St George Girls High School

Year 11

End of Preliminary Course Examination

2008

Mathematics

Time Allowed: 3 hours
(plus 5 minutes reading time)

Instructions

1. Attempt all 7 questions.
2. All questions are of equal value.
2. All necessary working must be shown.
3. Begin each question on a **new page**.
4. Marks will be deducted for careless work or poorly presented solutions.

Question 1 – (15 marks) **Marks**

a) Calculate to three significant figures $\sqrt{\dfrac{13.5^2 - 4.3^3}{29.2 \times 15.4}}$ 1

b) Expand and simplify $(3-2x)^2 - 4(3x+5)$ 2

c) Solve $2x^2 - 15 \geq -7x$. 2

d) Rationalise the denominator of $\dfrac{7+4\sqrt{3}}{7-4\sqrt{3}}$ 2

e) Simplify $\cot\theta \times \sin\theta$ 1

f) What is the exact value of cosec 240°? 2

g) Find the derivatives of: 5

(i) $y = 2\sqrt{x}$

(ii) $y = (3x-4)^4$

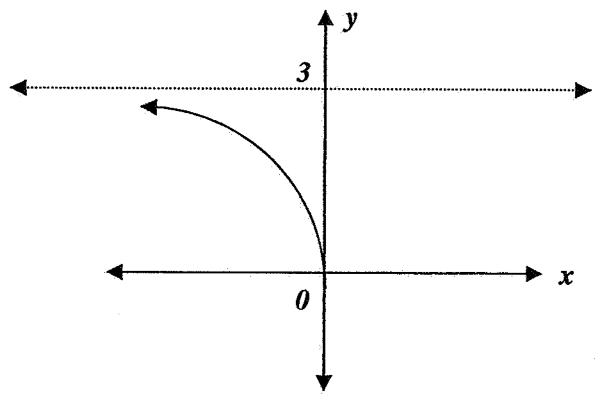
(iii) $y = \dfrac{2x-1}{2x+1}$

Question 2 – (15 marks) **Marks**

a) Simplify $\frac{2x}{x-3}+\frac{3x}{x-2}$ 2

b) Find the equation of the tangent to the curve $y=3x^2-2x+1$ at the point $P(-1, 6)$ 3

c) Using the graph drawn below. 4



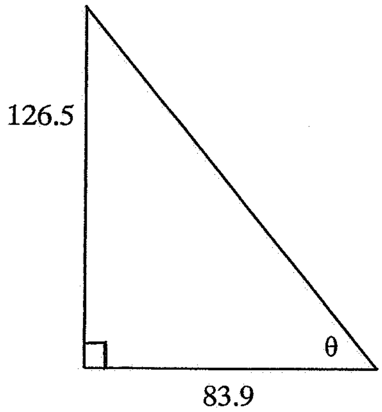
(i) Copy the graph onto your answer booklet and complete the graph if it is known to be odd.

(ii) Write down the domain and range of the completed function.

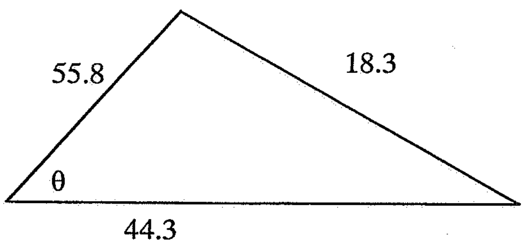
Question 2 (cont'd) **Marks**

d) Calculate θ correct to the nearest minute. 4

(i)



(ii)



e) Simplify $14x^{-2}y^{-1} \div 7^2xy^{-3}$ giving the answer in index form without negative indices. 2

Question 3 – (15 marks)

Marks

a) Show that the points $A(1, -1)$, $B(3, 5)$ and $C(4, 8)$ are collinear. 3

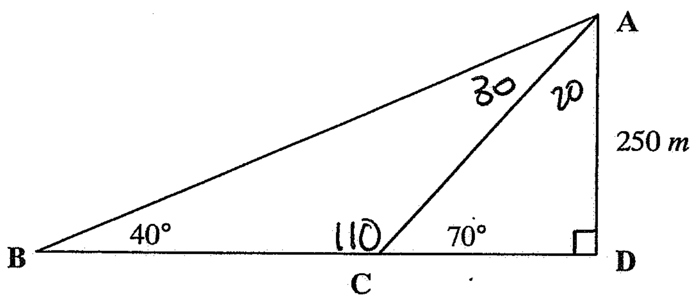
b) Using the sequence 8, 11, 14 … , find 5

(i) the formula for the nth term in its simplest form.

