Sydney Grammar School
Mathematics Department
Yearly Examinations 2005

FORM V

MATHEMATICS

Examination date

Friday 14th October 2005

Time allowed

2 hours

Instructions

All nine questions may be attempted.
All nine questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

Write your name, class and master clearly on each booklet.
Hand in the nine questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

5P: PKH   5Q: REN   5R: DS

Checklist

SGS booklets: 9 per boy. A total of 500 booklets should be sufficient.
Candidature: 37 boys.

Examiner

REN

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Simplify:

(i) $3\sqrt{3} - 7\sqrt{3}$ **1**

(ii) $4\sqrt{20}$ **1**

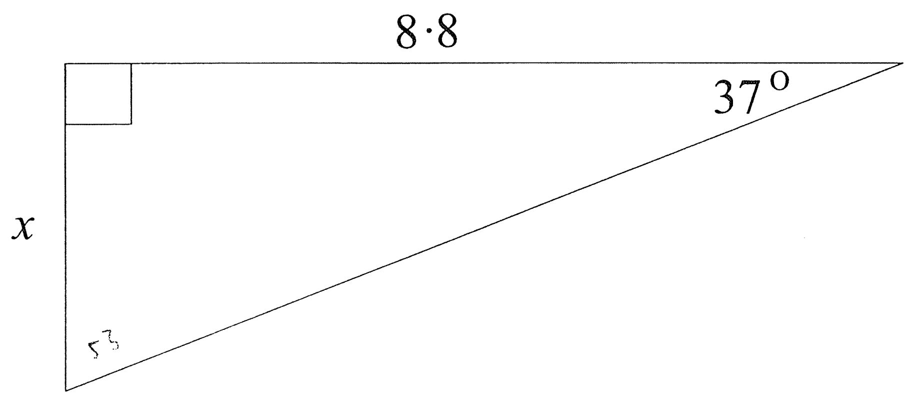
(iii) $\dfrac{\sqrt{28}}{\sqrt{7}}$ **1**

(b) (i) Solve the inequation $4x + 5 < 11$. **1**

(ii) Graph the solution on the number line. **1**

(c) Write down the exact value of $\cos 150°$. **1**

(d) **2**



For the diagram above, find the value of x correct to 2 significant figures.

(e) Evaluate $16^{\frac{3}{4}}$. **1**

(f) (i) If $2^x = 32$, find x. **1**

(ii) Simplify $(a^3)^4 \div a^{12}$. **2**

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) A series is given by $3 + 11 + 19 + 27 + \cdots$

(i) Show that the series is arithmetic. 1

(ii) Find the 200th term. 1

(iii) Find the sum of the first 200 terms. 1

(iv) Show that the nth term, T_n, is given by $T_n = 8n - 5$. 1

(v) Hence find the first term of the series that is greater than 1000. 2

(b) A geometric series is given by $7 + 14 + 28 + 56 + \cdots$

(i) Find the 25th term of the series. 1

(ii) Find the sum of the first 20 terms of the series. 1

(c) The 10th term of an arithmetic series can be written as $a + 9d$, where a is the first term and d is the common difference.

(i) Write a similar expression for the 15th term of the series. 1

(ii) If the 10th term of the series is 16 and the 15th term is -14, find a and d. 3

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate the following:

(i) $y = 5x^7$ 1

(ii) $y = 2x^{-6}$ 1

(iii) $y = 10x^{\frac{1}{2}}$ 1

(iv) $y = (4x - 7)^{10}$ 1

(b) (i) Use the formula $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to differentiate $f(x) = 2x^2$ from first principles. 3

(ii) Find the gradient of the curve $f(x) = 2x^2$ at the point where $x = -1$. 1

(c) (i) State the product rule for differentiation. 1

(ii) Use the rule to differentiate $y = 2x(x + 1)^9$.
(There is no need to fully factorise your answer.) 1

