SYDNEY BOYS HIGH

MOORE PARK, SURRY HILLS

2003

YEAR 11 YEARLY EXAMINATION

Mathematics

General Instructions

- Reading time — 5 minutes
- Working time — 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Start each Section in a new booklet

Total marks — 130

- Attempt all questions
- Questions are not of equal value, the mark value is shown beside each part.
- Hand up your paper in five parts:
  Section A, Questions 1 & 2;
  Section B, Questions 3 & 4;
  Section C, Questions 5 & 6;
  Section D, Questions 7 & 8;
  Section E, Questions 9 & 10

Examiner: R.Boros

Total marks – 130
Attempt Questions 1–10

Answer each Section in a SEPARATE writing booklet. Extra writing booklets are available upon request.

Section A Use a SEPARATE writing booklet. **Marks**

Question 1 (16 marks)

(a) Express each of the following as a rational number.

(i) $49^{-\frac{1}{2}} \times 27^{\frac{2}{3}}$ **1**

(ii) The quotient of $\sqrt{7}$ and $\sqrt{63}$ **1**

(iii) $\log_2 8$ **1**

(iv) $\dfrac{\sqrt{32} - \sqrt{8}}{3\sqrt{2}}$ **2**

(b) Find, correct to 2 decimal places, $\dfrac{(3{\cdot}24)^2}{5{\cdot}73 - 2{\cdot}84}$. **1**

(c) Solve for x, $\dfrac{2x}{x-5} = \dfrac{3}{5}$. **2**

(d) Factorise fully the expression $x^3 - x^2 - x + 1$. **2**

(e) Find the centre and radius of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$. **2**

(f) Solve the inequality $x^2 - 4x < 0$. **2**

(g) Given $v^2 = u^2 - 2aS$, $v = 2{\cdot}5$, $u = 2{\cdot}3$, and $a = 7$, find S correct to 3 significant figures. **2**

Question 2 (14 marks)

(a) Given the points $A(-5, 3)$, $B(1, -5)$, and $C(2, 2)$,

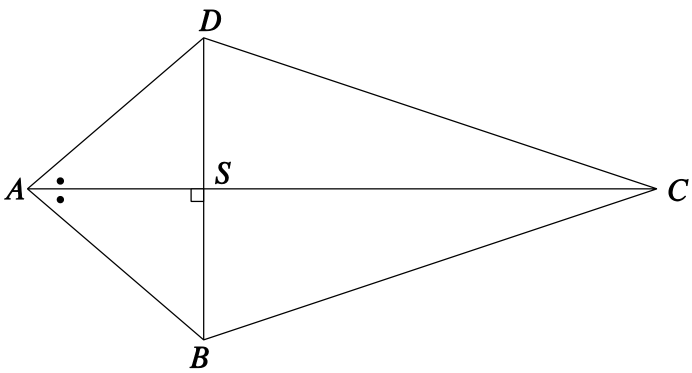
(i) Find the length of AB. **1**

(ii) Find the equation of the line AB written in general form. **2**

(iii) Find the perpendicular distance of C from the line AB. **2**

(iv) Hence or otherwise, calculate the area of the triangle ABC. **1**

(b)



In the above diagram, not to scale, $ABCD$ is a quadrilateral. The diagonals AC and DB intersect at right angles at point S. $D\hat{A}S = B\hat{A}S$.

(i) Prove that ΔASB is congruent to ΔASD. **2**

(ii) Hence prove that $DA = BA$. **1**

(c) Given $f(x) = x^2 + 3x + 2$,

(i) Evaluate $f(-3)$. **1**

(ii) Find a simple expression for $f(a+2)$. **2**

(d) Solve for x, $|2x - 1| \leq 5$. **2**

Section B Use a SEPARATE writing booklet. **Marks**

Question 3 (13 marks)

(a) Doug observes a clifftop A from his yacht at position P. He then sails 500 m towards the cliff to position Q. The angle of elevation to the clifftop from P is 5° and from Q is 8°.

