NORTH SYDNEY GIRLS HIGH SCHOOL

2011

YEARLY EXAMINATION

Preliminary Mathematics

General Instructions

Reading Time – 5 minutes
Working Time – 2 hours
Write using black or blue pen

Board-approved calculators may be used
All necessary working should be shown in every question.

Total Marks – 84

Attempt Questions 1 – 7
All questions are of equal value.
At the end of the examination, place your solution booklets in order and put this question paper on top. Submit ONE bundle. The bundle will be separated for marking so please ensure your name is written on each solution booklet.

Student Name: ________________________________

Teacher: ________________________________

QUESTION	MARK
1	/12
2	/12
3	/12
4	/12
5	/12
6	/12
7	/12
TOTAL	/84

Total Marks – 84
Attempt Questions 1 – 7
All questions are of equal value

Begin each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 Marks) Use a SEPARATE writing booklet.

(a) Evaluate $\dfrac{\cos 92°9'}{\tan 130°}$ correct to three significant figures. **2**

(b) Rationalise the denominator of $\dfrac{2\sqrt{3}}{3-\sqrt{7}}$. **2**

Express your answer in simplest form.

(c) Simplify $\dfrac{\frac{a}{b}+1}{1+\frac{b}{a}}$. **2**

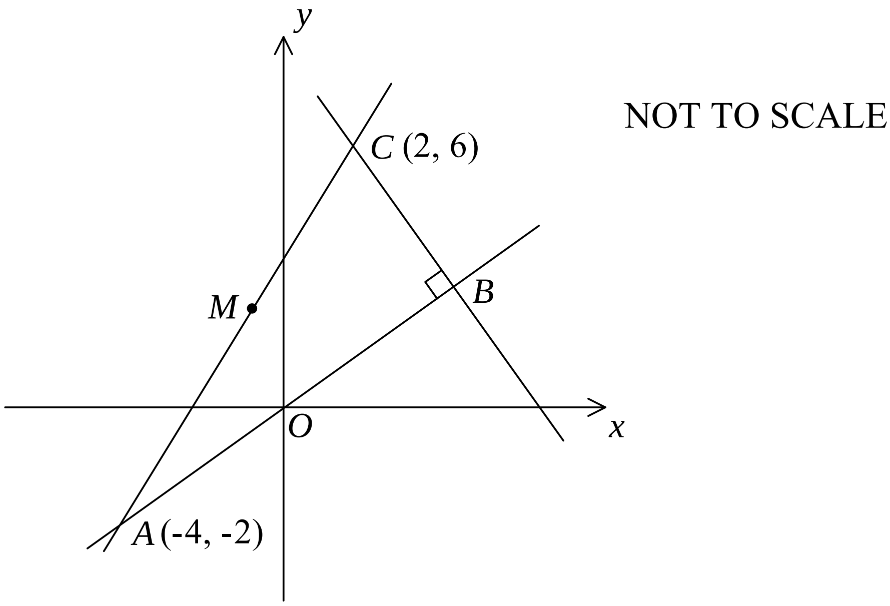
(d) Solve $|2x-1|=5$. **2**

(e) Simplify $\dfrac{\left(6x^3y\right)^2}{6x^2y^4}$. **2**

(f) Solve $x^2 \le 2x$. **2**

Question 2 (12 Marks) Use a SEPARATE writing booklet.

In the diagram below, A and C are the points $(-4,-2)$ and $(2,6)$ respectively.
Lines AB and BC are perpendicular and intersect at B.



(a) Find the equation of the line AB, given that it passes through the origin. **1**

(b) The line BC is perpendicular to AB. Find the equation of line BC. **2**

(c) Hence, or otherwise, find the coordinates of B. **2**

(d) Find the coordinates of M, the midpoint of AC. **1**

(e) Find the distance AM and hence show that $AM = MB$. **2**

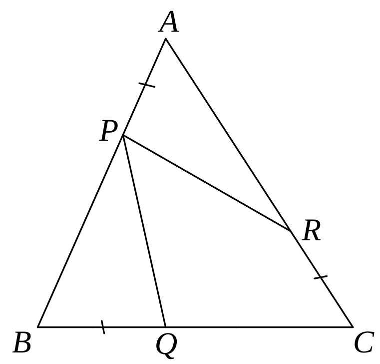
(f) Find the coordinates of point D that makes quadrilateral $ABCD$ a rectangle. **1**

(g) Explain why a circle can be drawn through A, B, C and D, with its centre at M. **1**

(h) Write down the equation of this circle. **2**

Question 3 (12 Marks) Use a SEPARATE writing booklet.

