NORTH SYDNEY GIRLS HIGH SCHOOL

2013

YEARLY EXAMINATION

Preliminary Mathematics

General Instructions

Reading Time – 5 minutes
Working Time – 2 hours
Write using black or blue pen

Board-approved calculators may be used
All necessary working should be shown in every question.

Total Marks – 84

Attempt Questions 1 – 7
All questions are of equal value.
At the end of the examination, place your solution booklets in order and place them inside your question paper.
Submit ONE bundle.

Student Name: ____________________ **Student Number:** ____________

Teacher: ____

QUESTION	MARK
1	/12
2	/12
3	/12
4	/12
5	/12
6	/12
7	/12
TOTAL	/84

Total Marks – 84
Attempt Questions 1 – 7
Questions are of equal value.

Begin each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 Marks) Use a SEPARATE writing booklet.

(a) Evaluate $\sqrt[3]{\dfrac{1.2\times10^4-3.3\times10^2}{2.48}}$ correct to three significant figures. **2**

(b) Find a and b such that $(2-5\sqrt{3})^2=a+b\sqrt{3}$. **2**

(c) Write $\dfrac{3-q}{3}-\dfrac{2+q}{4}$ as a single fraction in simplest form. **2**

(d) Write $\dfrac{8^x-18^x}{12^x-27^x}$ in simplest form. **2**

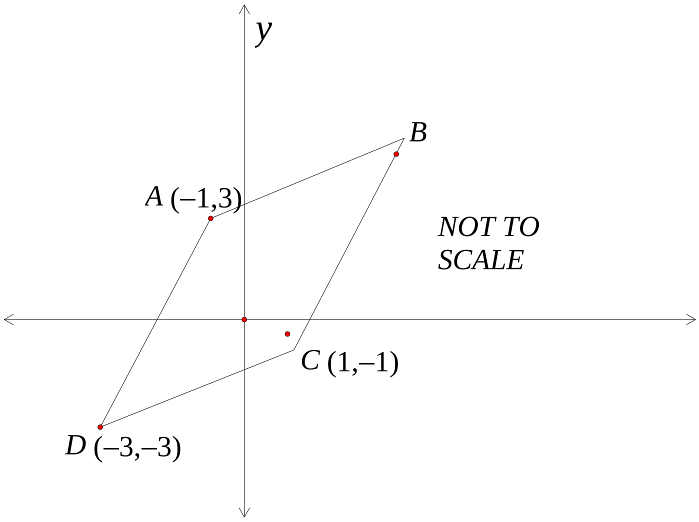
(e) Solve $3x^2-7x=11$. Leave your answer in exact form. **2**

(f) Solve $|x+2|\geq 5$. **2**

Question 2 (12 Marks) Use a SEPARATE writing booklet.

(a) Fully factorise $8x^3 - 27$. **2**

(b) In the diagram below, *ABCD* is a parallelogram and the points *A*, *C* and *D*, are $(-1,3)$, $(1,-1)$ and $(-3,-3)$ respectively.



(i) Find the length of the interval *DC*. **1**

(ii) Show that the equation of the line passing through *D* and *C* is $x - 2y - 3 = 0$. **2**

(iii) Determine the angle of inclination that the line through *DC* makes with *x*-axis to the nearest minute.. **1**

(iv) Find the midpoint of *AC*. **1**

(v) Hence or otherwise, find the co-ordinates of *B*. **2**

(vi) Find the perpendicular distance from *A* to *DC*. **2**

(vii) Find the area of the parallelogram *DABC*. **1**

Question 3 (12 Marks) Use a SEPARATE writing booklet.