(i) Draw a diagram to illustrate the above information and use the Sine Rule to calculate AQ correct to the nearest metre. **2**

(ii) Hence or otherwise find the distance QB correct to the nearest 10 metres. **2**

(b) Simplify the expression $\dfrac{\tan\theta}{\cot\theta} - \dfrac{\sec^2\theta}{1}$. **2**

(c) (i) Find θ given that $\sin\theta = \frac{1}{2}$ where $0^\circ \leq \theta \leq 180^\circ$. **1**

(ii) Hence find the exact values of $\tan\theta$ and $\sec\theta$. **2**

(d) Simplify $\sin\theta\cos(90^\circ - \theta) + \cos\theta\sin(90^\circ - \theta)$. **2**

(e) If $\sin\theta = \frac{8}{17}$ and θ is an acute angle, find the exact values of $\cos\theta$ and $\tan\theta$. **2**

Section B continued **Marks**

Question 4 (15 marks)

(a) Sketch the area defined by the inequality $y \leq x^2$. **2**

(b) Let α and β be the roots of the equation $x^2 - 5x + 2 = 0$. Find the values of:

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $(\alpha + 1)(\beta + 1)$ **1**

(c) State the domain and range of each of these functions.

(i) $x^2 + 3 = y$ **2**

(ii) $y = 3^x$ **2**

(d) Solve for x the equation $9^x - 9(3)^x = 0$. **3**

(e) The roots of the quadratic equation $px^2 - x + q = 0$ are -1 and 3. Find p and q. **3**

Section C Use a SEPARATE writing booklet. **Marks**

Question 5 (15 marks)

(a) A parabola has the equation $y = x^2 - 12x + 20$. Find

(i) where it cuts the x and y axes, **2**

(ii) its axis of symmetry and vertex, **2**

(iii) the focus, by first expressing it in the form $(x-h)^2 = 4a(y-k)$, **2**

(iv) the equation of the directrix. **1**

(b) For each of the quadratics below, evaluate the discriminant and state the relevance of this with regard to the roots of the equation.

(i) $x^2 + 3x + \frac{9}{4} = y$ **2**

(ii) $3x^2 - 2x = y + 5$ **2**

(c) Find the values of M for which the equation $4x^2 - Mx + 9 = 0$ has

(i) exactly one real root, **2**

(ii) real roots. **2**

Section C continued **Marks**

Question 6 (14 marks)

(a) Solve the following equations simultaneously, **2**
$4x - y = 3$ and $10x + 3y = 2$.

(b) Find x given $2\log_9\sqrt{3} + \log_9 81 = x$. **3**

(c) Find x correct to 3 decimal places given that $7^x = 15$. **2**

(d) If Ron invests \$500 at 12·5 % p.a. compound interest, how long would it take the investment to grow to a sum of \$1000. (Answer in years, correct to 2 decimal places.) **3**

(e) The r^{th} term of a series is $3 \times 2^{(r-4)}$. Determine which of the numbers 96, ¾, 256 belong to the series. **2**

(f) Evaluate $\sum_{n=-1}^{7} (2n + 3)$. **2**

Section D Use a SEPARATE writing booklet. **Marks**

Question 7 (10 marks)

(a) Let A (0, -2) and B (1, 0) be 2 fixed points and let P (x, y) be a variable point. **3**
Find the locus of P such that the length $(PA)^2$ equals the length $(PB)^2$.

(b) Find the locus of points 2 units away from the line $y = 3$. **2**

(c) Sketch the graph of $y = \sin\theta$ given $-180^\circ < \theta < 90^\circ$. **3**

(d) Find the values of a and b given that **2**

$$(ax - 3)^2 + b = 4x^2 - 12x + 15.$$

Question 8 (12 marks)

(a) Differentiate each of the following:

(i) $y = 2x^3 - 8$ **1**

(ii) $y = (2x - 1)^3$ **1**

(iii) $y = \dfrac{2x}{1-3x}$ **2**

(iv) $y = x^2\sqrt{x}$ **2**

(v) $y = \dfrac{7}{2x^3}$ **1**

(b) (i) Find the gradient of the normal to the curve $y = 1 - \frac{1}{2}x^2$ at the point (1, 3). **3**

