SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

Year 11 YEARLY EXAMINATIONS – August 2000

MATHEMATICS

Time allowed — Two Hours
Examiners: E.Choy, A.M.Gainford

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Use a new booklet for each question.
- If required, additional booklets may be obtained from the Examination Supervisor upon request.

Question 1. (18 Marks) (**Start a new booklet**.)

(a) Calculate $\sqrt{\dfrac{67}{4 \cdot 7 \times 2 \cdot 3}}$ correct to two decimal places.

(b) Simplify $x - 2(3 - x)$.

(c) Solve the equation $\dfrac{x}{3} - \dfrac{x+1}{2} = 4$.

(d) Simplify $\sqrt{32} - \sqrt{8}$.

(e) Find x if $\log_3 x = 4$.

(f) Find θ to the nearest minute if $0^0 \le \theta \le 90^0$ and $\cos\theta = 0 \cdot 613$.

(g) Solve the equation $3x^2 = 12$.

(h) Graph on a number line the solution of the inequality $|x - 2| < 3$.

(i) Simplify $\dfrac{(xy^2)^3}{x^3 y^2}$.

(j) Find the exact value of $\sin 135^0 + \tan 480^0$.
Express your answer as a single fraction with rational denominator.

(k) Given that $f(x) = x - \dfrac{1}{x}$:

(i) Find $f(4)$.

(ii) Show that $f(x)$ is an odd function.

Question 2. (18 Marks) (**Start a new booklet**.)

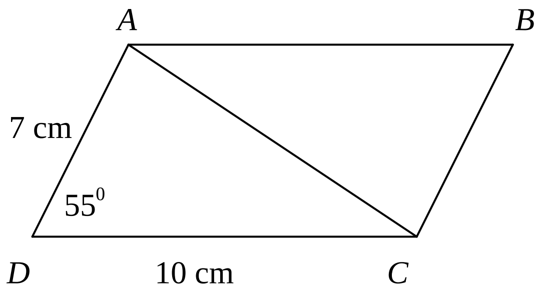
(a) Simplify $\frac{x^2 - y^2}{(x+y)^2}$.

(b) Sketch the graphs of the following:

(i) $y = (x-1)^2$ (ii) $y = \sqrt[3]{x}$

(c) Express the recurring decimal $0 \cdot 4232323\ldots$ as a common fraction.

(d) Given the parallelogram *ABCD*:

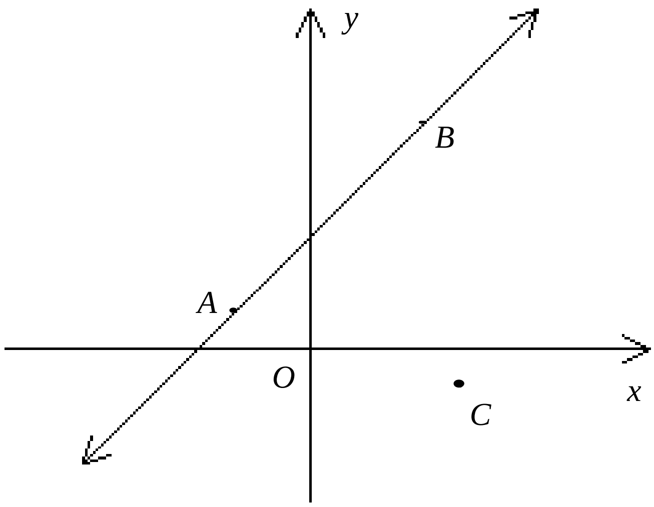


(i) Find the length of the diagonal *AC*, correct to 2 decimal places.

(ii) Calculate the area of the parallelogram, correct to 2 decimal places.

(e) State the natural domain and range of the function $f(x) = \sqrt{9 - x^2}$.

(f)



The diagram above shows the points *A*(-2,1), *B*(3,5), and *C*(4, -1).

Copy the diagram to your answer booklet.