(a) Differentiate with respect to x:

(i) $2x^3 - 7x + 5$. **1**

(ii) $\dfrac{5}{x^3} - 2\sqrt{x}$. **2**

(iii) $\left(1 - 8x^2\right)^5$. **2**

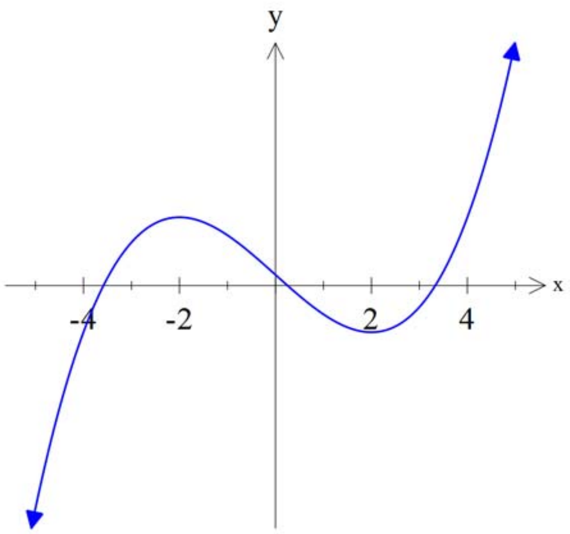
(b) (i) Given $f(x) = (x+1)(2+3x)^6$ find $f'(x)$. **2**

(ii) Find the values for x for which the tangent is horizontal. **2**

(c) Find the equation of the normal to the curve $y = 2x^3 + 4x - 5$ at the point where $x = 1$. **3**

Question 4 (12 Marks) Use a SEPARATE writing booklet.

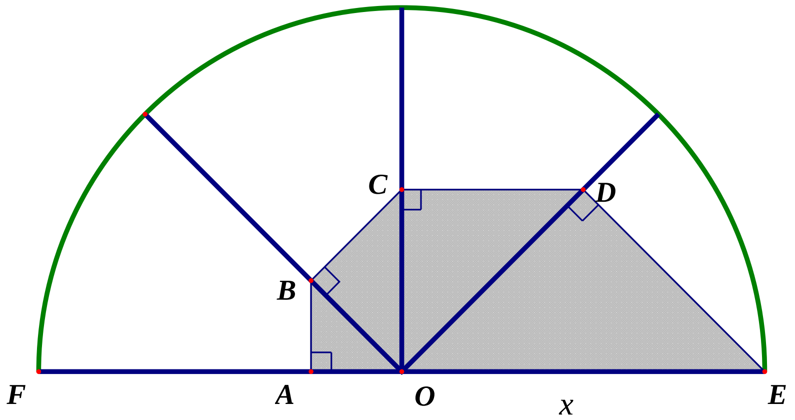
(a) Below is the graph of $y = f(x)$.



Sketch the graph of $y = f'(x)$. **2**

(b) Solve $2\cos^2\theta - 1 = 0$ for $0° \le \theta \le 360°$. **3**

(c) The diagram below shows a semicircle with radius x.
The semicircle is divided into four equal sectors.
$ED \perp OD$, $DC \perp OC$, $CB \perp OB$ and $BA \perp OA$.



(i) Explain briefly why ΔODE is an isosceles triangle. **1**

(ii) Calculate the exact length of OD in terms of x. **1**

(iii) Prove that $\Delta ODE \,|||\, \Delta OCD$. **2**

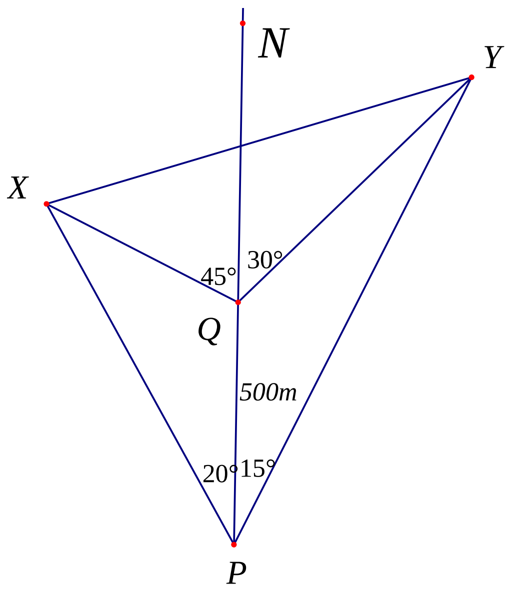
(iv) Calculate the ratio of the area of ΔOAB to the area of the shaded region $ABCDE$. **3**

Question 5 (12 Marks) Use a SEPARATE writing booklet.