(ii) the first term which is greater than 1000.

(iii) the value of the sum to twenty terms.

c) Given the following diagram. 7



(i) Show that AC = $\dfrac{250}{\sin 70^\circ}$

(ii) Prove that $B\hat{A}C = 30^\circ$

(iii) Calculate BC correct to three significant figures

(iv) Calculate the area of ΔBAC.

Question 4 – (15 marks) **Marks**

a) Solve simultaneously $2x - y = -1$ and $x + y = 4$ 2

b) Find the centre and radius of the circle by completing the squares of $x^2 - 4x + y^2 - 5 = 0$ 3

c) Solve for θ where $0^\circ \le \theta \le 360^\circ$: $\tan 2\theta = -\sqrt{3}$ 3

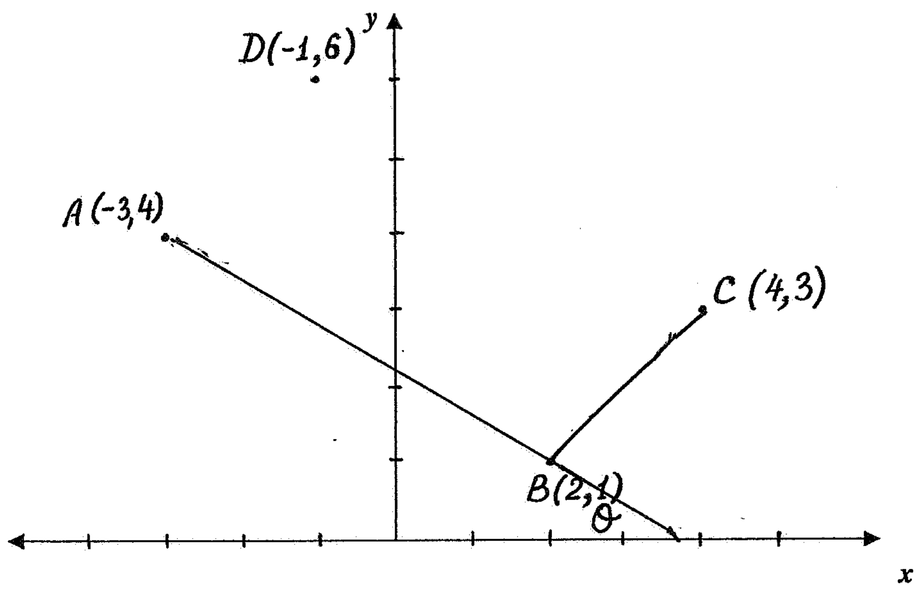
d) For a certain arithmetic sequence the third term is 3 and the tenth term is 38. Determine the common difference. 3

e) If $f(x) = x^2 + 2x$ 4

(i) Find $f(3+h)$ in its simplest form.

(ii) Evaluate $\lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$

Question 5 – (15 marks) **Marks**



ABCD is a parallelogram.

a) Find the length of *BC*. 2

b) Show that the equation of the line *BC* is $x - y - 1 = 0$. 3

c) Show that *AB* is parallel to *DC*. 2

d) Find the perpendicular distance from *A* to *BC*. 2

e) Calculate the area of the parallelogram *ABCD*. 1

f) What is the equation of the line through *A* perpendicular to *BC*? 3

g) The line through the points *A* and *B* makes an angle θ with the positive *x*-axis. Find θ to the nearest minute. 2

Question 6 – (15 marks) **Marks**

a) At the point (1, 2) on the curve $y = x^3 - 3x + 2$, the value of $\frac{dy}{dx}$ is zero.
What does this tell us about the tangent to the curve at this point? 1

b) Sketch the function $y = x(x-1)^2$ showing the x and y intercepts. 2

c) Solve for x:

(i) $8^x = \frac{1}{16}$ 2

(ii) $x = \log_4 16$ 1

(iii) $x = \log_5 20$ to 3 decimal places 2

d) Consider the series $6 + 12 + 24 + \ldots$ 4

(i) Find S_n, the sum of the first n terms in simplest form.