(i) Find the equation of the line through the points *A* and *B*.

(ii) Write the equation of the line through *C* perpendicular to *AB*.

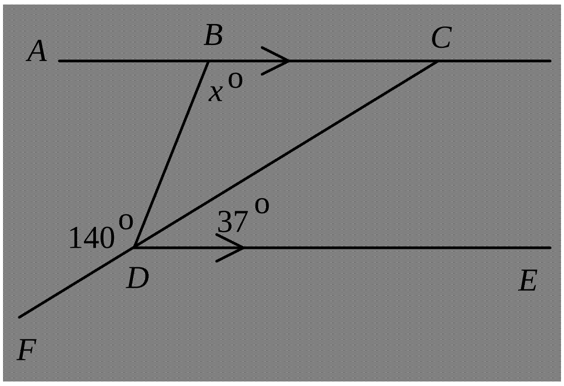
(iii) Hence or otherwise find the distance from *C* to *AB*.

Question 3. (18 Marks) (**Start a new booklet**.)

(a) (i) Find the points of intersection of the line $y = 4 - x$ and the circle $x^2 + y^2 = 16$.

(ii) Hence sketch the region where $y \geq 4 - x$ and $x^2 + y^2 < 16$ hold simultaneously.

(b)

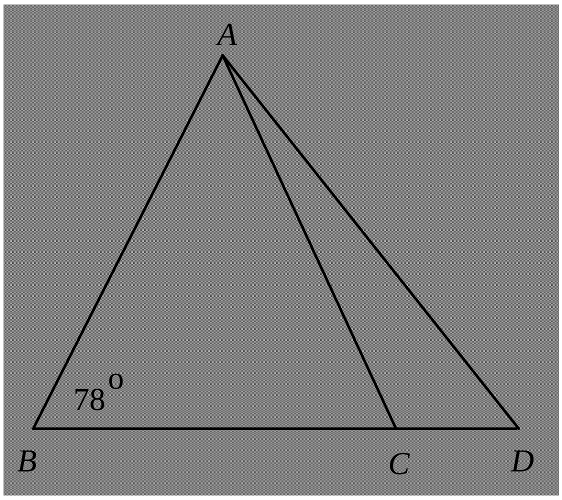


$AC \parallel DE$, CDF is a straight line.

Find the measure of x.

(c) $AB = AC = BD$.

Determine the size of $\angle ADB$ and $\angle DAC$ giving reasons.



(d) Solve the following equations:

(i) $x^4 - 13x^2 - 9 = 0$

(ii) $9^x - 8(3^x) - 9 = 0$

(e) For the parabola $y^2 - 6y - 2x + 7 = 0$ write down the

(i) equation of the axis of symmetry

(ii) coordinates of the vertex

(iii) equation of the directrix and coordinates of the focus.

(f) Show that the expression $x^2 - (k+2)x + (3k+6)$ is positive definite if $-2 < k < 10$.

Question 4. (18 Marks) (**Start a new booklet**.)

(a) Differentiate the following with respect to x:

(i) $-x^3 + 2x^2 + \frac{1}{2}$

(ii) $\sqrt{7x}$

(iii) $\frac{ax^3 - bx^2 + cx}{x^2}$

(b) (i) Use the product rule to find $\frac{dy}{dx}$ if $y = 2x(x+1)^8$.

(ii) Differentiate $y = \frac{5+t}{5-2t}$ by using the quotient rule.