(a) Sketch the graph of $y = f(x)$, where $f(x)$ is defined below. **4**

$$f(x)=\begin{cases}\dfrac{1}{x}, & \text{for} \quad x<-2\\ x+1, & \text{for} \quad -2\le x<0\\ 4-x^2, & \text{for} \quad x\ge 0\end{cases}$$

(b) A sailor at point *P* observes two buoys at *X* and *Y* on a bearing of $340°$ and $015°$ respectively.
He sails due North for 500 m to a point *Q* and then observers *X* and *Y* on a bearing of $315°$ and $030°$.



NOT TO SCALE

(i) Show that $XP = \dfrac{250\sqrt{2}}{\sin 25°}$. **2**

(ii) Show that $YP = \dfrac{250}{\sin 15°}$ **2**

(iii) Find the distance *XY* to the nearest metre. **2**

(iv) Find the bearing of *X* from *Y*. **2**

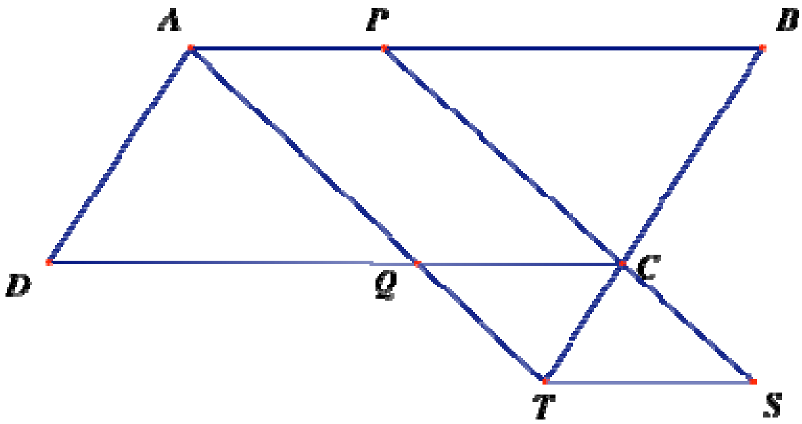
Question 6 (12 Marks) Use a SEPARATE writing booklet.

(a) Find the points of intersection of $2x - y - 5 = 0$ and $y = \dfrac{9}{x-3} - 6$ **3**

(b) Sketch the region of the number plane where the following hold simultaneously: **3**

$$(x-5)^2 + y^2 \le 25, \qquad y < x, \qquad \text{and} \qquad y \ge 0$$

(c) In the following diagram *ABCD* is a parallelogram.
$AD = AQ = PC$. AQ and BC are produced to intersect at T.
PC is produced to S such that $\angle PST = \angle SPB$.



(i) Prove that $\Delta DAQ \equiv \Delta BCP$. **3**

(ii) Show that $\angle AQD = \angle PCQ$. **1**

(iii) Hence, prove that *APST* is a parallelogram. **2**

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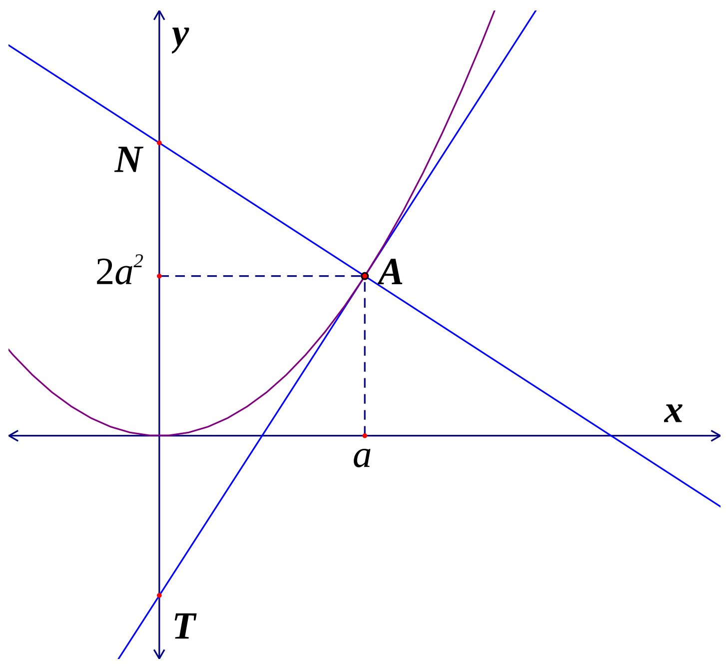
Question 7 is on the next page

Question 7 (12 Marks) Use a SEPARATE writing booklet.