(ii) How many terms are required for the sum to be 3066?

e) Using the sum of an infinite series express $0.1\dot{2}$ as a fraction in its simplest form. 3

Question 7 – (15 marks) Marks

a) Sketch any curve for which $\frac{dy}{dx}$ is positive. 1

b) The lines $3x+5y-9=0$ and $2x+y+1=0$ intersect at point B. Find the equation of the line through B and point $A(4, -2)$. 3

c) A circle has a diameter with endpoints $A(2, 4)$ and $B(8, 12)$. Show that $C(1, 11)$ lies on the circle. 3

d) If $\log_a 5 = m$ and $\log_a 2 = n$ find expressions, in simplest form, for: 3

i) $\log_a 10$

ii) $\log_a \left(\frac{2}{5}\right)$

iii) $\log_a \sqrt{2}$

e) Calculate x if $(2x-3)$, $(x+1)$, $(2-x)$ are the first three terms of an Arithmetic sequence. 2

f) Evaluate $\sum_{n=1}^{\infty} 16\left(\frac{1}{2}\right)^n$ 3

Question 8 – (15 marks) **Marks**

a) For the function $f(x) = x(3x-1)^4$ 3

(i) Find $f'(x)$

(ii) Evaluate $f'(1)$

b) Sketch the region bounded by $y > x - 2$ and $y \le 4 - x^2$ 3

c) Calculate the value of k if the interval joining the points $(1, k-1)$ to $(k, 2k-3)$ has a gradient of $\frac{3}{4}$. 2

d) On a number plane the points $O(0, 0)$, $P(8, 0)$, $Q(10, 5)$ and $R(2, 5)$ form a parallelogram. Show that the diagonals of the parallelogram bisect each other. 3

e) The numbers x, y and 9 are in arithmetic progression and x, y and 12 are in geometric progression. Hence find the values of x and y. 4

End of Paper

August 2008

Question 1

a) $\sqrt{\dfrac{102.743}{449.68}} = 0.477996\ldots$

$= 0.478$ (1)

b) $(3-2x)^2 - 4(3x+5)$ (1)

$= 9 - 2\times3\times2x + 4x^2 - 12x - 20$

$= 4x^2 - 24x - 11$ (1)

c) $2x^2 - 15 \geq -7x$

$2x^2 + 7x - 15 \geq 0$

$2x^2 + 10x - 3x - 15 \geq 0$

$2x(x+5) - 3(x+5) \geq 0$

$(2x-3)(x+5) \geq 0$

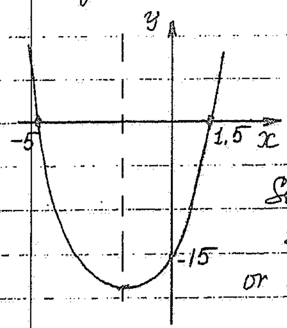
$\therefore$ zeros: $x = 1.5$ or $x = -5$ (1)

vertex: $x = \dfrac{1.5-5}{2}$

$x = -1.75$

$y = -21.125$

y-intercept is -15



Solution:

$x \leq -5$ (1)

or $x \geq 1.5$

d) $\dfrac{7+4\sqrt{3}}{7-4\sqrt{3}} \times \dfrac{7+4\sqrt{3}}{7+4\sqrt{3}}$ (1)

$= \dfrac{(7+4\sqrt{3})^2}{49-16\times3} = \dfrac{49+2\times7\times4\sqrt{3}+48}{1}$

$= 97 + 56\sqrt{3}$ (1)

e) $\cot\theta \times \sin\theta$

$= \dfrac{\cos\theta}{\sin\theta} \times \sin\theta = \cos\theta$ (1)

f) $\operatorname{cosec} 240° = \dfrac{1}{\sin 240°}$ (1)

$= \dfrac{1}{\sin(180°+60°)}$

$= \dfrac{1}{-\sin 60°}$

$= \dfrac{1}{-\frac{\sqrt{3}}{2}}$

$= -\dfrac{2}{\sqrt{3}}$ (1)

g)

(i) $\dfrac{dy}{dx} = \left(2x^{\frac{1}{2}}\right)' = x^{-\frac{1}{2}}$

$= \dfrac{1}{\sqrt{x}}$ (1)

(ii) Let $u = 3x-4$

then $y = u^4$

$\dfrac{dy}{du} = 4u^3 = 4(3x-4)^3$ (1)

$\dfrac{du}{dx} = 3$

$\dfrac{dy}{dx} = 12(3x-4)^3$ (1)

or using powers of a linear function formula

$\dfrac{dy}{dx} = 4\times3(3x-4)^3 = 12(3x-4)^3$ (1) (1)