(d) Differentiate $y = \dfrac{3x + 4}{x - 2}$. 2

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Find the perpendicular distance from the point $(2, -1)$ to the line $3x + 4y - 6 = 0$. [2]

(b) Solve the quadratic equation $x^2 - 5x - 7 = 0$.
(Write your solutions correct to 2 decimal places.) [2]

(c) Solve the following for x:

(i) $\log_{27} x = \frac{1}{3}$ [1]

(ii) $\log_x 100\,000 = 5$ [1]

(d) Simplify the following:

(i) $\log_a 1$ [1]

(ii) $\log_a a^6$ [1]

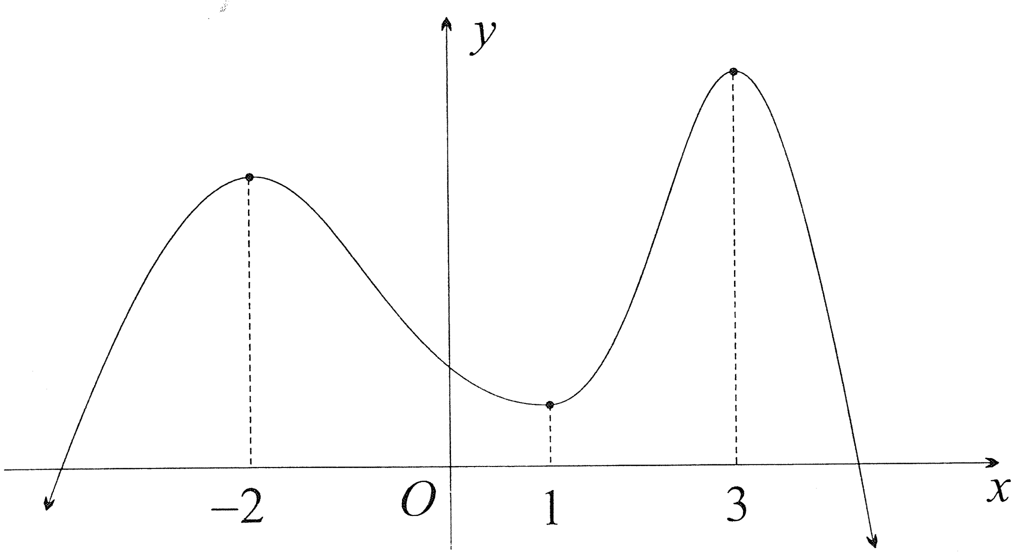
(iii) $\log_a \sqrt{a}$ [1]

(e) Use the change of base formula to calculate $\log_7 100$ correct to 1 decimal place. [1]

(f) Solve $3^x = 55$, giving your solution correct to 2 decimal places. [2]

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the graph of $y = f(x)$.

(i) Write down the values of x for which $f(x)$ is stationary. [1]

(ii) Write down the values of x for which $f(x)$ is

(α) increasing [1]

(β) decreasing [1]

Exam continues next page ...

(b) The equation of a curve is given by $y = x^3 - 3x^2 + 5$.

(i) Show that $\dfrac{dy}{dx} = 3x(x-2)$. **1**

(ii) Find the equation of the tangent to the curve $y = x^3 - 3x^2 + 5$ at the point $(1, 3)$. **2**

(iii) Find the value(s) of x for which the tangent is horizontal. **1**

(c) Solve $x^4 - 7x^2 + 12 = 0$. **3**

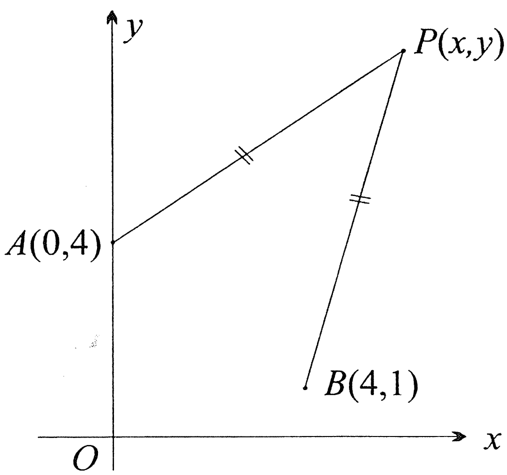
(d) (i) Find the discriminant of the quadratic equation $2x^2 - 7x + 2 = 0$. **1**