(a) (i) Prove that $\dfrac{\tan x}{\sec x+1}+\dfrac{\tan x}{\sec x-1}=2\operatorname{cosec} x$. **2**

(ii) Hence solve $\dfrac{\tan x}{\sec x+1}+\dfrac{\tan x}{\sec x-1}=4$ for $0\le x\le 360°$. **2**

(b) In the diagram below, the function $y=2x^2$ has been drawn with a point $A(a,2a^2)$ such that $a>0$.
The tangent at the point $A(a,2a^2)$ intersects the *y*-axis at *T* and the normal at *A* intersects the *y*-axis at *N*.



(i) Show that the point *T* has coordinates $(0,-2a^2)$. **2**

(ii) Show that the point *N* has coordinates $\left(0,\dfrac{1+8a^2}{4}\right)$. **2**

(iii) Show that the area of ΔTAN is equal to $\dfrac{a}{8}+2a^3$. **2**

(iv) Find the area of ΔTAN when it is an isosceles triangle. **2**

End of paper.

QUESTION 1

$a)$

$$\sqrt[3]{\frac{1.2\times10^4-3.3\times10^2}{2.48}}$$

$$=16.75739056$$

$$=16.8$$

$b)$

$$(2-5\sqrt{3})^2$$

$$=2^2-2\times2\times5\sqrt{3}+\left(5\sqrt{3}\right)^2$$

$$=4-20\sqrt{3}+75$$

$$=79-20\sqrt{3}$$

$$\therefore a=79 \qquad b=-20$$

$c)$

$$\frac{3-q}{3}-\frac{2+q}{4}$$

$$\frac{4(3-q)-3(2+q)}{12}$$

$$=\frac{12-4q-6-3q}{12}$$

$$=\frac{6-7q}{12}$$

$d)$

$$\frac{8^x-18^x}{12^x-27^x}$$

$$\frac{2^{3x}-(3^{2x})(2^x)}{(2^{2x})(3^x)-(3^{3x})}$$

$$\frac{(2^x)\left(2^{2x}-3^{2x}\right)}{(3^x)\left(2^{2x}-3^{2x}\right)}=\frac{2^x}{3^x}$$

$e)$

$$3x^2-7x=11$$

$$3x^2-7x-11=0$$

$$x=\frac{7\pm\sqrt{(-7)^2-4(3)(-11)}}{2(3)}$$

$$x=\frac{7\pm\sqrt{49+132}}{6}$$

$$x=\frac{7\pm\sqrt{181}}{6}$$

$f)$

$$|x+2|\ge5$$

$$x+2\ge5 \qquad OR \qquad -(x+2)\ge5$$

$$x\ge3 \qquad\qquad -(x+2)\ge5$$

$$-x-2\ge5$$

$$-x\ge7$$

$$x\le-7$$

QUESTION 2

$a)$

$$8x^3-27$$

$$(2x-3)\left(4x^2+6x+9\right)$$

$b)i)$

$$d_{DC}=\sqrt{(-3--1)^2+(-3-1)^2}$$

$$d_{DC}=\sqrt{4+16}=\sqrt{20}=2\sqrt{5}$$

$b)ii)$

$$m_{DC}=\frac{(-3--1)}{(-3-1)}=\frac{-2}{-4}=\frac{1}{2}$$

Equation of DC

$$y-(-1)=\frac{1}{2}(x-1)$$

$$2y+2=x-1$$

$$0=x-2y-3$$

$b)iii)$

Let θ be angle of inclination.

