2}+1}{1}$

$= 3+2\sqrt{2}$

$\therefore p = 3$ and $q = 2$

e) $kx^2 - (3k-1)x + k$

$b^2 - 4ac < 0$ and $k > 0$

$a = k$
$b = -3k+1$
$c = k$

$(-3k+1)^2 - 4(k)(k) < 0$

$9k^2 - 6k + 1 - 4k^2 < 0$

$5k^2 - 6k + 1 < 0$

$(5k-1)(k-1) < 0$

$\left\{\frac{1}{5} < k < 1\right\}$.

f) $\frac{\cos A}{1+\sin A} + \frac{1+\sin A}{\cos A} = 2\sec A$

LHS

$= \frac{\cos^2 A + (1+\sin A)^2}{(1+\sin A)\cos A}$

$= \frac{\cos^2 A + 1 + 2\sin A + \sin^2 A}{(1+\sin A)\cos A}$

$= \frac{\cos^2 A + \sin^2 A + 1 + 2\sin A}{(1+\sin A)(\cos A)}$

$= \frac{1 + 1 + 2\sin A}{(1+\sin A)\cos A}$

$= \frac{2(1+\sin A)}{(1+\sin A)\cos A}$

$= \frac{2}{\cos A}$

$= 2\sec A$

$=$ RHS

Question 2

i) $-15\pi(2-\pi x)^2$

ii) $2e^{2x} - 2e^{-2x}$

OR

$2(e^{k}+e^{-k})(e^{k}-e^{-k})$

iii) $2x\log_e x$

Let $u = 2x$ and $u' = 2$
$v = \log_e x$ $v' = \frac{1}{x}$

$\frac{dy}{dx} = uv' + vu'$

$= x + 2\ln x$

iv) $\frac{2x+4}{x-1}$

let $u = 2x+4$ and $u' = 2$
$v = x-1$ and $v' = 1$

Quotient Rule

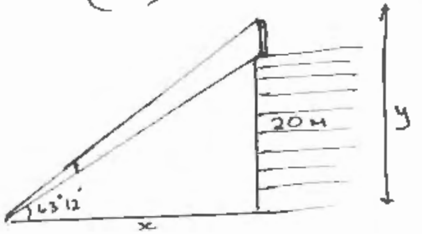
$\frac{vu' - uv'}{v^2}$

$= \frac{2(x-1) - (2x+4)}{(x-1)^2}$

$= \frac{2x-2-2x-4}{(x-1)^2}$

$= \frac{-6}{(x-1)^2}$

b)



Method 1

$\frac{20}{x} = \tan 63°12'$

$x = \frac{20}{\tan 63°12'}$

$\frac{y}{x} = \tan 65$

$y = x\tan 65$

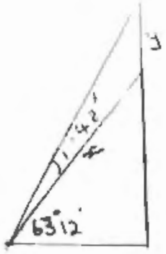
$= \frac{20\tan 65}{\tan 63°12'}$

$y = 22\text{m}$

Method 2

$\sin 63°12' = \frac{20}{x}$

$x = \frac{20}{\sin 63°12'}$



Using sine rule

$\frac{y}{\sin 1°48'} = \frac{20}{\sin 63°12' \sin 25}$

$= 1.6$

$\therefore$ Total height $= 20 + 1.6$

$= 21.6$

$= 22\text{m}$.

c) $(x, -2)$ and $4x - 3y + 2 = 0$

$d = \frac{|ax+by+c|}{\sqrt{a^2+b^2}}$

$8 = \frac{|4x+6+2|}{\sqrt{3^2+4^2}}$

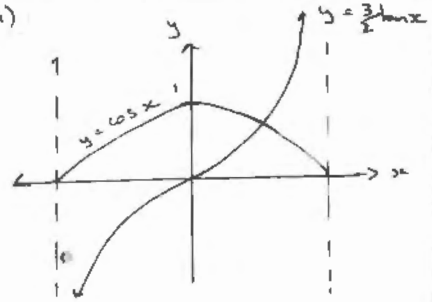
$8 = \frac{|4x+8|}{5}$

$|4x+8| = 40$

$4x+8 = 40$ or $-4x-8 = 40$

$x = 8$ $x = -12$

d(i)



graph

(ii)