(iii) If $f(x) = 3x + \frac{1}{x^3}$, find

(α) $f'(2)$

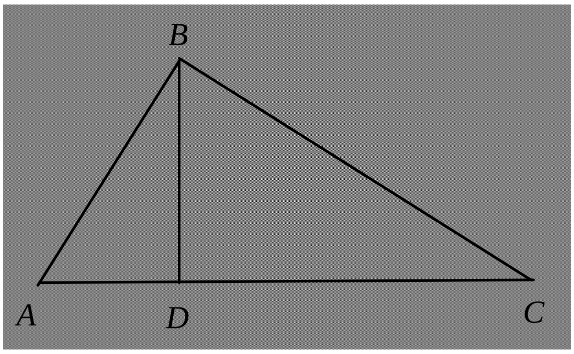
(β) $f''(2)$

(c) (i) For the curve $y = \frac{1}{x^2}$, find the gradient of the tangent to the curve at the point $\left(2, \frac{1}{4}\right)$. Also find the gradient of the normal to the curve at this point.

(ii) Given that $f(x)$ is defined as below, find the value of $f(-3) + f(4) + f(-1)$.

$$f(x) = \begin{cases} -5 & \text{for } x \le -3 \\ 2x & \text{for } -3 < x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$$

(d)



In the diagram *ABC* and *ABD* are right-angled triangles. $\angle ADB = \angle ABC = 90^0$.

Copy the diagram to your answer booklet.

(i) Prove that $\Delta ABD \,|||\, \Delta ACB$.

(ii) Hence find *AB* if *AD* = 4 cm and *DC* = 5 cm.

(e) Find the value or values of *k* that will make the equation $x^2 + 16x - 4k = 0$ have:

(i) equal roots

(ii) two distinct real roots

(iii) roots which are reciprocals of one another

(iv) the sum of roots equal to their product.

Question 5. (18 Marks) (**Start a new booklet**.)

(a) The point $P(x,y)$ moves in the plane so that its distance from a point $A(-2,4)$ is always twice its distance from the point $B(4,1)$.

(i) Write down an expression connecting *PA* and *PB*.

(ii) Hence find the locus of *P*.

(b) Consider the function $y = \dfrac{1}{|x-1|}$.

(i) What is the natural domain of the function?

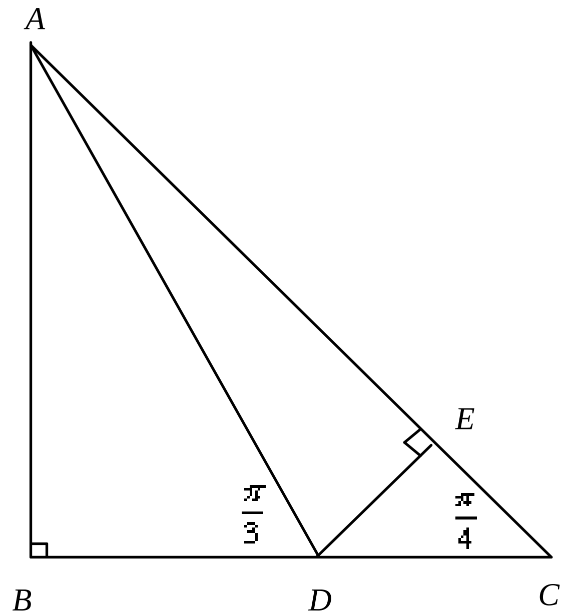
(ii) Write down the equations of the two branches of the function, and sketch its graph.

(c) Solve the equation $8\cos^2 x° = 2\sin x° + 7$ where $0^0 \le x° \le 360^0$. Give your answer correct to the nearest minute.

(d) Prove the identity

$$\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} = 2\sec^2\theta$$

(e) The diagram shows a right-angled triangle *ABC*, whose angle *ACB* is $\dfrac{\pi}{4}$. Line *AD* meets *BC* at *D* such that angle *ADB* is $\dfrac{\pi}{3}$, and length *BD* is one unit. Line *DE* meets line *AC* at right angles.



Find in **exact** form the:

(i) length *AB*

(ii) length *DC*

(iii) length *DE*

(iv) $\angle DAC$ in terms of π, and

hence show that $\sin\dfrac{\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

This is the end of the paper.

