SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

September
2006
Year 11 Yearly Examination

Mathematics (2 Unit Continuers)

General Instructions

- Reading Time – 5 Minutes
- Working time – 1 ½ Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- EACH SECTION IS TO BE RETURNED IN A SEPARATE BUNDLE. Section A (Questions 1 & 2) and Section B (Questions 3 & 4)
- All necessary working should be shown in every question.

Total Marks – 60

- Attempt questions 1 – 4

Examiner: *F Nesbitt*

SECTION A

QUESTION 1

(a) Show that $\frac{1}{\sqrt{7}+2}-\frac{1}{\sqrt{7}-2}$ is rational. 2

(b) Find $\lim_{x \to 5} \frac{x^2-5}{x-5}$ 1

(c) Differentiate:

(i) $24x(2x^2-1)$ 2

(ii) $\frac{x-5}{2x+3}$ 2

(iii) $(7x^2-6x)^3$ 2

(d) Simplify fully: $\log_3 \sqrt{27} - \log_2 \frac{1}{8} + \log_5 125$ 2

(e) Solve the inequation $|3x-1| \le 5$ and graph the solution on a number line. 2

(f) Solve for m: $2^{2m} - 3 \times 2^m - 10 = 0$ 2

QUESTION 2

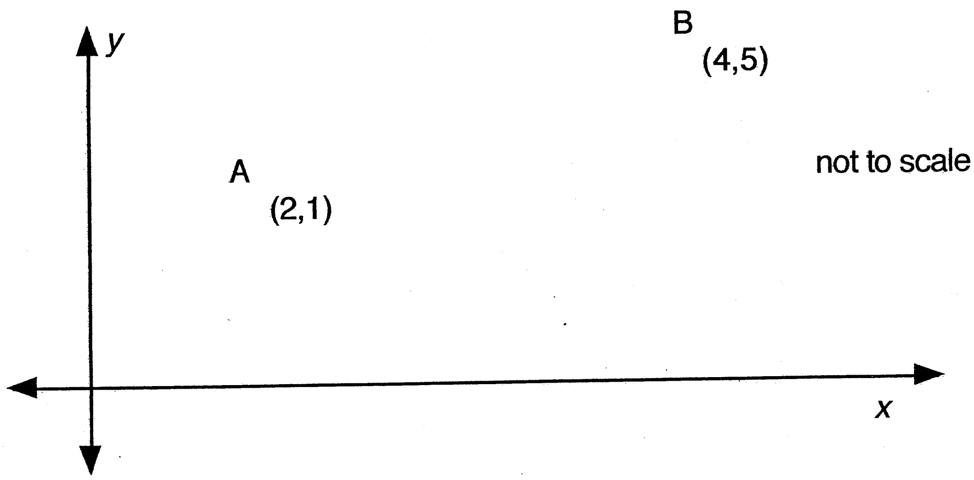
(a) For the parabola $(x+2)^2 = 8(y-1)$, find:

(i) the coordinates of its vertex

(ii) the coordinates its focus

(ii) the equation of its directrix. 3

(b)



In the diagram above, The points A and B have coordinates (2,1) and (4,5).

(i) Find M, the midpoint of AB. 1

(ii) Find the Gradient of AB. 1

(iii) Find the equation of the perpendicular bisector of AB. 2

(iv) This perpendicular through M [in part (iii)] meets the y axis at D.

Find the coordinates of D 1

(v) Find the perpendicular distance from D to AB 2

(vi) Find the area of the triangle ADB 2

(c) Find from first principles, the gradient of the curve

$x^2 + x$ at the point where $x=1$ 3

SECTION B - START A NEW BOOKLET

QUESTION 3

(a) For the curve $2x^3 + 3x^2 - 12x - 2$:

(i) Find any turning points.

(ii) Determine the nature of each turning point.

(iii) Sketch the curve $-3 \le x \le 2$ showing all relevant features.

(iv) Find the minimum value of the curve in the given domain. 6

(b) In the arithmetic series $2 + 5 + 8 + 11 + \ldots\ldots.$, find:

(i) the 15th term.

(ii) the sum of the first 15 terms. 4

(c) a, b and 10 form an arithmetic sequence.

(i) Write an equation in a and b using the above information:

a, b and 18 form a geometric sequence.