(ii) Explain why the quadratic equation has real roots. **1**

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) Find the centre and radius of the circle $x^2 + y^2 - 2x + 8y + 8 = 0$. **2**

(b)



The diagram above shows the points $A(0, 4)$ and $B(4, 1)$ with $P(x, y)$ equidistant from A and B.

(i) Find the equation of the locus of the point P. **3**

(ii) Describe the locus in words. **1**

(c) The equation of a parabola is given by $x^2 = 16y$.

(i) Write down the co-ordinates of the focus. **1**

(ii) Write down the equation of the directrix. **1**

(d) A parabola has a focus at $(-1, -1)$ and directrix $y = 3$.

(i) On a number plane diagram sketch the graph of the parabola with the given focus and directrix. (Mark the focus, vertex and directrix on your diagram.) **2**

(ii) Write down the co-ordinates of the vertex. **1**

(iii) Write down the equation of the parabola. **1**

Exam continues overleaf ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Write down the formula for the limiting sum of a geometric series (assuming the limiting sum exists). 1

(ii) A geometric series is given by $27 + 18 + 12 + 8 + \cdots$

(α) Show that the series has a limiting sum. 1

(β) Find the limiting sum. 1

(b) By writing the recurring decimal $0{\cdot}\dot{1}\dot{8}$ as an infinite series, express $0{\cdot}\dot{1}\dot{8}$ as a fraction in simplest form. 2

(c) Let α and β be the roots of the quadratic equation $2x^2 - 4x + 1 = 0$. Without finding α and β, evaluate the following:

(i) $\alpha + \beta$ 1

(ii) $\alpha\beta$ 1

(iii) $4\alpha^2\beta + 4\alpha\beta^2$ 1

(iv) $\alpha^2 + \beta^2$ 2

(d) Find the values of x for which the function $y = 3x^2 - 8x + 1$ is increasing. 2

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Find the vertex and focus of the parabola $y^2 - 4y - 8x - 20 = 0$. 2

(ii) Sketch the parabola, showing the vertex and focus. 1

(b) Find the values of k for which the quadratic expression $x^2 - (k+1)x + (k+1)$ is positive definite. 3

(c) (i) If $f(x) = \sqrt{2x^2 + 1}$, show that $f'(x) = \dfrac{2x}{\sqrt{2x^2 + 1}}$. 1

(ii) Hence find the equation of the normal to the curve $y = \sqrt{2x^2 + 1}$ at the point $(2, 3)$. 2

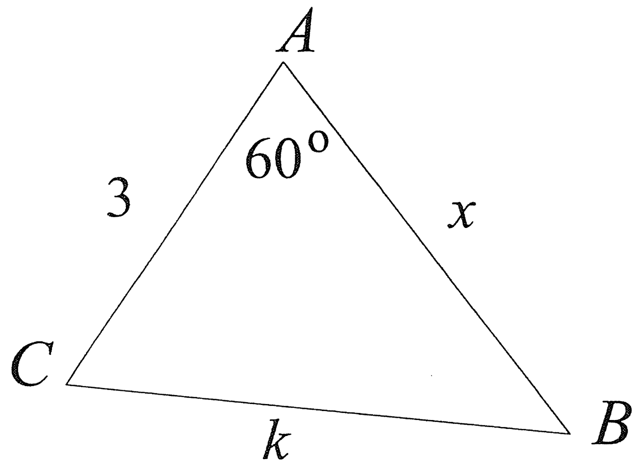
(d) If the function $y = ax^2 + bx - 5$ has a stationary point at $(2, 1)$, find a and b. 3

Exam continues next page ...