$$\tan\theta=m_{DC}=\frac{1}{2}$$

$$\tan^{-1}\frac{1}{2}=\theta$$

$$\theta=26.56505188$$

$$\theta=26°34'$$

$b)iv)$

$Midpo\,\mathrm{int}_{AC}$

$$\left(\frac{-1+1}{2},\frac{3+-1}{2}\right)$$

Midpoint is $(0,1)$

$b)v)$

$\text{Midpoint}_{AC}=\text{Midpoint}_{DB}$

$$0=\frac{-3+x_B}{2}$$

$$x_B=3$$

$$1=\frac{-3+y_B}{2}$$

$$y_B=5$$

Co-ordinates of B are $(3,5)$

$b)vi)$

Equation of DC

$$0=x-2y-3$$

Point $A(-1,3)$

$$d=\left|\frac{(1)(-1)+(-2)(3)+(-3)}{\sqrt{(1)^2+(-2)^2}}\right|$$

$$d=\left|\frac{-1-6-3}{\sqrt{5}}\right|=\frac{10}{\sqrt{5}}$$

$b)vii)$

$$Area_{PARALLELOGRAM}=Perp\ Height\times Base$$

$$Area_{ABCD}=DC\times\frac{10}{\sqrt{5}}$$

$$=2\sqrt{5}\times\frac{10}{\sqrt{5}}=20\text{ units}^2$$

QUESTION 3

$a)$

$$i)\quad \frac{d}{dx}2x^3-7x+5=6x^2-7$$

$$ii)\ \frac{d}{dx}\left(\frac{5}{x^3}-2\sqrt{x}\right)=\frac{d}{dx}\left(5x^{-3}-2x^{\frac{1}{2}}\right)$$

$$=-15x^{-4}-x^{-\frac{1}{2}}$$

$$=-\frac{15}{x^4}-\frac{1}{\sqrt{x}}$$

$$iii)\quad \frac{d}{dx}\left(1-8x^2\right)^5=5\left(1-8x^2\right)^4(-16x)$$

$$=-80x\left(1-8x^2\right)^4$$

$b)i)$

$f(x)=(x+1)(2+3x)^6$

$f'(x)=u'v+v'u$

$f'(x)=1\times(2+3x)^6+6(2+3x)^5\times 3(x+1)$

$f'(x)=(2+3x)^6+18(2+3x)^5(x+1)$

$ii)$

Horizontal tangent when $f'(x)=0$

$0=(2+3x)^5\left[(2+3x)+18\left(x+1\right)\right]$

$0=(2+3x)^5\left(20+21x\right)$

$x=-\frac{2}{3} \qquad x=-\frac{20}{21}$

$c)$

$y=2x^3+4x-5$

$\frac{dy}{dx}=6x^2+4$

$At x=1$

$\frac{dy}{dx}=6\left(1\right)^2+4$

$=10$

$\therefore m_{Tangent}=10$ when $x=1$

$\therefore m_{Normal}=\frac{-1}{10}$ when $x=1$

and $y=2\left(1\right)^3+4\left(1\right)-5=1$

Equation of Normal

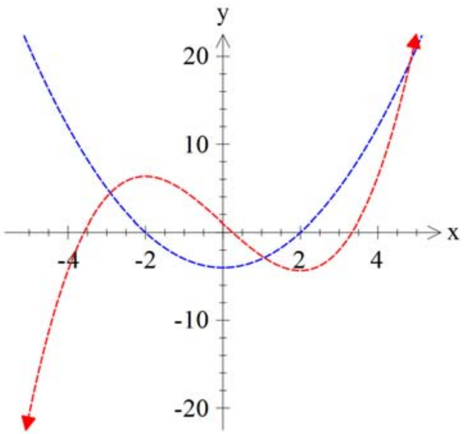
$y-1=-\frac{1}{10}\left(x-1\right)$

$10y-10=-x+1$

$x+10y-11=0$

QUESTION 4

a)



b)

$2\cos^2\theta-1=0$ for $0°\le\theta\le 360°$.