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AUGUST 2000

YEARLY EXAMINATION

YEAR 11

Mathematics

Sample Solutions

Question 1

(a) $2\cdot 49$

(b) $x-6+2x=3x-6$

(c) $2x-(3x+3)=24$

$-x-3=24$

$\therefore x=-27$

(d) $4\sqrt{2}-2\sqrt{2}=2\sqrt{2}$

(e) $x=3^4=81$

(f) $\theta=\cos^{-1}(0\cdot 613)=52°12'$

(g) $x^2=4$

$\therefore x=\pm 2$

(h) $-3<x-2<3$

$\therefore -1<x<5$

(i) $\dfrac{x^3y^6}{x^3y^2}=y^4$

(j) $\dfrac{1}{\sqrt{2}}-\sqrt{3}=\dfrac{1-\sqrt{6}}{\sqrt{2}}=\dfrac{\sqrt{2}-\sqrt{12}}{2}=\dfrac{\sqrt{2}-2\sqrt{3}}{2}$

(k) (i) $f(4)=4-\dfrac{1}{4}=3\tfrac{3}{4}$

(ii) $f(-x)=(-x)-\dfrac{1}{(-x)}$

$=-x+\dfrac{1}{x}$

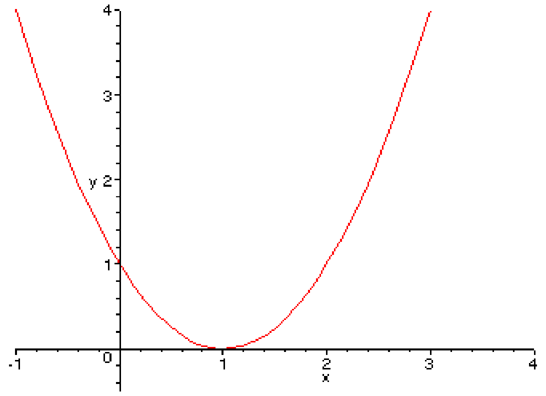
$=-\left(x-\dfrac{1}{x}\right)$

$=-f(x)$

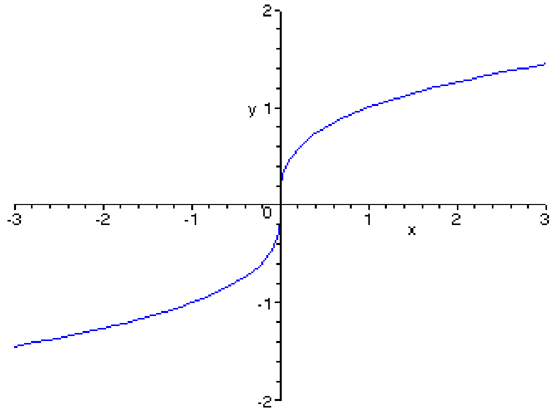
Question 2

(a) $\dfrac{(x+y)(x-y)}{(x+y)^2}=\dfrac{x-y}{x+y}$

(b) (i)



(ii)



(c) $0\cdot4\dot{2}\dot{3}=\dfrac{423-4}{990}=\dfrac{419}{990}$

(d) (i) $AC^2=7^2+10^2-2\times7\times10\times\cos55°$

$\therefore AC\approx8\cdot29$

(ii) $7\times10\times\sin55°\approx57\cdot34$

(e) D: $-3\le x\le3$

R: $0\le y\le3$

Question 2 continued

(f) (i)

$$m_{AB} = \frac{5-1}{3+2} = \frac{4}{5}$$

$$\therefore y-5 = \frac{4}{5}(x-3)$$

$$\therefore 5y-25 = 4x-12$$

$$\therefore 4x-5y+13 = 0$$

(ii)

$$y+1 = -\frac{5}{4}(x-4)$$

$$\therefore 4y+4 = -5x+20$$

$$\therefore 5x+4y-16 = 0$$

(iii)