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) Find the point(s) on the curve $y = \dfrac{1}{x}$ where the gradient is equal to $-\frac{1}{4}$. **3**

(b)



The diagram above shows $\triangle ABC$ with $AC = 3, AB = x$, $BC = k$ and $\angle BAC = 60°$.

(i) Use the cosine rule to show that $k^2 = x^2 - 3x + 9$. **2**

(ii) Explain why the length of side BC must be at least $\frac{3\sqrt{3}}{2}$. **2**

(c) An arithmetic series has first term a and common difference d.

(i) Write down algebraic expressions for T_n and T_{n+1}. **1**

(ii) If the sum of the first n terms of the series is equal to twice the sum of the next n terms, show that $d = \dfrac{2a}{1 - 5n}$. **4**

END OF EXAMINATION

FORM 5 MATHEMATICS SOLUTIONS

Q1

(a) (i) $3\sqrt{3} - 7\sqrt{3} = -4\sqrt{3}$ ✓

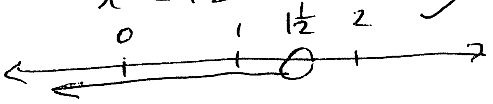
(ii) $4\sqrt{20} = 8\sqrt{5}$ ✓

(iii) $\frac{\sqrt{28}}{\sqrt{7}} = \sqrt{4} = 2$ ✓

(b) (i) $4x < 6$ ✓

$x < 1\frac{1}{2}$ ✓

(ii)



(c) $\cos 150^\circ = -\frac{\sqrt{3}}{2}$ ✓

(d) $\tan 37^\circ = \frac{x}{8.8}$ ✓

$x = 8.8 \times \tan 37^\circ$ ✓

$x \doteq 6.6$

(e) $16^{\frac{3}{4}} = 8$ ✓

(f) (i) $2^x = 32$

$x = 5$ ✓

(ii) $a^{12} \div a^{12} = 1$ ✓✓

Q2

(a) (i) $11 - 3 = 19 - 11 = 8$

AP with $d = 8$ ✓

(ii) $T_{200} = 3 + 199 \times 8$ ✓

$= 1595$

(iii) $S_{200} = \frac{200}{2}(3 + 1595)$

$= 159\ 800$ ✓

(iv) $T_n = 3 + (n-1)8$

$= 3 + 8n - 8$ ✓

$= 8n - 5$

(v) $8n - 5 > 1000$ ✓

$8n > 1005$

$n > 125\frac{5}{8}$

$n = 126$

$T_{126} = 8 \times 126 - 5$ ✓

$= 1003$

(b) (i) $T_{25} = 7 \times 2^{24}$ ✓

$= 117\ 440\ 512$

(ii) $S_{20} = \frac{7(2^{20} - 1)}{2 - 1}$ ✓

$= 7\ 340\ 025$

(c) (i) $a + 14d$ ✓

(ii) $a + 9d = 16$ ✓

$a + 14d = -14$

$5d = -30$

$d = -6$ ✓

$a = 16 + 54 = 70$ ✓

$a = 70$

$d = -6$

Q3

(a) (i) $y' = 35x^6$ ✓

(ii) $y' = -12x^{-7}$ ✓

(iii) $y' = 5x^{-\frac{1}{2}}$ ✓

(iv) $y' = 40(4x-7)^9$ ✓

(b) (i) $f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$ ✓

$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h}$

$= \lim_{h \to 0} \frac{h(4x + 2h)}{h}$

$= \lim_{h \to 0} 4x + 2h$ ✓

$= 4x$ ✓

(ii) $f'(-1) = 4 \times (-1) = -4$

gradient $= -4$ ✓

(c) (i) $y' = vu' + uv'$ ✓

(ii) $u = 2x \qquad v = (x+1)^9$

$u' = 2 \qquad v' = 9(x+1)^8$

$y' = 2(x+1)^9 + 18x(x+1)^8$ ✓

(d) $u = 3x+4 \qquad v = x-2$

$u' = 3 \qquad v' = 1$

$y' = \frac{vu' - uv'}{v^2}$

$= \frac{3(x-2) - 1 \times (3x+4)}{(x-2)^2}$ ✓

$= \frac{3x - 6 - 3x - 4}{(x-2)^2}$

$= -\frac{10}{(x-2)^2}$ ✓

Q4

(a) $d = \left| \frac{3 \times 2 + 4 \times (-1) - 6}{\sqrt{3^2 + 4^2}} \right|$ ✓