(a) The diagram below shows an equilateral triangle, ΔABC, with $AP = BQ = CR$.

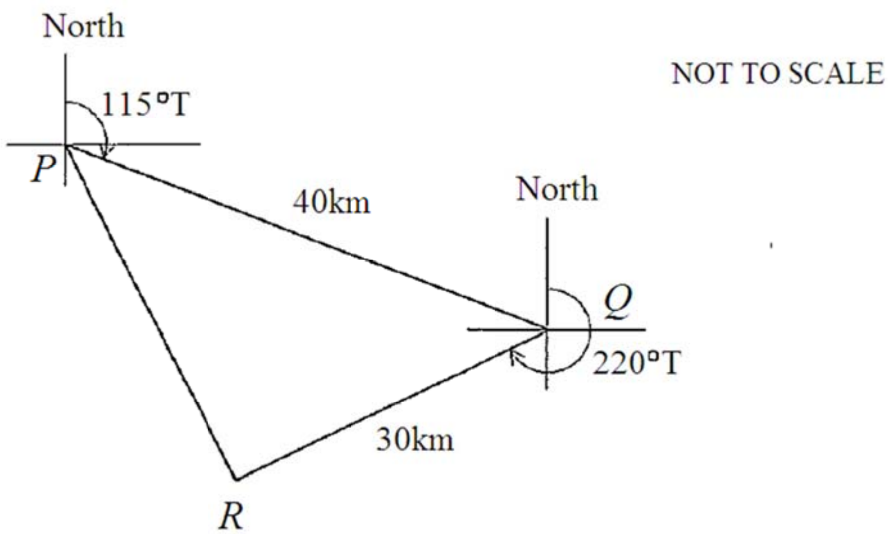


(i) Prove that ΔAPR and ΔBQP are congruent. **3**

(ii) Show that $\angle BPQ + \angle APR = 120°$. **2**

(iii) Hence prove that ΔPQR is equilateral. **2**

(b) The diagram below shows a lighthouse at Q and 40 km from P.
The bearing of the lighthouse, at Q, from P is 115°T.
R is a headland, 30 km from Q and on a bearing of 220°T from Q.



(i) Find the size of $\angle PQR$, correct to the nearest degree. **1**

(ii) Find the distance from P to R, correct to 2 decimal places. **2**

(iii) Find the bearing of R from P. **2**

Question 4 (12 Marks) Use a SEPARATE writing booklet.

(a) Differentiate with respect to x:

(i) $x^5 - 2x^{\frac{3}{2}}$ **1**

(ii) $(1-3x)^{-2}$ **2**

(iii) $\frac{x}{2}(2x-5)^3$ **2**

(iv) $\frac{x-2}{4-x^3}$ **2**

(b) Given the function $f(x) = x^2 - 3x - 3$, find the values of b for which $f'(b) = f(b)$. **2**

(c) (i) Find $\frac{dy}{dx}$ where $y = \frac{x-\sqrt{x}}{\sqrt{x}}$. **2**

(ii) For $x > 0$, explain why every tangent has a positive gradient. **1**

Question 5 (12 Marks) Use a SEPARATE writing booklet.

(a) Simplify $\frac{\sin(90° - \theta)}{1+\cot^2(90° - \theta)}$. **2**

(b) Find the exact value of $\sec 210°$. **2**

(c) Show that $\frac{\cos^3\theta}{\sin\theta} + \sin\theta\cos\theta = \cot\theta$. **2**

(d) Solve $2\sin^3\theta = \sin\theta$ for $0° \le \theta \le 360°$. **3**

(e) A function is defined as $f(x) = x + \frac{1}{x}$.