$\cos x = \frac{3}{2}\tan x$

$2\cos x = \frac{3\sin x}{\cos x}$

$2\cos^2 x = 3\sin x$

$2(1-\sin^2 x) = 3\sin x$

$2 - 2\sin^2 x = 3\sin x$

$0 = 2\sin^2 x + 3\sin x - 2$

let $x = \sin x$

$0 = 2x^2 + 3x - 2.$

$0 = (2x-1)(x+2)$

$x = \frac{1}{2}$ or $x = -2$

$\sin x = \frac{1}{2}$ or $\sin x = -2$. No solution

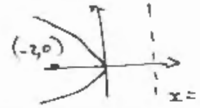
$x = 30°$

$= \frac{\pi}{6}$

$\therefore$ pt of intersection is $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

Question 3

a) $y^2 = -8x$



b) $(\alpha + 2\alpha) = 3\alpha = -\frac{b}{a} = r+3$

$\therefore \alpha = \frac{r+3}{3}$ ——→ ①

$\alpha \cdot 2\alpha = 2\alpha^2 = \frac{c}{a} = 2r+2.$

$\alpha = \pm\sqrt{r+1}$ —— ②

Solve simultaneously

$\frac{r+3}{3} = \sqrt{r+1}$

$r+3 = 3\sqrt{r+1}$

$(r+3)^2 = 9(r+1)$

$r^2+6r+9 = 9r+9$

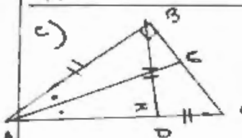
$r^2 - 3r = 0$

$r(r-3) = 0$

$r = 0$ or $r - 3 = 0$

$\therefore r = 0$ or 3

c)



i) $\angle ADB = \angle BAD = x°$ (equal sides of isosceles triangle)

$\angle ABD = 180 - 2x$ (angle sum of $\triangle ABD$)

$\therefore \angle DBC = 90 - (180 - 2x)$

$= 2x - 90$

ii)

$x = 60°$

iii) $\angle BAD = 60°$

$\therefore \angle EAD = 30°$ — from part (i)

$\angle DBC = \angle DCB$ (equal angles are opposite equal sides in $\triangle DBC$)

$= 2x - 90$

$= 2(60) - 90$ from part (ii)

$= 30°$

$\therefore \angle EAC$ and $\angle DCB$ are both $30°$ — form the base angles of an isosceles $\triangle$.

d) $y = x^3 - 3x^2 - 9x + 1$

(i) TP $\frac{dy}{dx} = 0$

$3x^2 - 6x - 9 = 0$

$3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

$x = 3$ or $x = -1$

when $x = 3 \Rightarrow y = -26$

when $x = -1 \Rightarrow y = 6$

$\therefore$ TP occur at $(3,-26)$ and $(-1,6)$

Nature

at $x = 3$

x	2	3	4
$\frac{dy}{dx}$	-3	0	5
	\ (+ve)	–	/ (-ve)

OR $\frac{d^2y}{dx^2} = 12 > 0$

$\therefore$ relative minimum point

at $x = -1$

x	-2	-1	0
$\frac{dy}{dx}$	5	0	-3
	/ (+ve)	–	\ (-ve)

OR $\frac{d^2y}{dx^2} = -12 < 0$

$\therefore$ relative minimum point.

(ii) Point of inflexion

$\frac{d^2y}{dx^2} = 0.$

$6x - 6 = 0$

$x = 1$

at $x = 1$ $y = -10$

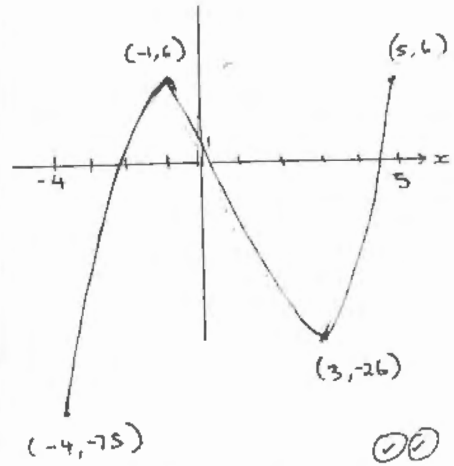
Test

x	0	1	2
$\frac{d^2y}{dx^2}$	-6	–	+6

Concavity changes.