(ii) Write a second equation in a and b using the above information

(iii) Solve the equations in (i) and (ii) to find 2 sets of values for a and b. 5

QUESTION 4

(a) Find the value(s) of k if the roots of the equation $x^2 + kx + 36 = 0$ are real and distinct. 2

(b) In an "organic" fruit shop only 60% of the oranges and 80% of the apples are really organic. George bought 2 apples and one orange. What is the probability that at least one of the three pieces of fruit was really organic? 2

(c) A closed cylindrical can is made from 100π square centimetres of metal. If h is the height and r is the radius,

(i) show that $h = \dfrac{50}{r} - r$ 2

(ii) Hence find an expression for the volume in terms of r. 1

(iii) Find the maximum possible volume of the can and show why it is a maximum. Answer to the nearest centimetre. 4

(d) From a point O, an observer can see a lighthouse L on a bearing of 285°. An oil rig R can also be seen by the observer on a bearing of 215°. The lighthouse is 12 km from the oil rig and on a bearing of 012° from the oil rig.

(i) Find the distance, to the nearest kilometre, of the oil rig from the observer. 3

(ii) Find the bearing of the oil rig from the lighthouse. 1

END OF PAPER

SECTION A

Question 1

(a) $\frac{1}{(\sqrt{7}+2)} \times \frac{(\sqrt{7}-2)}{(\sqrt{7}-2)} - \frac{1}{(\sqrt{7}-2)} \times \frac{(\sqrt{7}+2)}{(\sqrt{7}+2)}$

$= \frac{\sqrt{7}-2}{7-4} - \frac{(\sqrt{7}+2)}{7-4}$

$= \frac{\sqrt{7}-2-\sqrt{7}-2}{3}$

$= -\frac{4}{3}$ which is rational

$\therefore \frac{1}{\sqrt{7}+2} - \frac{1}{\sqrt{7}-2}$ is rational

(b) $\lim_{x \to 5} \frac{x^2-5}{x-5}$

we can't factorise x^2-5 into factors that will cancel with $x-5$.

Therefore, the limit does not exist $\left(\frac{20}{0}\right)$.

(c)(i) let $y = 24x(2x^2-1)$

$y = 48x^3 - 24x$

$y' = 144x^2 - 24$

(ii) let $y = \frac{x-5}{2x+3}$ $\quad v = 2x+3$, $v' = 2$ $\quad u = x-5$, $u' = 1$

$y' = \frac{(2x+3).1 - 2(x-5)}{(2x+3)^2}$

$y' = \frac{2x+3-2x+10}{(2x+3)^2}$

$y' = \frac{13}{(2x+3)^2}$

(iii) let $y = (7x^2-6x)^3$

$y' = 3(7x^2-6x)^2(14x-6)$

$y' = 6(7x-3)(7x^2-6x)^2$

(d) $\log_3 \sqrt{27} - \log_2 \frac{1}{8} + \log_5 125$

$= \log_3 3^{3/2} - \log_2 2^{-3} + \log_5 5^3$

$= \frac{3}{2}\log_3 3 + 3\log_2 2 + 3\log_5 5$

(since $\log_a a = 1$)

$= \frac{3}{2} + 3 + 3$

$= \frac{15}{2}$

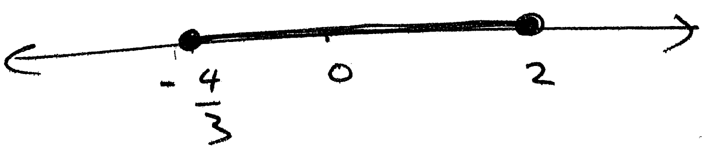
(e) $|3x - 1| \leq 5$

$3x - 1 \leq 5$ **or** $-(3x-1) \leq 5$

$3x \leq 6$ $\quad$ $-3x + 1 \leq 5$

$-3x \leq 4$

$x \leq 2$ $\quad$ $x \geq -\frac{4}{3}$



(f) $2^{2m} - 3 \times 2^m - 10 = 0$

$(2^m)^2 - 3 \times 2^m - 10 = 0$

let $a = 2^m$

$a^2 - 3a - 10 = 0$

× | -10
+ | -3
-5, 2

$(a - 5)(a + 2) = 0$

$a - 5 = 0$ $\quad$ $a + 2 = 0$

$a = 5$ $\quad$ $a = -2$

$2^m = 5$ $\quad$ $2^m = -2$ (since $2^m > 0$)