$= \left| \frac{6 - 4 - 6}{5} \right|$

$= \frac{4}{5}$ ✓

(b) $x = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 1 \times (-7)}}{2}$ ✓

$= \frac{5 \pm \sqrt{53}}{2}$

$\doteqdot 6.14$ or -1.14 ✓

(c) (i) $x = 27^{\frac{1}{3}} = 3$ ✓

(ii) $x^5 = 100\,000$

$x = 10$ ✓

(d) (i) $\log_a 1 = 0$ ✓

(ii) $\log_a a^6 = 6$ ✓

(iii) $\log_a a^{\frac{1}{2}} = \frac{1}{2}$ ✓

(e) $\log_7 100 = \frac{\log_{10} 100}{\log_{10} 7}$

$\doteqdot 2.4$ ✓

(f) $x = \log_3 55$ ✓

$= \frac{\log_{10} 55}{\log_{10} 3}$

$\doteqdot 3.65$ ✓

Q5

(a) (i) $x = -2,\ x = 1$ or $x = 3$

(ii) (α) $x < -2$ or $1 < x < 3$

(β) $-2 < x < 1$ or $x > 3$

(b) (i) $\frac{dy}{dx} = 3x^2 - 6x = 3x(x-2)$

(ii) At $(1,3)$, $\frac{dy}{dx} = 3 \times 1 \times (1-2)$
$= -3$

Eqn of tangent $y - 3 = -3(x-1)$
$y - 3 = -3x + 3$
$y = -3x + 6$

(iii) tangent horiz when $3x(x-2) = 0$
$x = 0$ or $x = 2$

(c) Let $a = x^2$
$a^2 - 7a + 12 = 0$
$(a-4)(a-3) = 0$
$a = 4$ or 3
$x = \pm 2$ or $\pm\sqrt{3}$

(d) (i) $\Delta = (-7)^2 - 4 \times 2 \times 2$
$= 33$

(ii) Since $\Delta \geq 0$ equation has real roots.

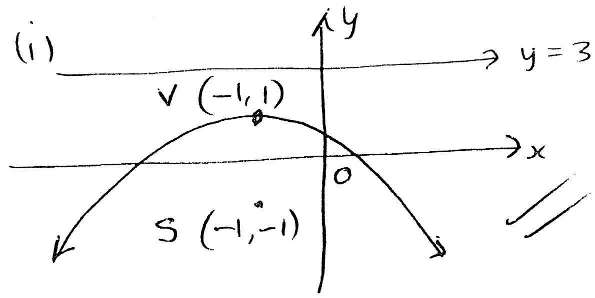
(b) (i) $AP = BP$
$AP^2 = BP^2$
$(x-0)^2 + (y-4)^2 = (x-4)^2 + (y-1)^2$
$x^2 + y^2 - 8y + 16 = x^2 - 8x + 16 + y^2 - 2y +$
$8x - 6y - 1 = 0$

(ii) Perp. bisector of AB (accept "straight line")

(c) (i) $(0, 4)$

(ii) $y = -4$

(d) (i)



(ii) Vertex : $(-1, 1)$

(iii) Equation of parabola :
$(x+1)^2 = -8(y-1)$

Q6

(a) $x^2 - 2x + 1 + y^2 + 8y + 16 = -8 + 17$
$(x-1)^2 + (y+4)^2 = 9$

Centre $(1, -4)$
Radius $= 3$

Q7

(a) (i) $S = \frac{a}{1-r}$

(ii) (α) $r = \frac{18}{27} = \frac{2}{3}$

Since $\left|\frac{2}{3}\right| < 1$ there is a limiting sum.