(ii) Find the point on $y = 1 - \frac{1}{2}x^2$ where the tangent to the curve is parallel to this normal. **2**

Section E Use a SEPARATE writing booklet. **Marks**

Question 9 (11 marks)

(a) Evaluate the following limits:

(i) $\lim_{x \to 5} \left(\dfrac{x-5}{x^2 - 25} \right)$ **2**

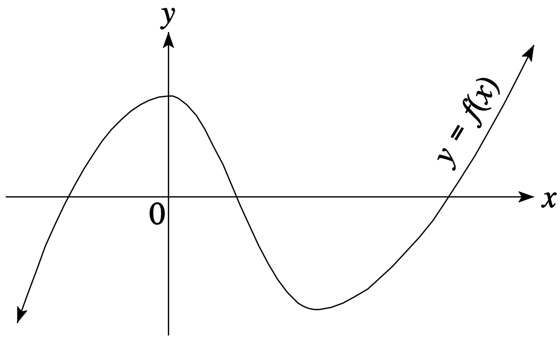
(ii) $\lim_{x \to \infty} \left(\dfrac{7 - 2x - 3x^2}{5x^2 + 3} \right)$ **3**

(b) Two cars depart town A at the same time. Car X travels at 60 km/h on a bearing of 345°T, whilst car Y travels at 100 km/h on a bearing of 085°T. How far apart would the cars be after 3 hours? Answer correct to one decimal place. **3**

(c) Differentiate $y = x^2 + x$ from first principles. **3**

Question 10 (10 marks)

(a) The diagram below shows $y = f(x)$. **3**



Copy or trace this diagram into your answer booklet and sketch a possible graph for $y = f'(x)$ on the same set of axes.

(b) A farmer wishes to make a rectangular enclosure using a river as one boundary and 400 m of fencing on the other three sides.

(i) Find the maximum possible area of the enclosure. **3**

(ii) What are the dimensions of this enclosure? **1**

(c) Find x correct to 3 decimal places, given that $\log_7 6 - 2\log_7 3 = x$. **3**

END OF THE PAPER

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2003

YEARLY EXAMINATION

YEAR 11

Mathematics

Sample Solutions

[16 marks]

Question (1)

(a) $49^{-1/2} \times 27^{2/3}$

(i) $= \frac{1}{7} \times (3^3)^{2/3}$

$= \boxed{9/7}$ (1)

(ii) $\frac{\sqrt{7}}{\sqrt{63}} = \boxed{\frac{1}{3}}$ (1)

(iii) $\log_2 8 = 3\log_2 2$

$= \boxed{3.}$ (1)

(iv) $\frac{2\sqrt{2}(2-1)}{3\sqrt{2}} = \boxed{\frac{2}{3}.}$ (2)

(b). $\frac{(3.24)^2}{5.73-2.84} = \boxed{3.63}$ (2d.p.) (1)

(c) $\frac{2x}{x-5} = \frac{3}{5}$

$10x = 3x - 15$

$7x = -15$

$\boxed{x = -15/7.}$ (2)

(d) $x^2(x-1) - (x-1)$

$= \boxed{(x-1)^2(x+1).}$ (2)

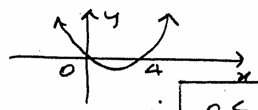
(e) $x^2 - 6x + 9 + y^2 + 4y + 4 = 25$ (2)

$\therefore (x-3)^2 + (y+2)^2 = 25$

Centre $(3, -2)$	$r = 5$

(f) $x^2 - 4x < 0$

$x(x-4) < 0$ (2)



$\therefore \boxed{0 < x < 4.}$

(g) $(2.5)^2 = (2.3)^2 - 14(s).$ (2)

$14s = 0.96$

$\therefore \boxed{s = -0.0686.}$ (3s.f.)