$m = \log_2 5$ $\quad$ $2^m = -2$ has no solution

$m \approx 2.322$

Question 2

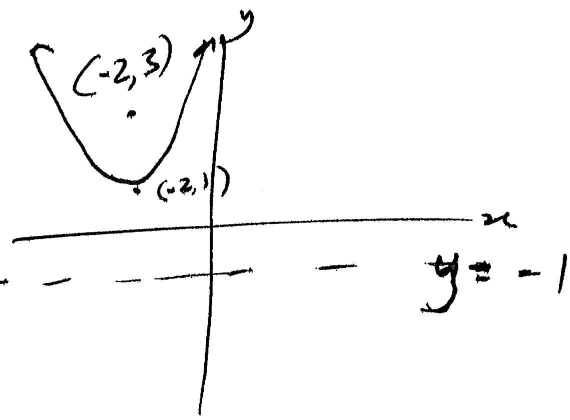
(a) $(x+2)^2 = 8(y-1)$

is in the form

$(x-h)^2 = 4a(y-k)$

where (h, k) is vertex

a is focal length



(i) vertex is $(-2, 1)$

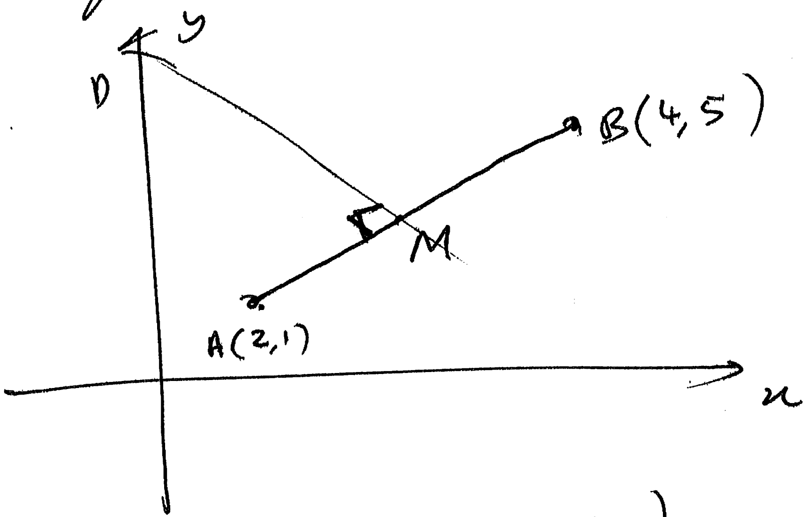
(ii) $4a = 8$

$a = 2$

focus is $(-2, 3)$

(iii) equation of directrix is $y = -1$

(b)



(i) $M\left(\frac{2+4}{2}, \frac{1+5}{2}\right)$

$M(3,3)$

(ii) $m = \frac{y_2 - y_1}{x_2 - x_1}$

$m_{AB} = \frac{5-1}{4-2}$

$m_{AB} = \frac{4}{2}$

$m_{AB} = 2$

(iii) $m_2 = -\frac{1}{2}$ (perpendicular lines $m_1 \times m_2 = -1$)

$y - y_1 = m(x - x_1)$

$y - 3 = -\frac{1}{2}(x - 3)$

$2y - 6 = -x + 3$

$x + 2y - 9 = 0$

(iv) let $x = 0$

$2y - 9 = 0$

$2y = 9$

$y = \frac{9}{2}$

$\therefore D\left(0, \frac{9}{2}\right)$

(v) The perpendicular distance from D to AB is the length DM.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$DM = \sqrt{(3-0)^2 + \left(3 - \frac{9}{2}\right)^2}$

$DM = \sqrt{9 + \frac{9}{4}}$

$DM = \sqrt{\frac{45}{4}}$

$DM = \frac{3\sqrt{5}}{2}$

(vi) $AB = \sqrt{(4-2)^2 + (5-1)^2}$

$AB = \sqrt{4 + 16}$

$AB = \sqrt{20}$

$AB = 2\sqrt{5}$

$\text{Area} = \frac{1}{2} \times AB \times DM = \frac{1}{2} \times 2\sqrt{5} \times \frac{3\sqrt{5}}{2} = \frac{15}{2} \text{ units}^2$

(c) $y = x^2 + x$

Let $f(x) = x^2 + x$

$f(x+h) = (x+h)^2 + (x+h)$

$= x^2 + 2xh + h^2 + x + h$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{x} + h - (\cancel{x^2} + \cancel{x})}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(2x + h + 1)}{\cancel{h}}$