(i) Simplify $\left[f(x)\right]^2 - f(x^2)$. **2**

(ii) Determine whether $f(x)$ is even, odd or neither. **1**

Question 6 (12 Marks) Use a SEPARATE writing booklet.

(a) (i) Find the equation of the line, L, that passes through the point of intersection, A, of the lines. **3**

$$l_1 : 5x + y - 6 = 0 \text{ and } l_2 : 2x - 3y + 1 = 0$$

and also passes through the point $B(3, -2)$.

(ii) Find the perpendicular distance from the origin to L. Leave your answer in exact form. **2**

(b) Consider the function $f(x) = x^2\sqrt{4x-3}$.

(i) State the domain of the function $f(x)$. **1**

(ii) Show that $f'(x) = \dfrac{2x(5x-3)}{\sqrt{4x-3}}$. **3**

(iii) Find the equation of the tangent to $f(x)$ at the point $P(3, 27)$. **2**

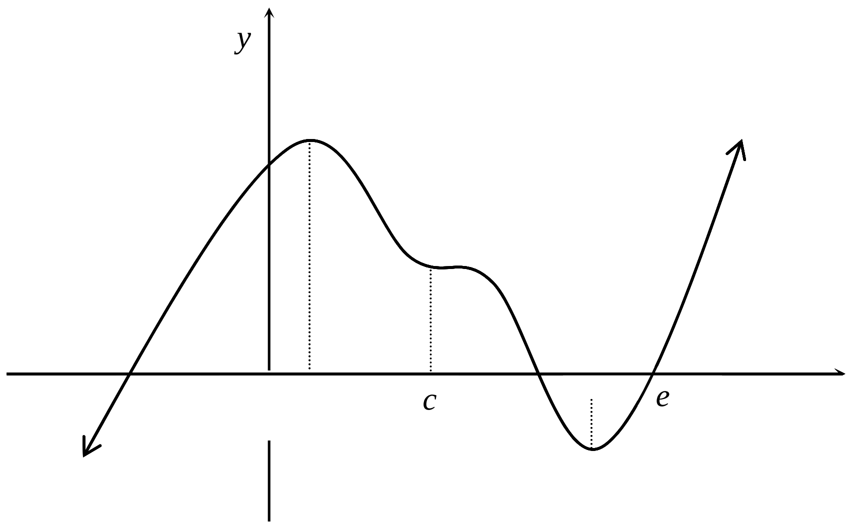
(iv) State the values of x where the tangents to $f(x)$ are horizontal. **1**

Question 7 (12 Marks) Use a SEPARATE writing booklet.

(a) On a number plane, shade the region satisfying the following inequalities. **3**

$$y > x^2 + 1 \text{ and } y < \sqrt{16 - x^2}\,.$$

(b) The diagram below shows the graph of $y = f(x)$.

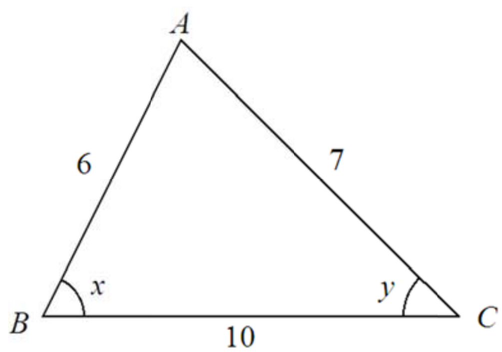


Copy this graph into your answer booklet.

(i) Find x, where $\dfrac{dy}{dx} > 0$. **2**

(ii) Sketch the graph $y = f'(x)$. **3**

(c) The diagram below shows ΔABC with $AC = 7$, $BC = 10$ and $AB = 6$.
$\angle ABC = x$ and $\angle ACB = y$.