Question (2). [14 marks]

(a) (i) $AB = \sqrt{(-6)^2 + 8^2}$

$\boxed{= 10}$ (1)

(ii) $m = \frac{-8}{6} = -4/3$

$y - 3 = -4/3(x+5)$ (2)

$3y - 9 = -4x - 20$

$\therefore \boxed{4x + 3y + 11 = 0}$

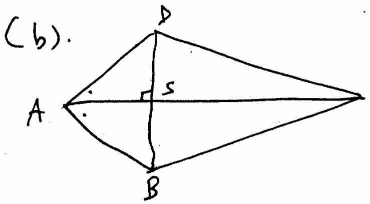
(iii) (2)

$d = \frac{|8 + \cancel{8}^{6} + 11|}{\sqrt{25}} = \boxed{\cancel{27/5}}$ 5

(iv) $\therefore A = \frac{1}{2} \times 10 \times \cancel{\frac{27}{5}}$ 5

$= \boxed{25 \text{ sq units.}}$ (1)

(b). (2)



$\angle DAS = \angle BAS$ (given)

$\angle DSA = \angle BSA$ (st. line BD)

AS is common

$\therefore \Delta ASB \equiv \Delta ASD$ (AAS)

$\therefore DA = BA$

(Corresp. sides of cong. Δ's are equal) (1)

(c) $f(x) = x^2 + 3x + 2$

$\boxed{\text{(i) } f(-3) = 2}$ (1)

(ii) $(a+2)^2 + 3(a+2) + 2$

$\boxed{= a^2 + 7a + 12.}$ (2)

$(a+3)(a+4)$

(d) $|2x - 1| \le 5$

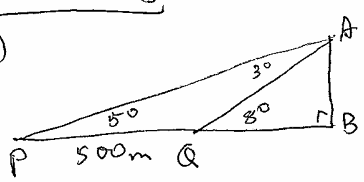
$-5 \le 2x - 1 \le 5$ (2)

$-4 \le 2x \le 6$

$\boxed{-2 \le x \le 3}$

Question 3

(a)



[1]

(i) $\angle PAQ = 3°$ (ext. angle thm)

$\frac{AQ}{\sin 5°} = \frac{500}{\sin 3°}$

$AQ = \frac{500 \sin 5°}{\sin 3°}$

$\doteq 833\,m$ [1]

(ii) $\cos 8° = \frac{QB}{AQ}$

$\therefore QB = AQ \cos 8°$

$\doteq 820\,m$ (nearest 10m) [2]

(b) $\frac{\tan\theta}{\cot\theta} - \sec^2\theta$

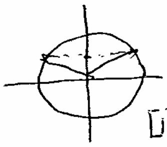
$= \tan^2\theta - \sec^2\theta$

$= \tan^2\theta - (1 + \tan^2\theta)$

$= -1$ [2]

(c) (i) $\sin\theta = \frac{1}{2} \quad 0° \leq \theta \leq 180°$

$\theta = 30°$ or $150°$



[1]

(ii) $\tan\theta = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

$\sec\theta = \frac{2}{\sqrt{3}}$ or $-\frac{2}{\sqrt{3}}$ [2]

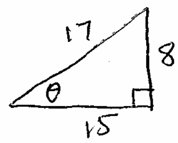
(d) $\sin\theta . \cos(90-\theta) + \cos\theta \sin(90-\theta)$

$= \sin\theta . \sin\theta + \cos\theta . \cos\theta$

$= \sin^2\theta + \cos^2\theta$ [2]

$= 1$

(e) $\sin\theta = \frac{8}{17}$

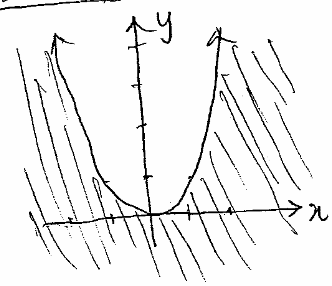


$\cos\theta = \frac{15}{17}$ [2]

$\tan\theta = \frac{8}{15}$

Question 4

(a)



[2

(b) $x^2 - 5x + 2 = 0$ [1

(i) $\alpha + \beta = 5$ [1]