$= 2x + (0) + 1$

$= 2x + 1$

$\therefore y' = 2x + 1$

when $x = 1$

$m_T = 2(1) + 1$

$= 3$

2006 Mathematics Continuers 2-Unit Yearly: **Section B** solutions

3. (a) For the curve $y = 2x^3 + 3x^2 - 12x - 2$:

(i) Find any turning points. [2]

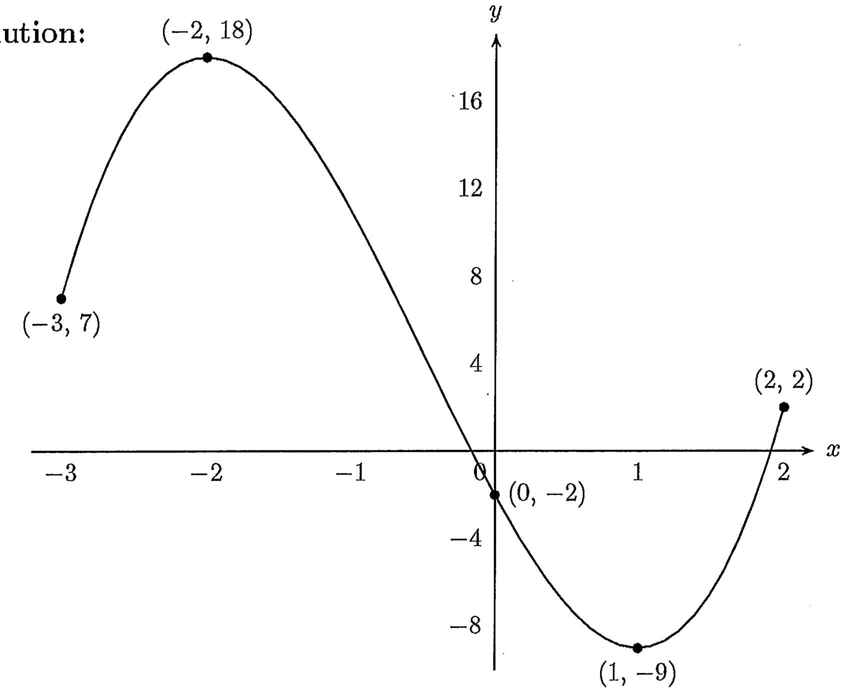
Solution:
$$\begin{aligned}\frac{dy}{dx} &= 6x^2 + 6x - 12,\\ &= 6(x^2 + x - 2),\\ &= 6(x+2)(x-1),\\ &= 0 \text{ when } x = -2,\ 1.\end{aligned}$$
$\therefore$ Turning points are at $(-2,\ 18)$ and $(1,\ -9)$.

(ii) Determine the nature of each turning point. [1]

Solution:
$$\begin{aligned}\frac{d^2y}{dx^2} &= 12x + 6,\\ &= -18 \text{ when } x = -2,\\ &= 18 \text{ when } x = 1.\end{aligned}$$
$\therefore$ Maximum at $(-2,\ 18)$ and minimum at $(1,\ -9)$.

(iii) Sketch the curve $-3 \le x \le 2$ showing all relevant features. [2]

Solution:



(iv) Find the minimum value of the curve in the given domain. [1]

Solution: The minimum value is -9.

(b) In the arithmetic series $2 + 5 + 8 + 11 + \ldots$, find:

(i) the 15[th] term, [2]

Solution: $a = 2, \quad d = 3.$
$$\therefore U_{15} = 2 + (15 - 1)3, \\ = 44.$$

(ii) the sum of the first 15 terms. [2]

Solution:
$$S_{15} = \frac{15}{2}(2 + 44), \\ = 345.$$

(c) a, b, and 10 form an arithmetic sequence.

(i) Write an equation in a and b using the above information. [1]

Solution:
$$10 - b = b - a, \\ a = 2b - 10.$$

a, b, and 18 form a geometric sequence.