(i) Express the size of $\angle BAC$ in terms of x and y. **1**

(ii) Hence, show that $\sin x + \sin y = \frac{13}{10}\sin(x + y)$ **3**

End of paper

Solution

Question 1.

a) $\dfrac{\cos 92°9'}{\tan 130°} = 0.031479\ldots$

$\therefore\ 0.0315$ (to 3 sig fig)

b) $\dfrac{2\sqrt{3}}{3-\sqrt{7}} \times \dfrac{3+\sqrt{7}}{3+\sqrt{7}}$

$= \dfrac{6\sqrt{3}+2\sqrt{21}}{9-7}$

$= 3\sqrt{3}+\sqrt{21}$

c) $\dfrac{\;\dfrac{a+b}{b}\;}{\;\dfrac{a+b}{a}\;} = \dfrac{a}{b}$

d) $2x-1 = \pm 5$

$2x-1=5, \qquad 2x-1=-5$

$\therefore x=3, \qquad x=-2$

e) $\dfrac{(6x^3y)^2}{6x^2y^4} = \dfrac{6^{\cancel{2}}x^{\cancel{6}\,4}\cancel{y^2}}{\cancel{6}\cancel{x^2}y^{\cancel{4}\,2}}$

$= \dfrac{6x^4}{y^2}$

f) $x^2 \le 2x$

$x^2-2x \le 0$

$x(x-2) \le 0$

$\therefore 0 \le x \le 2$

Question 2.

a) $m_{AO} = \frac{2}{4} = \frac{1}{2}$

$\therefore\ y = \frac{1}{2}x$

b) $m_{BC} = -2$ ($\because BC \perp AB$)

$C(2, 6)$

$\therefore$ eq. of the line BC is

$y - 6 = -2(x-2)$

$y = -2x + 10$

c) B is the intersection of AB & BC

$\left.\begin{array}{l} y = \frac{1}{2}x \\ y = -2x+10 \end{array}\right\}$ $\therefore$ solve simultaneously.

$\frac{1}{2}x = -2x + 10$

$\frac{5}{2}x = 10$

$\therefore x = 4$

$\therefore y = 2$

$\therefore B(4, 2)$

d) $M(-1, 2)$

e) $d_{AM} = \sqrt{(-4+1)^2 + (-2-2)^2}$

$= \sqrt{9 + 16}$

$= 5$ units

$M(-1, 2)$

$B(4, 2)$

$\therefore d_{MB} = 4 + 1$

$= 5$ units

$\therefore AM = MB$

f) $D(-6, 2)$ $DM = MB$ Diagonals are equal and bisect each other in rectangle.

g) $AM = MC = MB = MD = 5$ units

radii of the circle, centre M.

h) radius $= 5$ units centre $M(-1, 2)$

$\therefore (x+1)^2 + (y-2)^2 = 25$

Question 3. (a) (i) $AB = AC$ (sides of the equilateral $\triangle$)

$AP = RC$ (given)

$BP = AR$ (by subtraction)

In $\triangle APR$ & $\triangle BQP$

$AR = BP$ (proven above)

$AP = BQ$ (given)

$\angle PAR = \angle QBP$ (angles of equilateral $\triangle$)

$\therefore \triangle APR \equiv \triangle BQP$ (SAS)

(ii) $\angle APR = \angle PQB$ (matching angles of cong $\triangle$s)

In $\triangle BPQ$

$\angle BPQ + \angle PQB + 60^\circ = 180^\circ$ ($\angle$ sum of $\triangle$)

$\angle BPQ + \angle PQB = 120^\circ$

$\therefore \angle BPQ + \angle APR = 120^\circ$

(iii) $\angle QPR = 180^\circ - (\angle BPQ + \angle APR)$ ($\angle$s on a straight line)

$\therefore \angle QPR = 60^\circ$

$PQ = PR$ (matching sides of cong $\triangle$s in part (i))

$\therefore \triangle PQR$ is isos with apex angle 60°

$\therefore \triangle PQR$ is equilateral.