$\cos^2\theta=\frac{1}{2}$

$\cos\theta=\pm\frac{1}{\sqrt{2}}$

Relalted angle

$\cos^{-1}\frac{1}{\sqrt{2}}=45°$

Solutions are in all four quadrants

$x=45°$

$x=180°-45°=135°$

$x=180°+45°=225°$

$x=360°-45°=315°$

c)i)

The semi circle is divided into four equal sectors so

$\angle EOD=180\div 4=45°$

$\angle EOD+\angle EDO+\angle DEO=180°$

$\left(Angle \text{ sum of } \Delta ODE\right)$

$\therefore \angle DEO=45°$

$\therefore \Delta ODE$ is an isosceles triangle

(Two equal angles)

$c)ii)$

$x^2=\left(OD\right)^2+\left(ED\right)^2$

$x^2=\left(OD\right)^2+\left(OD\right)^2$

$x^2=2\left(OD\right)^2$

$\frac{x^2}{2}=\left(OD\right)^2$

$\frac{x}{\sqrt{2}}=OD$

$c)iii)$

$\angle EOD=\angle DOE=180\div 4=45°$

$\left(\begin{array}{c} The\ semi\ circle\ is\ divided \\ into\ four\ equal\ sectors\end{array}\right)$

$\angle EDO=\angle DCO=90°$

$(DC\perp CO \text{ and } ED\perp DO)$

$\therefore EDO\,|||\,\angle DCO(equiangular)$

$c)iv)$

$\frac{x^2}{2}=2\left(OC\right)^2$

$\frac{x^2}{4}=\left(OC\right)^2$

$\frac{x}{2}=OC$

Similarily $\frac{x}{2\sqrt{2}}=OB$ and $\frac{x}{4}=OA$

Area of ΔABO

$\frac{1}{2}\left(\frac{x}{4}\right)^2=\frac{x^2}{32}$

Area of $ABCDE$

$=\frac{1}{2}\left(\left(\frac{x}{4}\right)^2+\left(\frac{x}{2\sqrt{2}}\right)^2+\left(\frac{x}{2}\right)^2+\left(\frac{x}{\sqrt{2}}\right)^2\right)$

$=\frac{1}{2}\left(\left(\frac{x^2}{16}\right)+\left(\frac{x^2}{8}\right)+\left(\frac{x^2}{4}\right)+\left(\frac{x^2}{2}\right)\right)$

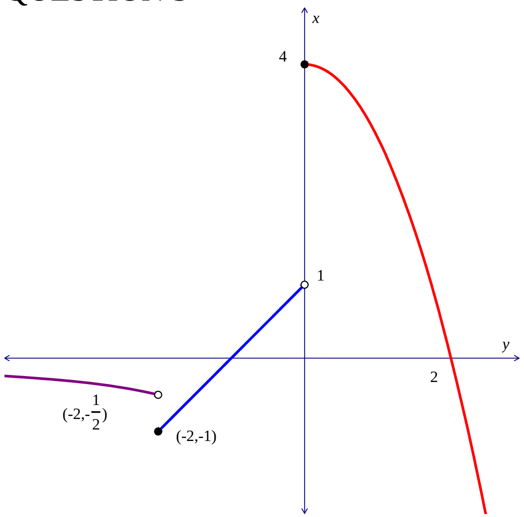
$=\frac{x^2+2x^2+4x^2+8x^2}{32}=\frac{15x^2}{32}$

Area of ABO : Area of $ABCDE$

$\frac{x^2}{32}:\frac{15x^2}{32}$

$1:15$

QUESTION 5



b)i)

$\angle XQP = 180° - 45° = 135°$

$\angle QXP = 180° - 135° - 20 = 25°$

Using the sine rule in ΔQPY

$$\frac{XP}{\sin 135°} = \frac{500}{\sin 25°}$$

$$XP = \frac{500}{\sin 25°} \times \frac{1}{\sqrt{2}}$$

$$XP = \frac{250\sqrt{2}}{\sin 25°}$$

bii)

$\angle YQP = 180° - 30° = 150°$

$\angle QYP = 180° - 150° - 15° = 15°$

Using the sine rule in ΔQPY

$$\frac{YP}{\sin 150°} = \frac{500}{\sin 25°}$$

$$YP = \frac{500}{\sin 25°} \times \frac{1}{2}$$

$$YP = \frac{250}{\sin 25°}$$

b)iii)