(ii) Write a second equation in a and b using the above information. [1]

Solution:
$$\frac{18}{b} = \frac{b}{a}, \\ b^2 = 18a.$$

(iii) Solve the equations in (i) and (ii) to find 2 sets of values for a and b. [3]

Solution:
$$\begin{aligned} b^2 &= 18(2b - 10), \\ b^2 - 36b + 180 &= 0, \\ (b - 6)(b - 30) &= 0. \\ \therefore b &= 6 \text{ or } 30, \\ a &= 2 \text{ or } 50. \end{aligned}$$

4. (a) Find the value(s) of k if the roots of the equation $x^2 + kx + 36 = 0$ are real and distinct. [2]

Solution:
$$\begin{aligned} \Delta &> 0, \\ i.e.,\ k^2 - 4 \times 36 &> 0, \\ k^2 &> 12^2, \\ \therefore\ k &< -12, \quad k > 12. \end{aligned}$$

(b) In an "organic" fruit shop only 60% of the oranges and 80% of the apples are really organic. George bought two apples and one orange. What is the probability that at least one of the three pieces of fruit was really organic? [2]

Solution:
$$\begin{aligned} \text{P(at least one)} &= \text{P(not none)}, \\ &= 1 - \tfrac{1}{5} \times \tfrac{1}{5} \times \tfrac{2}{5}, \\ &= \frac{123}{125}. \end{aligned}$$

(c) A closed cylindrical can is made from 100π square centimetres of metal. If h is the height and r is the radius,

(i) show that $h = \dfrac{50}{r} - r$. [2]

Solution: Surface area,
$$\begin{aligned} 100\pi &= 2\pi r^2 + 2\pi rh, \\ 50 &= r^2 + rh, \\ rh &= 50 - r^2, \\ h &= \frac{50}{r} - r. \end{aligned}$$

(ii) Hence find an expression for the volume in terms of r. [1]

Solution: Volume,
$$\begin{aligned} V &= \pi r^2 h, \\ &= \pi r^2 \left(\frac{50}{r} - r \right), \\ &= (50\pi r - \pi r^3),\ \text{or} \\ &= (50 - r^2)\pi r. \end{aligned}$$

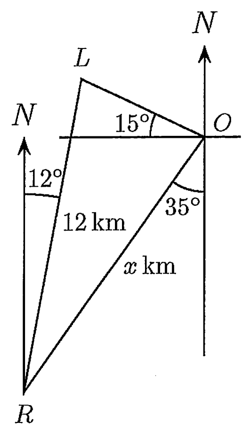
(iii) Find the maximum possible volume of the can and show why it is a maximum. Answer to the nearest cubic centimetre. [4]

Solution:
$$\begin{aligned}\frac{dV}{dr} &= (50 - 3r^2)\pi,\\ &= 0 \text{ when } r^2 = \frac{50}{3},\\ \textit{i.e.} \text{ when } r &= 5\sqrt{\tfrac{2}{3}}\,\text{cm.}\\ \frac{d^2V}{dr^2} &= -6\pi r,\\ &< 0 \text{ when } r > 0.\end{aligned}$$
$\therefore$ Max. volume when $r^2 = \frac{50}{3}$ cm.
$$\begin{aligned}\text{Max. volume} &= (50 - \tfrac{50}{3}) \times 5\sqrt{\tfrac{2}{3}} \times \pi\,\text{cm}^3,\\ &\approx 427{\cdot}516609974\,\text{cm}^3 \text{ by calculator},\\ &\approx 428\,\text{cm}^3.\end{aligned}$$

(d) From a point O, an observer can see a lighthouse L on a bearing of 285°T. An oil rig R can also be seen by the observer on a bearing of 215°T. The lighthouse is 12 km from the oil rig and on a bearing of 012°T from the oil rig.

(i) Find the distance, to the nearest kilometre, of the oil rig from the observer. [3]

Solution:



$$\begin{aligned}\angle LOR &= (15 + 90 - 35)°,\\ &= 70°.\\ \angle ORL &= (35 - 12)°,\\ &= 23°.\\ \angle RLO &= (180 - 70 - 23)°,\\ &= 87°.\\ \frac{x}{\sin 87°} &= \frac{12}{\sin 70°},\\ x &= \frac{12 \sin 87°}{\sin 70°},\\ &\approx 12{\cdot}75263225 \text{ by calculator.}\end{aligned}$$
$\therefore$ Rig is about 13 km from the observer.

(ii) Find the bearing of the oil rig from the lighthouse. [1]

Solution:
$$\begin{aligned}\text{Bearing} &= 12° + 180°,\\ &= 192°\text{T}.\end{aligned}$$