(ii) $\alpha\beta = 2$

(iii) $(\alpha+1)(\beta+1) = \alpha\beta + (\alpha+\beta) + 1$

$= 2 + 5 + 1$

$= 8$ [1

(c) (i) $y = x^2 + 3 \quad D = \{x : x \in \mathbb{R}\}$

$R = \{y : y \geq 3\}$

(ii) $y = 3^x \quad D = \{x : x \in \mathbb{R}\}$

$R = \{y : y > 0\}$

Question 4 (Cont'd)

(d) $9^x - 9(3)^x = 0$

$(3^x)^2 - 9(3^x) = 0$

Let $u = 3^x$

$u^2 - 9u = 0$

$u(u-9) = 0$

$\therefore u = 0$ or $u = 9$

$\therefore 3^x = 0$ or $3^x = 9$

No soln $\quad x = 2$ [3]

(e) $px^2 - x + q = 0$

Roots -1 and 3

$\alpha + \beta = \frac{1}{p} \quad \alpha\beta = \frac{q}{p}$

$\therefore \frac{1}{p} = 2 \quad \frac{q}{p} = -3$

$p = \frac{1}{2} \quad 2q = -3$

$q = \frac{-3}{2}$

[3]

$\boxed{2}$ 5. (a) (i) On the x-axis, $y = 0$; so $(x-10)(x-2) = 0$.
On the y-axis, $x = 0$; so $y = 20$.
$\therefore$ The curve meets the x-axis at 2, 10, and the y-axis at 20.

$\boxed{2}$ (ii) The midpoint of 2 and 10 is 6, so the axis of symmetry is $x = 6$.
When $x = 6$, $y = -16$, so the vertex is $(6, -16)$.

$\boxed{2}$ (iii)
$$\begin{aligned} x^2 - 12x + 36 &= y - 20 + 36, \\ (x-6)^2 &= 4 \times \tfrac{1}{4} \times (y + 16). \end{aligned}$$
$\therefore$ The focus is at $(6, -15\frac{3}{4})$.

$\boxed{1}$ (iv) The directrix is $y = -16\frac{1}{4}$.

$\boxed{2}$ (b) (i) $\triangle = 9 - 9 = 0 \Longrightarrow$ one real root.

$\boxed{2}$ (ii) $\triangle = 4 + 60 = 64 \Longrightarrow$ two real, rational roots.

$\boxed{2}$ (c) (i)
$$\begin{aligned} \triangle = M^2 - 144 &= 0, \\ M^2 &= 144. \end{aligned}$$
$\therefore$ One root when $M = \pm 12$.

$\boxed{2}$ (ii)
$$\begin{aligned} M^2 - 144 &\geq 0, \\ M^2 &\geq 144, \\ |M| &\geq 12. \end{aligned}$$
$\therefore\ M \leq -12,\ M \geq 12.$

$\boxed{2}$ 6. (a)
$$\begin{aligned} 4x - y &= 3, \quad \cdots\cdots(1) \\ 10x + 3y &= 2. \quad \cdots\cdots(2) \end{aligned}$$
$y = 4x - 3$ from (1), then substitute in (2).
$$\begin{aligned} 10x + 3(4x - 3) &= 2, \\ 10x + 12x - 9 &= 2, \\ 22x &= 11, \\ x &= \tfrac{1}{2}, \\ y &= -1. \end{aligned}$$

$\boxed{3}$ (b)
$$\begin{aligned} \log_9 3 + \log_9 9^2 &= x, \\ \log_9 9^{\frac{1}{2}} + \log_9 9^2 &= x, \\ \tfrac{1}{2}\log_9 9 + 2\log_9 9 &= x, \\ \tfrac{1}{2} + 2 &= x, \\ \textit{i.e.,} \quad x &= 2\tfrac{1}{2} \text{ (or } \tfrac{5}{2}). \end{aligned}$$

$\boxed{2}$ (c)
$$\begin{aligned} \log 7^x &= \log 15, \\ x \log 7 &= \log 15, \\ x &= \frac{\log 15}{\log 7}, \\ &\approx 1{\cdot}392. \end{aligned}$$