$\angle YPX = 15° + 20° = 35°$

Using the cosine rule in ΔXPY

$$(YX)^2 = (YP)^2 + (XP)^2 - 2(XP)(YP)\cos(\angle XPY)$$

$$(YX)^2 = \left(\frac{250}{\sin 15°}\right)^2 + \left(\frac{250\sqrt{2}}{\sin 25°}\right)^2 - 2\left(\frac{250}{\sin 15°}\right)\left(\frac{250\sqrt{2}}{\sin 25°}\right)\cos(35)$$

$$(YX)^2 = \left(\frac{250}{\sin 15°}\right)^2 + \left(\frac{250\sqrt{2}}{\sin 25°}\right)^2 - 2\left(\frac{250}{\sin 15°}\right)\left(\frac{250\sqrt{2}}{\sin 25°}\right)\cos(35)$$

$$(YX)^2 = 309007.3595$$

$$YX = 555.884304m$$

$$YX \approx 556m$$

b)iv)

Construct $N_Y S_Y$ through Y such that

$N_Y S_Y \parallel PN$

$\angle NPY = \angle PYS_Y = 15$(alternate angles)

$Bearing_{\text{of } X \text{ from } Y} = 180° + 15° + \angle PYX$

Using the cosine rule in ΔXPY

$$\frac{(PX)^2 - (XY)^2 - (PY)^2}{2(XY)(PY)} = \cos(\angle PYX)$$

$$\frac{\left(\frac{250\sqrt{2}}{\sin 25°}\right)^2 - (556)^2 - \left(\frac{250}{\sin 15°}\right)^2}{2(556)\left(\frac{250}{\sin 15°}\right)} = \cos(\angle PYX)$$

$\angle PYX = 59°40'$

$Bearing_{\text{of } X \text{ from } Y} = 180° + 15° + 59°40'$

$= 254°40'$

a)

$y = \dfrac{9}{x-3} - 6$ *and* $2x - y - 5 = 0$

$\dfrac{9}{x-3} - 6 = 2x - 5$

$\dfrac{9}{x-3} = 2x + 1$

$9 = (2x+1)(x-3)$

$9 = 2x^2 - 5x - 3$

$0 = 2x^2 - 5x - 12$

$0 = (2x+3)(x-4)$

$x = 4 \qquad x = -\dfrac{3}{2}$

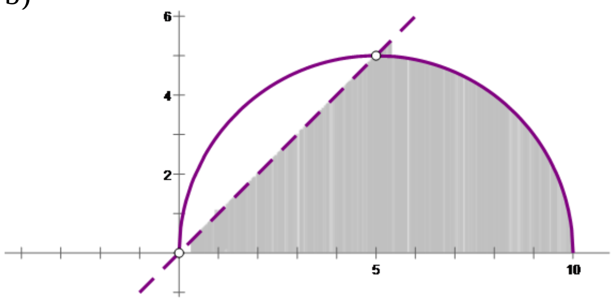
$y = \dfrac{9}{4-3} - 6 \qquad y = \dfrac{9}{-\frac{3}{2} - 3} - 6$

$y = 3 \qquad y = -8$

Points of intersection

(4,3) and $(-\frac{3}{2}, 6)$

b)



i

In ΔADQ and ΔBCP

$AD = BC$ (Opposite sides in quadrilateral)

$\therefore AD = AQ = PC = BC$

$\angle ADQ = \angle CBP$ (Opposite angles in quadrilateral)

$\therefore \angle ADQ = \angle AQD$ (Equal angles opposite equal sides)

$\angle PBC = \angle BPC$ (Equal angles opposite equal sides)

$\therefore \angle BPC = \angle AQD$

$\therefore \Delta ADQ \equiv \Delta BCP$ (AAS)

ii

$AD = BC$ (Opposite sides in quadrilateral)

$\angle BPC = \angle AQD$ (From part i)

$AB \parallel DC$ (opposite side of quadrilateral)

$\angle BPC = \angle PCQ$ (Alternate angles in parallel lines.)