(b) (i) $\angle PQR = 25^\circ + 50^\circ = 75^\circ$

(ii) Cosine Rule

$x^2 = 30^2 + 40^2 - 2 \times 30 \times 40 \times \cos 75^\circ$

$\therefore x = 43.35$ km (to 2 dp)

(iii) Let $\angle RPQ = \theta$

$$\frac{\sin\theta}{30} = \frac{\sin 75}{43.35}$$

$$\sin\theta = \frac{\sin 75^\circ \times 30}{43.35}$$

$\theta = 42^\circ$ (to the nearest degrees)

$\therefore$ bearing from R from P is $157^\circ T$

Question 4. (a) (i) $\frac{d}{dx}\left(x^5 - 2x^{\frac{3}{2}}\right) = 5x^4 - 3x^{\frac{1}{2}}$

(ii) $\frac{d}{dx}(1-3x)^{-2} = -2(1-3x)^{-3} \times -3$

$= 6(1-3x)^{-3}$

(iii) $\frac{d}{dx}\ \frac{x}{2}(2x-5)^3 = \frac{1}{2}(2x-5)^3 + \frac{x}{2} \times 6(2x-5)^2$

$= \frac{(2x-5)^3}{2} + \frac{6x(2x-5)^2}{2}$

$= \frac{(2x-5)^2(8x-5)}{2}$

(iv) $\frac{d}{dx}\left(\frac{x-2}{4-x^3}\right) = \frac{(4-x^3) - (x-2)(-3x^2)}{(4-x^3)^2}$

$= \frac{4 - x^3 + 3x^3 - 6x^2}{(4-x^3)^2}$

$= \frac{2x^3 - 6x^2 + 4}{(4-x^3)^2}$

(b) $f(x) = x^2 - 3x - 3$

$f'(x) = 2x - 3$

$\therefore f(b) = b^2 - 3b - 3$

$f'(b) = 2b - 3$

$f'(b) = f(b)$

$\therefore 2b - 3 = b^2 - 3b - 3$

$b^2 - 5b = 0$

$b(b-5) = 0$

$\therefore b = 0,\ 5$

(c) (i) $y = \dfrac{x - \sqrt{x}}{\sqrt{x}}$

$y = \dfrac{x}{\sqrt{x}} - 1$

$y = \sqrt{x} - 1$

$\therefore \quad y' = \frac{1}{2} x^{-\frac{1}{2}}$

$y' = \dfrac{1}{2\sqrt{x}}$

(ii) $x > 0$

$\sqrt{x} > 0$

$\dfrac{1}{2\sqrt{x}} > 0$

$\therefore y' > 0$

$\therefore$ Has a positive gradient.

Question 5. (a). $\dfrac{\sin(90^\circ - \theta)}{1 + \cot^2(90^\circ - \theta)} = \dfrac{\cos\theta}{1 + \tan^2\theta}$

$= \dfrac{\cos\theta}{\sec^2\theta}$

$= \cos^3\theta$

(b) $\sec 210^\circ = \dfrac{1}{\cos(180+30)^\circ}$

$= \dfrac{1}{-\cos 30^\circ}$

$= -\dfrac{2\sqrt{3}}{3}$

(c) $LHS = \dfrac{\cos^3\theta + \sin^2\theta\cos\theta}{\sin\theta}$

$= \dfrac{\cos\theta(\cos^2\theta + \sin^2\theta)}{\sin\theta}$

$= \dfrac{\cos\theta}{\sin\theta}$

$= \cot\theta$

$= RHS$

(d) $2\sin^3\theta = \sin\theta$

$2\sin^3\theta - \sin\theta = 0$

$\sin\theta\,(2\sin^2\theta - 1) = 0$

$\therefore \sin\theta = 0,$ | $\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = 0^\circ, 180^\circ, 360^\circ,$ | $\theta = 45^\circ, 135^\circ, 225^\circ, 315^\circ$

$\therefore \theta = 0^\circ, 45^\circ, 135^\circ, 180^\circ, 225^\circ, 315^\circ, 360^\circ$