(iii) Let $u = 2x-1$ and $v = 2x+1$

then $\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 2$ (1)

Using the quotient rule:

$\dfrac{dy}{dx} = \dfrac{(2x+1)\times2 - (2x-1)\times2}{(2x+1)^2} = \dfrac{4x+2-4x+2}{(2x+1)^2}$

$= \dfrac{4}{(2x+1)^2}$ (1)

End of Question 1

Question 2

a) $\frac{2x}{x-3} + \frac{3x}{x-2}$

$= \frac{2x(x-2) + 3x(x-3)}{(x-3)(x-2)}$ (1)

$= \frac{2x^2-4x+3x^2-9x}{(x-3)(x-2)}$

$= \frac{5x^2-13x}{(x-3)(x-2)}$ (1)

or $= \frac{5x^2-13x}{x^2-5x+6}$

b) $f'(x) = 6x-2$ (1)

$f'(-1) = -6-2 = -8$

Using point-gradient formula

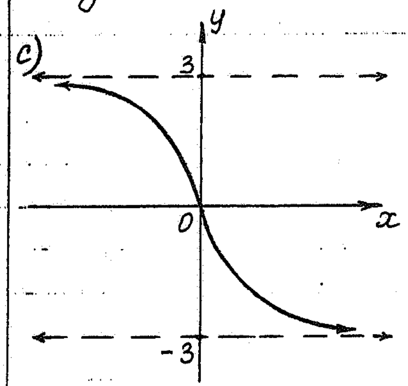
$m = -8$ $P(-1,6)$

$y-6 = -8(x+1)$ (1)

$y = -8x-8+6$

$y = -8x-2$ (1)

c)



(1) For the correct curve

(1) For the asymptote $y=-3$

Domain: all real x (1)

Range: $-3 < y < 3$ (1)

d)

(i) $\tan\theta = \frac{126.5}{83.9}$ (1)

$\theta = \tan^{-1}\left(\frac{126.5}{83.9}\right)$

$\theta = 56.4460267...^\circ$

$\approx 56°27'$ (1)

(ii) $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$\cos\theta = \frac{55.8^2+44.3^2-18.3^2}{2\times55.8\times44.3}$ (1)

$= \frac{4741.24}{4943.88}$

$\theta = \cos^{-1}\left(\frac{4741.24}{4943.88}\right)$

$= 16.461179...$

$\approx 16°28'$ (1)

e) $14x^{-2}y^{-1} \div 7^2xy^{-3}$

$= \frac{14}{x^2y} \div \frac{7^2x}{y^3}$ (1)

$= \frac{14}{x^2y} \times \frac{y^3}{49x} = \frac{2y^2}{7x^3}$ (1)

End of Question 2

Question 3

a)

(1) gradient of $AB = \frac{5+1}{3-1} = \frac{6}{2} = 3$

(1) gradient of $BC = \frac{8-5}{4-3} = \frac{3}{1} = 3$

gradient of AB = gradient of BC

Point B is common

∴ Points $A(1,-1)$, $B(3,5)$ & $C(4,8)$ are collinear. (1)

b) 8, 11, 14 ...

(i) $a=8$ $d=3$

$T_n = a+(n-1)d$

$T_n = 8+3(n-1)$

$T_n = 8+3n-3$

$= 3n+5$ (1)

(ii)

$3n+5 > 1000$ (1)

$3n > 1000-5$

$n > 995 \div 3$

$n > 331.67$

The first term which is greater than 1000 is the 332nd term (1) and its value is

$T_{332} = 8+331\times3$

$= 1001$

(iii)

$S_n = \frac{1}{2}n(2a+(n-1)d)$ (1)

$S_{20} = \frac{1}{2}\times20(2\times8+19\times3)$

$= 10\times38 = 730$ (1)

c)

(i) $\sin 70° = \frac{250}{AC}$ (opposite / hypotenuse)

$\therefore AC = \frac{250}{\sin 70°}$ (1)

(ii) In the ΔABC

$\angle ACB = 180°-70° = 110°$

(angles on a straight line add up to 180°) (1)

$\angle BAC = 180°-110°-40°$

$= 30°$

(Angle sum of a triangle is 180°) (1)

(iii)

$\tan 40° = \frac{250}{BD}$

$BD = \frac{250}{\tan 40°}$ (1/2)