$\therefore \angle AQD = \angle PCQ$

iii)

$\angle AQD = \angle PCQ$ (Part ii)

$\therefore AT \parallel PS$ (Corresponding angles are equal)

$\angle TSP = \angle BPC$ (Given)

$\therefore AB \parallel TS$ (Alternate angles are equal)

$\therefore APST$ is a parallelogram (Opposite sides are parallel)

QUESTION 7

a)i

$$\frac{\tan x}{\sec x + 1} + \frac{\tan x}{\sec x - 1} = 2\operatorname{cosec} x$$

$$LHS = \frac{\tan x}{\sec x + 1} + \frac{\tan x}{\sec x - 1}$$

$$= \frac{\tan x(\sec x + 1) + \tan x(\sec x - 1)}{\sec^2 x - 1}$$

$$= \frac{2\sec x \tan x}{\tan^2 x}$$

$$= \frac{2\sec x}{\tan x} = 2 \times \frac{1}{\cos x} \times \frac{\cos x}{\sin x}$$

$$= \frac{2}{\sin x} = 2\operatorname{cosec} x = RHS$$

a)ii)

$$\frac{\tan x}{\sec x + 1} + \frac{\tan x}{\sec x - 1} = 4$$

$$2\operatorname{cosec} x = 4$$

$$\operatorname{cosec} x = 2$$

$$\sin x = \frac{1}{2}$$

x lies in the first and second quadrant

$x = 30°$

$x = 150°$

b)*i*)

$y = 2x^2$

$y' = 4x$

When $x = a$ $y' = 4a$

Equation of tangent:

$y - 2a^2 = 4a(x - a)$

When $x = 0$

$y - 2a^2 = 4a(0 - a)$

$y = 2a^2 - 4a^2 = -2a^2$

Co-ordinates of T are $(0, -2a^2)$

b)*ii*)

Equation of normal:

$$m_{NORMAL} = -\frac{1}{4a}$$

$$y - 2a^2 = -\frac{1}{4a}(x - a)$$

When $x = 0$

$$y - 2a^2 = -\frac{1}{4a}(0 - a)$$

$$y = \frac{1}{4} + 2a^2$$

$$y = \frac{1 + 8a^2}{4}$$

Co-ordinates of N are $\left(0, \frac{1 + 8a^2}{4}\right)$

$b)iii)$

Area of ΔNTA

If the base is NT

then the perpendicular height is a

$$Area = \frac{1}{2} \times NT \times a$$

$$Area = \frac{1}{2} \times \left(\frac{1+8a^2}{4} + 2a^2 \right) \times a$$

$$Area = \left(\frac{1+16a^2}{8} \right) \times a$$

$$Area = \frac{a+16a^3}{8} = \frac{a}{8} + 2a^3$$

$b)iii)$

First find the value of so that ΔNTA is isosceles

$Method 1$

ΔNTA is isosceles when

$\angle NTA = \angle TNA = 45$

If $\angle NTA = 45$ then $m_{AT} = \tan 45° = 1$

$m_{AT} = 4a = 1$

$$a = \frac{1}{4}$$

$Method 2$

ΔNTA is isosceles when

$AT = AN$

$$a^2 + \left(\frac{1+2a^2}{4} - 2a \right)^2 = a^2 + \left(-2a - 2a \right)^2$$

$$a^2 + \left(\frac{1}{4} \right)^2 = a^2 + \left(-4a^2 \right)^2$$

$$a^2 + \frac{1}{16} = a^2 + 16a^4$$

$$\frac{1}{16} = 16a^4$$

$$\frac{1}{256} = a^4$$

$$a = \pm \frac{1}{4}$$

$$a = \frac{1}{4}$$

$Mathod 3$

ΔNTA is isosceles when

y-value of $A(a, 2a^2)$ is the midpoint of NT

$$2a^2 = \frac{1}{2}\left(\frac{1+8a^2}{4} - 2a^2 \right)$$

$$4a^2 = \frac{1+8a^2}{4} - 2a^2$$

$$16a^2 = 1 + 8a^2 - 8a^2$$

$$a^2 = \frac{1}{16}$$

$$a = \pm \frac{1}{4}$$

$$a = \frac{1}{4}$$

$$Area of\ \Delta NTA = \frac{a}{8} + 2a^3$$

$$= \frac{1}{4} \times \frac{1}{8} + 2\left(\frac{1}{4} \right)^3$$

$$= \frac{1}{32} + \frac{1}{32} = \frac{1}{16}$$