(e) $f(x) = x + \frac{1}{x}$

(i) $\left[f(x)\right]^2 = \left(x + \frac{1}{x}\right)^2$

$= x^2 + 2 + \frac{1}{x^2}$

$f(x^2) = x^2 + \frac{1}{x^2}$

$\therefore \left[f(x)\right]^2 - f(x^2) = \left(x^2 + 2 + \frac{1}{x^2}\right) - \left(x^2 + \frac{1}{x^2}\right)$

$= 2$

(ii) $f(x) = x + \frac{1}{x}$

$f(-x) = -x + \frac{1}{-x}$

$= -\left(x + \frac{1}{x}\right)$

$= -f(x)$ $\therefore$ ODD

Question 6 (a) $l_1: 5x+y-6=0 \qquad l_2: 2x-3y+1=0$

(i) $5x+y-6+k(2x-3y+1)=0$

Sub $(3,-2)$

$15-2-6+k(6+6+1)=0$

$7+13k=0 \qquad \therefore k=-\frac{7}{13}$

$\therefore (5x+y-6)-\frac{7}{13}(2x-3y+1)=0$

$13(5x+y-6)-7(2x-3y+1)=0$

$65x+13y-78-14x+21y-7=0$

$51x+34y-85=0$

$\therefore 3x+2y-5=0$

Alternatively Solve simultaneously

Sub $l_1: y=6-5x$ into l_2

$\therefore 2x-3(6-5x)+1=0$

$17x=17$

$\therefore x=1$

$\therefore y=1 \qquad \therefore A(1,1)$

eq. of the line is

$y-1=-\frac{3}{2}(x-1)$

$\therefore 3x+2y-5=0$

(ii) $(0,0)$ to $3x+2y-5=0$

$D=\frac{|0+0-5|}{\sqrt{9+4}}=\frac{5}{\sqrt{13}}$ units

(b) $f(x) = x^2\sqrt{4x-3}$

(i) all real x, $x \geqslant \frac{3}{4}$

(ii) $f'(x) = 2x(4x-3)^{\frac{1}{2}} + x^2 \times \frac{1}{2} \times 4 \times (4x-3)^{-\frac{1}{2}}$