$\tan 70° = \frac{250}{CD}$

$CD = \frac{250}{\tan 70°}$ (1/2)

$BC = BD - CD$

$= \frac{250}{\tan 40°} - \frac{250}{\tan 70°} = 206.9458...$

≈ 207 m (1)

(iv) Area $= \frac{1}{2}bc\sin A$

Area $= \frac{1}{2} AC\times BC\times\sin\angle ACB$

$= \frac{1}{2}\times\frac{250}{\sin 70}\times207\times\sin 110°$ (1)

$= \frac{1}{2}\times266\times207\times\sin 110°$

$= 25870.6775...$ (1)

≈ 25871 m² (to the nearest m)

Question 4

a) $2x - y = -1 \quad (1)$

$x + y = 4 \quad (2)$

(1)+(2)

$3x = 3$

$x = 1$ ①

Substitute x into (2)

$1 + y = 4$

$y = 3$ ①

$\therefore\ x = 1$ and $y = 3$

b)

$x^2 - 4x + y^2 - 5 = 0$

$x^2 - 4x + 4 + y^2 = 5 + 4$

$(x-2)^2 + y^2 = 9$ ①

Centre $O(2, 0)$ ①

Radius $r = \sqrt{9} = 3$ ①

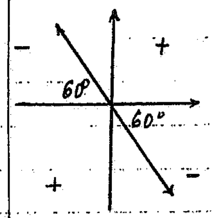
c) $\tan 2\theta = -\sqrt{3} \quad 0° \le \theta \le 360°$

Let $u = 2\theta$

Then $\tan u = -\sqrt{3}$

The restriction on u:

$0° \le u \le 720°$ ①



From the diagram: ①

$u = 120°, 300°, 480°$ or $660°$

$\theta = \frac{u}{2} = 60°, 150°, 240°$ or $330°$ ①

d)

$T_n = a + (n-1)d$

$T_3 = a + 2d = 3$ ①

$T_{10} = a + 9d = 38$ ①

Two simultaneous equations:

$a + 2d = 3 \quad (1)$

$a + 9d = 38 \quad (2)$

subtract (1) from (2):

$7d = 35$

$d = 5$ ①

e) $f(x) = x^2 + 2x$

(i) $f(3+h) = (3+h)^2 + 2(3+h)$

$= 3^2 + 2h \times 3 + h^2 + 2 \times 3 + 2h$

$= 15 + 8h + h^2$ ①

(ii) $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} =$

$\lim_{h \to 0} \frac{15 + 8h + h^2 - (9+6)}{h}$ ①

$= \lim_{h \to 0} \frac{8h + h^2}{h} = \lim_{h \to 0} \frac{h(8+h)}{h}$

$= \lim_{h \to 0} (8 + h) = 8$ ①

|End of question 4|

Question 5

a) $BC = \sqrt{(3-1)^2 + (4-2)^2}$ ①

$= \sqrt{4 + 4}$

$= \sqrt{8} = 2\sqrt{2}$ ①

b) $m = \frac{3-1}{4-2} = 1$ ①

Using point-gradient formula:

$m = 1, \ B(2, 1)$

$(y - 1) = 1(x - 2)$ ①

$y = x - 1$

$-x + y + 1 = 0$

$x - y - 1 = 0$ ①

c) gradient of $AB = \frac{1-4}{2+3} = \frac{-3}{5}$

gradient of $DC = \frac{3-6}{4+1} = \frac{-3}{5}$ ①

$\therefore$ AB // DC (as their gradients are equal) ①

d) $A(-3, 4)$

BC: $x - y - 1 = 0$ +①

$p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ ①

$p = \frac{|-3 \ -4 - 1|}{\sqrt{1^2 + (-1)^2}} = \frac{8}{\sqrt{2}}$

$= \frac{8\sqrt{2}}{2} = 4\sqrt{2}$

e) $A = BC \times h$

$A = 2\sqrt{2} \times 4\sqrt{2}$ ①

$A = 16$ units^2

f) gradient $= -\frac{1}{\text{gradient } BC}$

$m = -\frac{1}{1} = -1$ ①

$A(-3, 4)$

Using point-gradient formula:

$y - 4 = -1(x + 3)$ ①

$y - 4 = -x - 3$

$x + y + 1 = 0$ ①

g) gradient $AB = \frac{-3}{5}$ (from part c)