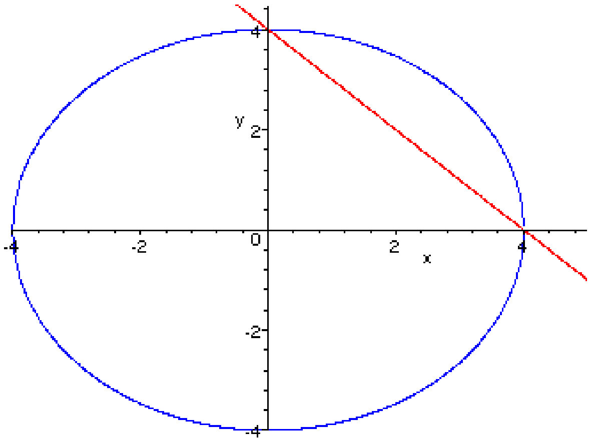
$$d_{\perp} = \frac{|Ax_c + By_c + C|}{\sqrt{A^2+B^2}}$$

$$= \frac{|4\times 4 + (-5)\times(-1) + 13|}{\sqrt{4^2+(-5)^2}}$$

$$= \frac{34}{\sqrt{41}} = \frac{34\sqrt{41}}{41}$$

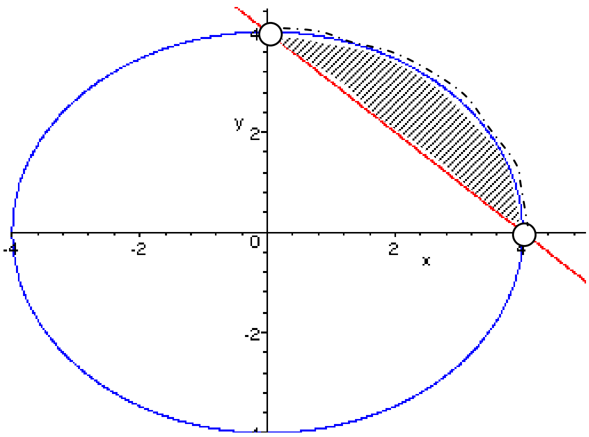
Question 3

(a) (i)



Clearly the points of intersection are $(0,4)$ and $(4,0)$

(ii)



NB the boundary of the circle, in the first quadrant, is dashed and open circles at $(0,4)$ and $(4,0)$.

(b) $\angle BDC = 40°$ (straight line)

$x = 180 - (40 + 37) = 103$ (cointerior angles)

(c) (i) $\angle ADB = \dfrac{180-78}{2} = 51°$ (base angles of isosceles triangle, $AB = BD$)

(ii) $\angle BAC = 180 - 2\times 78 = 24°$ (base angles of isosceles triangle, $AB = BC$)

$$\begin{aligned}\angle DAC &= \angle BAD - \angle BAC \\ &= \angle ADB - \angle BAC \\ &= 51 - 24 = 17°\end{aligned}$$

Question 3 continued

(d) (i) Let $m = x^2$

$\therefore m^2 - 13m - 9 = 0$

$\therefore m = \dfrac{13 \pm \sqrt{205}}{2}$

$\therefore x = \pm\sqrt{\dfrac{13 \pm \sqrt{205}}{2}}$

(ii) Let $n = 3^x$

$\therefore n^2 - 8n - 9 = 0$

$\therefore (n-9)(n+1) = 0$

$\therefore n = 9, -1$

$\therefore 3^x = 9 \qquad (3^x \neq -1, \because 3^x > 0)$

$\therefore x = 2$

(e) $y^2 - 6y = 2x - 7$

$\therefore y^2 - 6y + 9 = 2x - 7 + 9$

$\therefore (y-3)^2 = 2(x+2)$

NB This is the parabola on its side!