3 (d)
$$\begin{aligned}\$1\,000 &= \$500\left(1+\tfrac{12\cdot5}{100}\right)^n,\\ 2 &= (1\cdot125)^n,\\ \log 1\cdot125^n &= \log 2,\\ n\log 1\cdot125 &= \log 2,\\ n &= \frac{\log 2}{\log 1\cdot125}\\ n &\approx 5\cdot88 \text{ years.}\end{aligned}$$

2 (e) $96 \div 3 = 32, \; = 2^5.$ $\quad \frac{3}{4} \div 3 = \frac{1}{4}, \; = 2^{-2}.$ $\quad 256 \div 3 = 85\frac{1}{3},$ Not a power of 2.

$\therefore$ 96, $\frac{3}{4}$ belong to the series.

2 (f)
$$\begin{aligned}\sum_{n=-1}^{7}(2n+3) &= 1+3+5+7+9+11+13+15+17,\\ &= 81.\end{aligned}$$

QUESTION 7

(a) $A(0,-2)$ $B(1,0)$ $(PA)^2=(PB)^2$

$(x-0)^2+(y+2)^2=(x+1)^2+(y-1)^2$

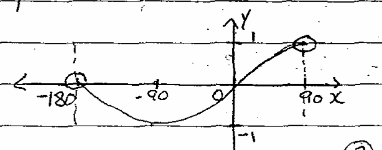
$x^2+y^2+4y+4=x^2 2x+1+y^2$

$4y+2x+3=0$

$2x+4y+3=0$ ③

(b) $y=5$ or $y=1$ ②

(c) $y=\sin\theta$ $-180°<\theta<90°$



③

(d) $(ax-3)^2+b=4x^2-12x+15$

$a^2x^2-6ax+9+b=4x^2-12x+15$

equating coefficients

$a^2=4$ $-6ax=-12x$ $9+b=15$

$a=\pm2$ $a=2$ $b=6$

$a=2$, $b=6$

or

$(ax-3)^2+b=4x^2-12x+9+6$

$(ax-3)^2+b=(2x-3)^2+6$

$a=2$ $b=6$ ②

QUESTION 8

(a)(i) $y=2x^3-8$

$y'=6x^2$ ①

(ii) $y=(2x-3)^3$

$y'=3(2x-3)^2\times2$

$=6(2x-3)^2$ ①

(iii) $y=\frac{2x}{1-3x}$

$y'=\frac{v\,du-u\,dv}{v^2}$

$=\frac{(1-3x)\times2-2x\times-3}{(1-3x)^2}$

$=\frac{2}{(1-3x)^2}$ ②

(iv) $y=x^2\sqrt{x}$

$=x^{5/2}$

$y'=\frac{5}{2}x^{3/2}$

$=\frac{5}{2}\sqrt{x^3}$ ②

(v) $y=\frac{7}{2x^3}=\frac{7}{2}x^{-3}$

$y'=-\frac{21}{2}x^{-4}$ ①

$=-\frac{21}{2x^4}$

(b)(i) $y=1-\frac{1}{2}x^2$

$y'=-x$

when $x=1$ grad of tangent $=-1$

" normal $=1$ ③

(ii) $y'=-x$

$-x=1$ when $x=-1$

point $(-1,\frac{1}{2})$ ②

Question 9.

(a) (i)

$$\lim_{x\to 5}\frac{x-5}{x^2-25}$$

$$=\lim_{x\to 5}\frac{x-5}{(x-5)(x+5)}$$

$$=\lim_{x\to 5}\frac{1}{(x+5)}$$

$$=\frac{1}{10}$$

(ii)

$$\lim_{x\to\infty}\frac{7-2x-3x^2}{5x^2+3}$$

$$=\lim_{x\to\infty}\frac{\frac{7}{x^2}-\frac{2}{x^2}-\frac{3x^2}{3x^2}}{\frac{5x^2}{5x^2}+\frac{3}{x^2}}$$

$$=-\frac{3}{5}$$

(b) In 3 hours car X travels 180km.
In 3 hours car Y travels 300km.