$\tan \theta = -\frac{3}{5}$ ①

$\theta = 180 - \tan^{-1}\left(\frac{3}{5}\right)$

$\theta = 180° - 30°58'$

$= 149°2'$ ①

|End of question 5|

Question 6

a) The tangent to the curve at this point is horizontal. (1)

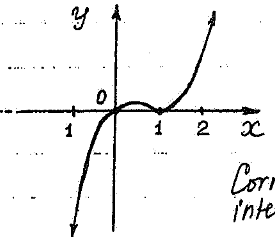
b) $y = x(x-1)^2$

x-intercepts: $x=0$ and $x=1$ (zeros)

y-intercept: $y=0$

Sign of the function near zeros:

x	-1	0	$\frac{1}{2}$	1	2
y	-4	0	$\frac{1}{8}$	0	2
sign	$-$	0	$+$	0	$+$



Correct intercepts (1)

Correct direction (1)

c) (i) $8^x = \frac{1}{16}$

$2^{3x} = (2^4)^{-1}$ (1)

$3x = -4$

$x = -\frac{4}{3}$ (1)

(ii) $x = \log_4 16$

$x = \log_4 4^2$

$= 2\log_4 4 = 2$ (1)

(iii) $x = \log_5 20$

$= \frac{\log_{10} 20}{\log_{10} 5} = 1.86135\ldots$ (1)

1.861 (Correct to 3 d.p.) (1)

d) (i) $S_n = \frac{a(r^n-1)}{r-1}$ (1)

$a = 6$, $r = 2$

$\therefore S_n = \frac{6(2^n-1)}{2-1}$

$= 6(2^n-1)$ (1)

(ii) $6(2^n-1) = 3066$ (1)

$2^n - 1 = 511$

$2^n = 512$

$n = \log_2 512$

$n = \frac{\log_{10} 512}{\log_{10} 2}$

$n = 9$ (1)

$\therefore$ 9 terms are required for the sum to be 3066.

e) $0.1\dot{2} = 0.1 + 0.02 + 0.002 + \ldots$ (1)

GP with $a = 0.02$ and $r = 0.1$

$0.1\dot{2} = 0.1 + \frac{a}{1-r}$ (1)

$= 0.1 + \frac{0.02}{0.9}$

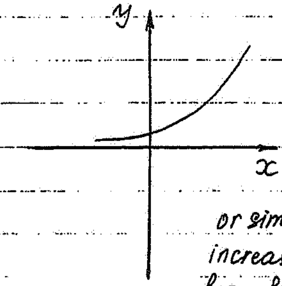
$= \frac{1}{10} + \frac{2}{90}$

$= \frac{9}{90} + \frac{2}{90} = \frac{11}{90}$ (1)

[End of question 6]

Question 7 — Marks

a)



(1)

or similar, increasing from left to right.

b) $3x + 5y - 9 + k(2x + y + 1) = 0$ (1)

Substituting $A(4,-2)$:

$3\times4 + 5\times(-2) - 9 + k(2\times4 - 2 + 1) = 0$

$12 - 10 - 9 + 7k = 0$

$7k = 7$

$k = 1$ (1)

Substituting k:

$3x + 5y - 9 + 1(2x + y + 1) = 0$

$5x + 6y - 8 = 0$ (1)

c) $AB = \sqrt{(8-2)^2 + (12-4)^2} = \sqrt{100} = 10$ units

AB is the diameter

$\therefore$ the radius $= 5$ units (1)

Let Q be the centre of the circle

$Q\left(\frac{8+2}{2}, \frac{12+4}{2}\right)$

$Q(5, 8)$ (1)

$QC = \sqrt{(1-5)^2 + (11-8)^2} = 5$ units

QC = the radius

$\therefore$ Point $C(1,11)$ lies on the circle. (1)

d) (i) $\log_a 10 = \log_a 2\times5$

$= \log_a 2 + \log_a 5$

$= n + m$ (1)

(ii) $\log_a\left(\frac{2}{5}\right) = \log_a 2 - \log_a 5$

$= n - m$ (1)

(iii) $\log_a \sqrt{2} = \log_a 2^{\frac{1}{2}}$

$= \frac{1}{2}\log_a 2$

$= \frac{n}{2}$ (1)

e) common difference $= T_2 - T_1$

$d = (x+1) - (2x-3)$

$= x - 2x + 1 + 3 = -x + 4$

$= 4 - x$ (1)