(i) Axis of symmetry is $y = 3$

(ii) Vertex is $(-2, 3)$

(iii) $a = 1/2$

Focus is $\left(-2 + \dfrac{1}{2}, 3\right) = \left(-\dfrac{3}{2}, 3\right)$

Directrix is $x = -2 - \dfrac{1}{2} = -\dfrac{5}{2}$

(f) $\Delta = \left[-(k+2)\right]^2 - 4(3k+6)$

$= k^2 + 4k + 4 - 12k - 24$

$= k^2 - 8k - 20$

$= (k+2)(k-10)$

Positive definite if $a > 0$ and $\Delta < 0$

$a = 1,\ \Delta < 0 \Rightarrow (k+2)(k-10) < 0$

$\therefore -2 < k < 10$

Question 4

(a) (i) $$\frac{d}{dx}\left(-x^3+2x^2+\frac{1}{2}\right)=-3x^2+4x$$

(ii) $$\frac{d}{dx}\left(\sqrt{7x}\right)=\frac{d}{dx}\left((7x)^{1/2}\right)$$
$$=\frac{1}{2}(7x)^{-1/2}\times 7$$
$$=\frac{7}{2\sqrt{7x}}$$
$$=\frac{\sqrt{7}}{2\sqrt{x}}=\frac{\sqrt{7}}{2}x^{-1/2}$$

(iii) $$\frac{d}{dx}\left(\frac{ax^3-bx^2+cx}{x^2}\right)=\frac{d}{dx}\left(ax-b+cx^{-1}\right)$$
$$=a-cx^{-2}$$
$$=a-\frac{c}{x^2}$$

(b) (i) $$\frac{dy}{dx}=(x+1)^8\frac{d}{dx}(2x)+2x\frac{d}{dx}\left((x+1)^8\right)$$
$$=2(x+1)^8+2x\times 8(x+1)^7$$
$$=2(x+1)^8+16x(x+1)^7$$
$$=2(x+1)^7[x+1+8x]$$
$$=2(9x+1)(x+1)^7$$

(ii) $$\frac{dy}{dt}=\frac{(5-2t)\frac{d}{dx}(5+t)-(5+t)\frac{d}{dx}(5-2t)}{(5-2t)^2}$$
$$=\frac{5-2t-(5+t)\times(-2)}{(5-2t)^2}$$
$$=\frac{5-2t+10+2t}{(5-2t)^2}$$
$$=\frac{15}{(5-2t)^2}$$

Question 4 continued

(b) (iii) $f(x)=3x+x^{-3} \Rightarrow f'(x)=3-3x^{-2}=3-\frac{3}{x^2},\ f''(x)=6x^{-3}=\frac{6}{x^3}$

(α) $f'(2)=3-\frac{3}{2^2}=3-\frac{3}{4}=2\tfrac{1}{4}$

(β) $f''(2)=\frac{6}{2^3}=\frac{3}{4}$

(c) (i) Let m be the gradient of the tangent and $m_{\perp}$ the gradient of the normal.

$y=x^{-2} \Rightarrow \frac{dy}{dx}=-2x^{-3}=-\frac{2}{x^3}$

$\therefore m=\left.\frac{dy}{dx}\right|_{x=2}=-\frac{2}{2^3}=-\frac{1}{4}$

$\therefore m_{\perp}=4$

(ii) $f(-3)+f(4)+f(-1)=[-5]+\left[4^2\right]+\left[2\times(-1)\right]$

$=-5+16-2=9$

(d) (i) $\angle BAD$ is common and $\angle ADB=\angle ABC=90°$
so $\Delta ABD \,|||\, \Delta ACB$ (equiangular).

(ii) $\frac{AB}{AD}=\frac{AC}{AB}$ (all sides are in the same ratio)

$\therefore \frac{AB}{4}=\frac{9}{AB} \Rightarrow AB^2=36$

$\therefore AB=6$

(e) $\Delta=16^2-4\times(-4k)=16(16+k)$

$\alpha+\beta=-16, \alpha\beta=-4k$

(i) $\Delta=0 \Rightarrow 16(16+k)=0$

$\therefore k=-16$

(ii) $\Delta>0 \Rightarrow 16(16+k)>0$

$\therefore k>-16$

(iii) $\alpha\beta=1 \Rightarrow -4k=1$

$\therefore k=-\frac{1}{4}$

(iv) $\alpha+\beta=\alpha\beta \Rightarrow -16=-4k$

$\therefore k=4$

Question 5

(a) (i) $PA = 2PB$.