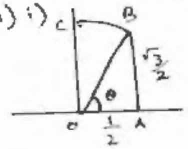
point of inflexion occurs at $(1,-10)$

iii)



Question 4

a) i)



$\tan\theta = \frac{\sqrt{3}}{2} \div \frac{1}{2}$

$= \sqrt{3}$

$\theta = \frac{\pi}{3}$

$\therefore \angle COB = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

ii) Total Area = Area of COB + Area of $\triangle$

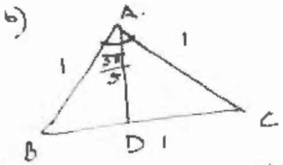
$= \frac{1}{2}r^2\theta + \frac{1}{2}bh$

$= \frac{1}{2}(1^2)\cdot\frac{\pi}{6} + \frac{1}{2}\times\frac{1}{2}\times\frac{\sqrt{3}}{2}$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$

$= \frac{2\pi + 3\sqrt{3}}{24}$

b)



$\angle ACB = \left(\pi - \frac{3\pi}{5}\right) \div 2$ (Equal angles are opposite equal sides in $\triangle ACB$)

$= \frac{2\pi}{5} \div 2$

$= \frac{\pi}{5}$

$\angle CAD$ and $\angle CDA$ {Equal angles are opposite equal sides in $\triangle ACD$}

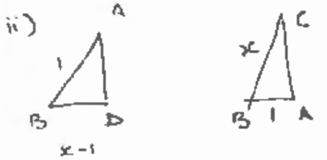
$= \left(\pi - \frac{\pi}{5}\right) \div 2$

$= \frac{4\pi}{5} \div 2$

$= \frac{4\pi}{10}$

$= \frac{2\pi}{5}$ ✓

ii)



ratio of similar triangles

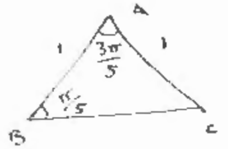
$\frac{x}{1} = \frac{1}{x-1}$

$x(x-1) = 1$

$x^2 - x = 1$

$x^2 - x - 1 = 0$ ✓

iii)



$x^2 - x - 1 = 0$

$a = 1,\ b = -1,\ c = -1$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{1 \pm \sqrt{5}}{2}$

Length is $\frac{1+\sqrt{5}}{2}$ ✓

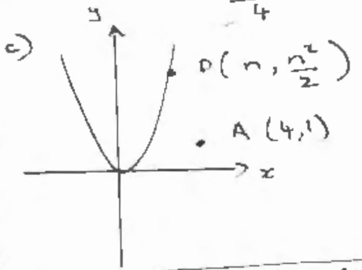
$\cos\frac{\pi}{5} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 + (1^2) - 1^2}{2\left(\frac{1+\sqrt{5}}{2}\right)\cdot 1}$ ✓

$= \frac{\frac{(1+\sqrt{5})^2}{2}}{(1+\sqrt{5})}$

$= \frac{(1+\sqrt{5})(1+\sqrt{5})}{4(1+\sqrt{5})}$

$= \frac{1+\sqrt{5}}{4}$ ✓

c)



$PA = \sqrt{(4-n)^2 + \left(1 - \frac{n^2}{2}\right)^2}$ ✓ 1

$= \sqrt{16 - 8n + n^2 + 1 - n^2 + \frac{n^4}{4}}$ ✓ 1

$= \sqrt{\frac{n^4}{4} - 8n + 17}$

$= \left(\frac{n^4}{4} - 8n + 17\right)^{\frac{1}{2}}$

ii) Minimum distance occurs when $\frac{d(PA)}{dn} = 0$

$0 = \frac{1}{2}\left(\frac{n^4}{4} - 8n + 17\right)^{-\frac{1}{2}} \times (n^3 - 8)$

$0 = \frac{n^3 - 8}{2\sqrt{\frac{n^4}{4} - 8n + 17}}$ ✓

$\therefore n^3 - 8 = 0$

$n^3 = 8$

$n = \sqrt[3]{8}$

$n = 2$ ✓

when $n = 2$

$PA = \left(\frac{2^4}{4} - 8(2) + 17\right)^{\frac{1}{2}}$ ✓

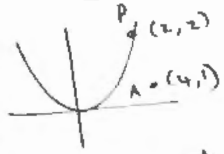
$= \sqrt{5}$ units

Test Minimum

x	1	2	3
$\frac{dy}{dx}$	$\frac{-7}{2\sqrt{10}}$	0	$\frac{19}{2\sqrt{53/4}}$
	\ -ve	0	/ +ve

This is a local minimum and with only one TP is also absolute minimum

iii) gradient of PA



$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{2-1}{2-4} = \frac{-1}{2}$

equation of tangent of parabola at $x = 2$

$y = \frac{1}{2}x^2$

$y' = x$

at $x = 2$ gradient is 2

$\therefore m_1 \cdot m_2 = 1$

$2 \times -\frac{1}{2} = -1$

true

$\therefore$ PA is $\perp$ to tangent of parabola at P ✓ 1