$d = T_3 - T_2$

$(2-x) - (x+1) = 4 - x$

$-2x + 1 = 4 - x$

$-x = 3$

$x = -3$ (1)

f) $\sum_{n=1}^{\infty} 16 \times \frac{1}{2}^n = 8 + 4 + 2 + \ldots$ (1)

$a = 8$, $r = \frac{1}{2}$

This is a geometric series with $-1 < r < 1$

$\therefore \sum_{n=1}^{\infty} 16 \times \left(\frac{1}{2}\right)^n = S_\infty = \frac{a}{1-r}$ (1)

$= \frac{8}{1-\frac{1}{2}}$

$= 8 \times 2$

$= 16$ (1)

End of question 7

Question 8

a)

(i) Let $y = x(3x-1)^4$,

$u = x$

$v = (3x-1)^4$

Then: $y = u \times v$

$u' = \frac{du}{dx} = 1$

$v' = \frac{dv}{dx} = 4 \times 3(3x-1)^3$

$= 12(3x-1)^3$ (1)

Using the product rule:

~~$=(3x-1)^4 \times 1 + x \times 12(3x-1)^3$~~

$= (3x-1)^4 + 12x(3x-1)^3$

$= (3x-1)^3(3x-1+12x)$

$= (3x-1)^3(15x-1)$ (1)

(ii)

$f'(1) = (3-1)^3(15-1)$

$= 8 \times 14 = 112$ (1)

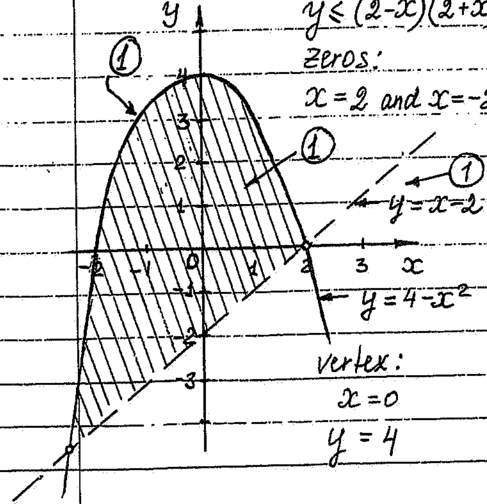
b)

$y > x - 2 \quad y \le 4 - x^2$

$y \le (2-x)(2+x)$

Zeros:

$x = 2$ and $x = -2$



vertex:

$x = 0$

$y = 4$

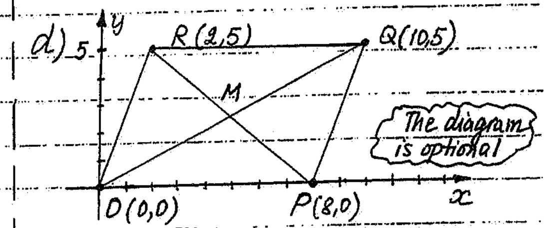
c) $\frac{(2k-3)-(k-1)}{k-1} = \frac{3}{4}$ (1)

$\frac{2k-3-k+1}{k-1} = \frac{3}{4}$

$\frac{k-2}{k-1} = \frac{3}{4}$

$k = 5$ (1)

d)



Let M be the point of intersection of the diagonals OQ and PR (1)

Midpoint of $OQ = \left(\frac{10+0}{2}, \frac{5}{2}\right) = (5, 2.5)$

Midpoint of $PR = \left(\frac{8+2}{2}, \frac{0+5}{2}\right) = (5, 2.5)$

Since two different straight lines can only have one common point, (1)

Midpoint of OQ = Midpoint of PR = M (1)

e)

Since $x, y, 9$ form an AP: $y - x = 9 - y$

$-x = 9 - 2y$

(1)   (1) $x = 2y - 9$

Since $x, y, 12$ form a GP: $\frac{y}{x} = \frac{12}{y}$

(1)   (2) $y^2 = 12x$

Substitute (1) into (2):

$y^2 = 12(2y-9)$

$y^2 - 24y + 108 = 0$

$y^2 - 18y - 6y + 108 = 0$

$y(y-18) - 6(y-18) = 0$

$(y-6)(y-18) = 0$

(1) $y = 6$ or $y = 18$ (1)

Substitute y into (1)

$y = 6 \quad x = 3$ or $y = 18 \quad x = 27$

End of Question 8