$= 2x(4x-3)^{\frac{1}{2}} + 2x^2(4x-3)^{-\frac{1}{2}}$

$= 2x(4x-3)^{-\frac{1}{2}}(4x-3+x)$

$= 2x(4x-3)^{-\frac{1}{2}}(5x-3)$

$= \dfrac{2x(5x-3)}{\sqrt{4x-3}}$

(iii) $f'(3) = \dfrac{6(12)}{3} = 24$

$\therefore y - 27 = 24(x-3)$

$y = 24x - 45$

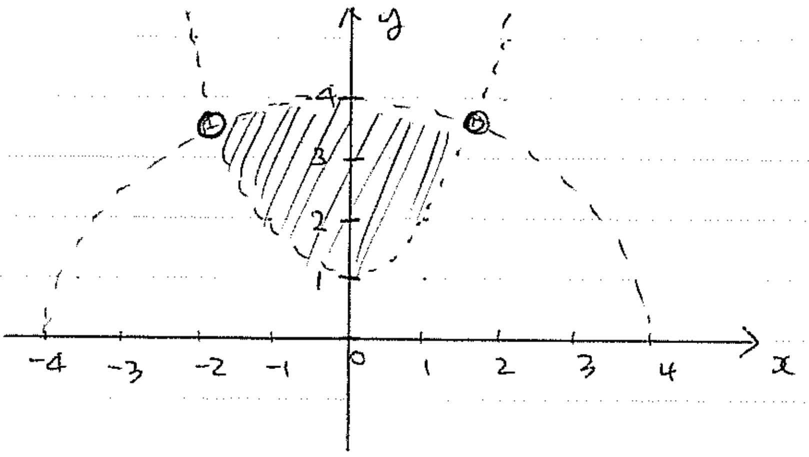
(iv) $f'(x) = 0$

$\dfrac{2x(5x-3)}{\sqrt{4x-3}} = 0$ $\quad (x \neq \frac{3}{4})$

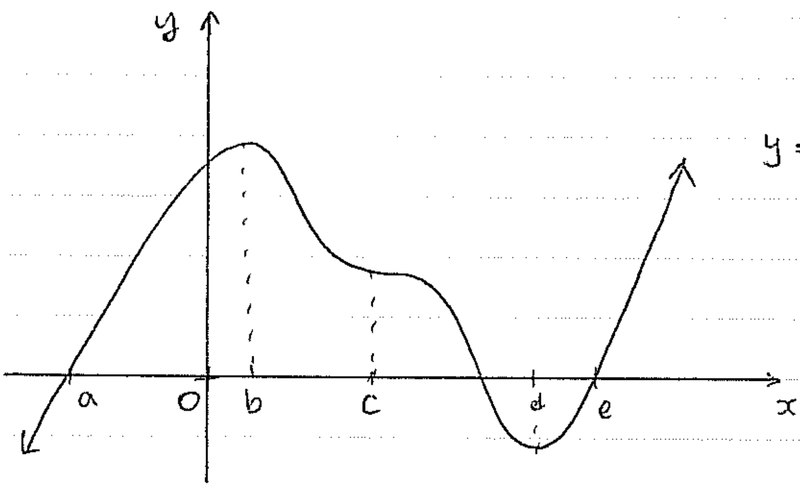
$\therefore x = 0, \frac{3}{5}$

BUT, these values are outside the domain from (i), so there are NO points with horizontal tangents.

Question 7. (a) $y > x^2+1$ $\quad y < \sqrt{16-x^2}$

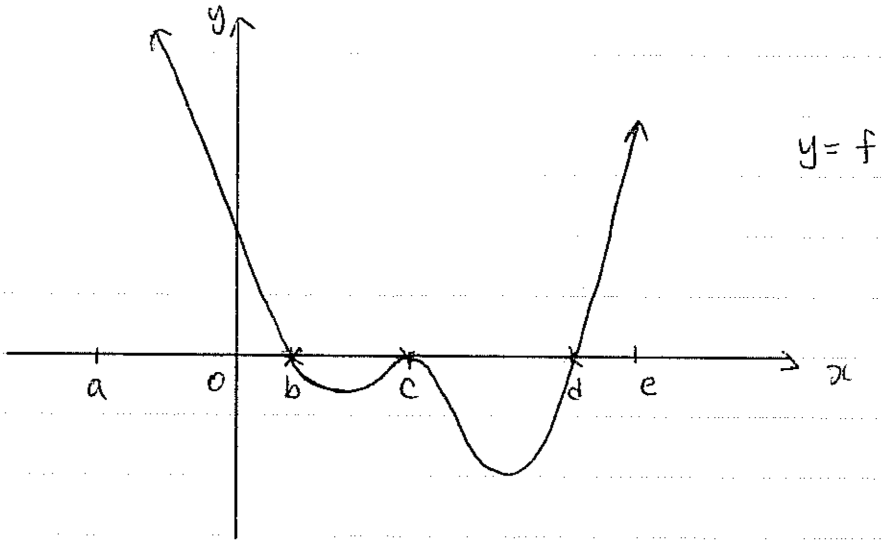


(b)



(i) $x < b$, $\quad x > d$

(ii)



(C) (i) $\angle BAC + x + y = 180^\circ$ (angle sum of Δ)

$\therefore \angle BAC = 180^\circ - (x+y)$

(ii) $\sin(180^\circ - (x+y)) = \sin(x+y)$

$\therefore$ In ΔABC

$$\frac{\sin x}{7} = \frac{\sin y}{6} = \frac{\sin(x+y)}{10}$$

Now, $\dfrac{\sin x}{7} = \dfrac{\sin(x+y)}{10}$

$$\sin x = \frac{7\sin(x+y)}{10}$$

$$\frac{\sin y}{6} = \frac{\sin(x+y)}{10}$$

$$\sin y = \frac{6\sin(x+y)}{10}$$

$$\begin{aligned} LHS &= \sin x + \sin y \\ &= \frac{7\sin(x+y)}{10} + \frac{6\sin(x+y)}{10} \\ &= \frac{13}{10}\sin(x+y) \\ &= RHS. \end{aligned}$$