(ii) $PA = \sqrt{(x+2)^2 + (y-4)^2}$, $PB = \sqrt{(x-4)^2 + (y-1)^2}$

$PA = 2PB \Rightarrow PA^2 = 4PB^2$

$\therefore (x+2)^2 + (y-4)^2 = 4\left[(x-4)^2 + (y-1)^2\right]$

$\therefore x^2 + 4x + 4 + y^2 - 8y + 16 = 4x^2 - 32x + 64 + 4y^2 - 8y + 4$

$\therefore 3x^2 - 36x + 3y^2 + 48 = 0$

$\therefore x^2 - 12x + y^2 + 16 = 0$

This is a **circle**.

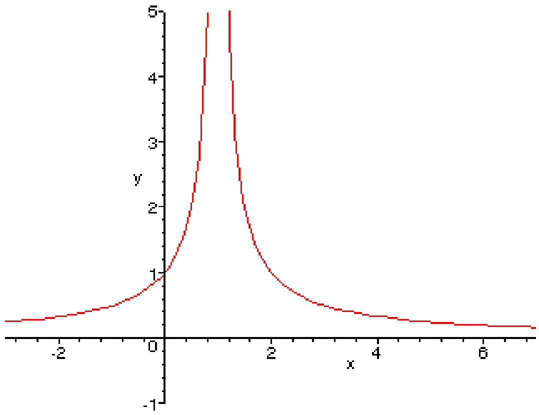
$x^2 - 12x + 36 + y^2 = -16 + 36$

$\therefore (x-6)^2 + y^2 = 20$

Centre $(6,0)$, radius $2\sqrt{5}$

(b) (i) $x \neq 1$ or $x \in \mathbb{R} \setminus \{1\}$

(ii)
$$y = \begin{cases} \dfrac{1}{x-1}, x > 1 \\ \dfrac{1}{1-x}, x < 1 \end{cases}$$



Question 5 continued

(c)

$$8\cos^2 x = 2\sin x + 7$$
$$\therefore 8\left(1-\sin^2 x\right) = 2\sin x + 7$$
$$\therefore 8\sin^2 x + 2\sin x - 1 = 0$$
$$\therefore \sin x = \frac{-2\pm\sqrt{36}}{16} = \frac{-2\pm 6}{16}$$
$$\therefore \sin x = -\frac{1}{2}, \frac{1}{4}$$
$$\therefore x = 14°29', 165°31', 210°, 330°$$

(d)

$$\begin{aligned}\text{LHS} &= \frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} \\ &= \frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)} \\ &= \frac{2}{1-\sin^2\theta} \\ &= \frac{2}{\cos^2\theta} \\ &= 2\sec^2\theta \\ &= \text{RHS}\end{aligned}$$

(e) (i) $\tan\frac{\pi}{3} = \frac{AB}{BD} = \frac{AB}{1} \Rightarrow AB = \sqrt{3}$

(ii) ΔABC is isosceles with $AB = BC$.

$DC = BC - BD = \sqrt{3} - 1$

(iii) ΔDEC is isosceles with $DE = EC$.

$$2DE^2 = \left(\sqrt{3}-1\right)^2$$
$$DE = \frac{\sqrt{3}-1}{\sqrt{2}}$$

(iv) (α) $\angle DAC = \angle BAC - \angle BAD = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

(β) $\sin\angle DAC = \sin\frac{\pi}{12} = \frac{DE}{AD}$

$$AD^2 = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$$
$$\therefore \sin\frac{\pi}{12} = \frac{\frac{\sqrt{3}-1}{\sqrt{2}}}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$