$$XY^2=AY^2+AX^2-2AY.AX.\cos A$$

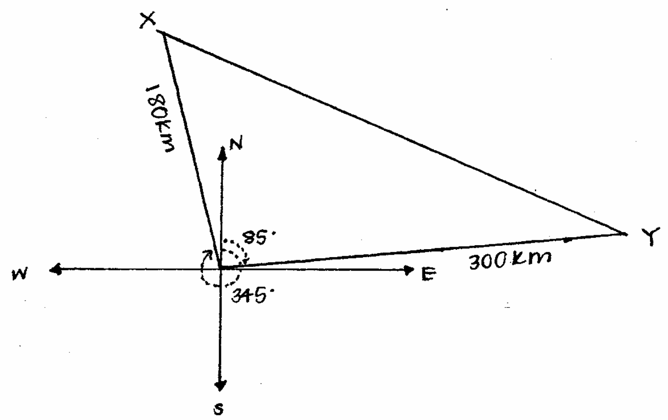
$$=300^2+180^2-2(300)(180)\cos 100^\circ$$

$$XY^2=AY^2+AX^2-2AY.AX.\cos A$$

$$=300^2+180^2-2(300)(180)\cos 100^\circ$$

$$XY=375.7$$

$\therefore XY=375.7$ km (1d.p.)



(c) Let $f(x) = x^2 + x$

Let $f(x+h) = (x+h)^2 + (x+h)$

$$f'(x) = \lim_{h\to\infty} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to\infty} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h}$$

$$= \lim_{h\to\infty} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$$

$$= \lim_{h\to\infty} \frac{2xh + h^2 + h}{h}$$

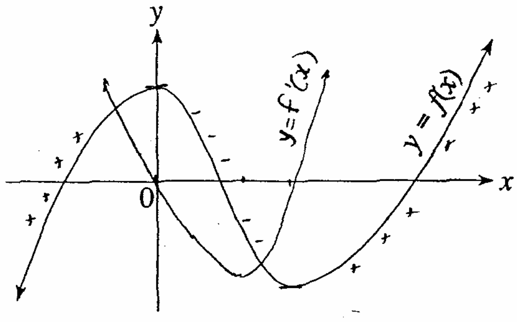
$$= \lim_{h\to\infty} \frac{h(2x + h + 1)}{h}$$

$$= 2x + 1$$

$\therefore f'(x) = 2x + 1$ by first principles.

Question 10.

(a)



(b) (i) Let $P = 2x + y$, where $P = 400$.

$400 = 2x + y$

$\therefore y = 400 - 2x$

$$\begin{aligned}\text{Area} &= xy \\ &= x(400 - 2x) \\ &= 400x - 2x^2\end{aligned}$$

Using the axis of symmetry formula, $x = -\dfrac{b}{2a}$, where $a = -2, b = 400$.

$$\begin{aligned}x &= -\frac{b}{2a} \\ &= -\frac{400}{2(-2)} \\ &= 100\end{aligned}$$

At $x = 100, y = 200$.

Hence the maximum possible area of this enclosure is $A = 20000\ \text{m}^2$

(ii) Hence the dimensions of this enclosure are 100m by 200m.

(c)

$$\log_7 6 - 2\log_7 3 = x$$
$$\log_7 6 - \log_7 3^2 = x$$
$$\frac{\log_{10} 6}{\log_{10} 7} - \frac{\log_{10} 9}{\log_{10} 7} = x$$
$$0.9207 - 1.1291 = x$$
$$x = -0\cdot 20845$$
$$\therefore x = -0\cdot 208 \text{ (3dp).}$$

or

$$\log_7\left(\frac{6}{3^2}\right) = x$$
$$\log_7\left(\frac{2}{3}\right) = x$$
$$7^x = \frac{2}{3}$$
$$x\log 7 = \log\frac{2}{3}$$
$$\therefore x = -0.208$$