(β) $S = \frac{27}{1 - \frac{2}{3}}$

$= 81$

(b) $0.\dot{1}\dot{8} = \frac{18}{100} + \frac{18}{10000} + \dots$

$r = \frac{1}{100}$, $a = \frac{18}{100}$

$S = \frac{\frac{18}{100}}{1 - \frac{1}{100}}$

$= \frac{18}{99}$

$= \frac{2}{11}$

(c) (i) $\alpha + \beta = 2$

(ii) $\alpha\beta = \frac{1}{2}$

(iii) $4\alpha\beta(\alpha+\beta) = 4 \times \frac{1}{2} \times 2$

$= 4$

(iv) $(\alpha+\beta)^2 - 2\alpha\beta = 2^2 - 2 \times \frac{1}{2}$

$= 3$

(d) $y' = 6x - 8$

Increasing when $6x - 8 > 0$

$x > \frac{4}{3}$

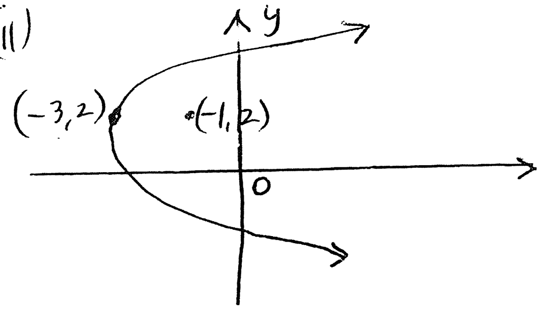
Q8

(a) (i) $y^2 - 4y + 4 = 8x + 24$

$(y-2)^2 = 8(x+3)$

Vertex : $(-3, 2)$
Focus : $(-1, 2)$

(ii)



(b) $\Delta = [-(k+1)]^2 - 4 \times 1 \times (k+1)$

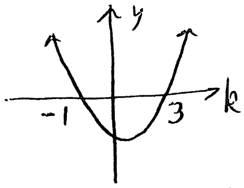
$= k^2 + 2k + 1 - 4k - 4$

$= k^2 - 2k - 3$

For pos. def. $k^2 - 2k - 3 < 0$

$(k-3)(k+1) < 0$

$-1 < k < 3$



(c) (i) $f(x) = (2x^2+1)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(2x^2+1)^{-\frac{1}{2}} \times 4x$

$= \frac{2x}{\sqrt{2x^2+1}}$

(ii) $f'(2) = \frac{2 \times 2}{\sqrt{2 \times 2^2 + 1}} = \frac{4}{3}$

grad of normal $= -\frac{3}{4}$

Eqn of normal : $y - 3 = -\frac{3}{4}(x-2)$

$4y - 12 = -3x + 6$

$3x + 4y - 18 = 0$

(d) $y' = 2ax + b$

At $(2,1)$, $y' = 0$

$4a + b = 0 \quad \dots (1)$

Also, $1 = 4a + 2b - 5$ (since $(2,1)$ on curve)

$4a + 2b = 6$

$2a + b = 3 \quad \dots (2)$

(1) − (2) : $2a = -3$

$a = -\frac{3}{2}$

$b = 6$

Q9

(a) $y' = -\frac{1}{x^2}$ ✓

$-\frac{1}{x^2} = -\frac{1}{4}$

$x^2 = 4$ ✓

$x = 2$ or -2

when $x = 2$, $y = \frac{1}{2}$

when $x = -2$, $y = -\frac{1}{2}$

Points : $(2, \frac{1}{2})$ ✓

$(-2, -\frac{1}{2})$

(b) (i) $k^2 = 3^2 + x^2 - 2\times 3\times x \times \cos 60^\circ$ ✓

$k^2 = 9 + x^2 - 2\times 3\times x \times \frac{1}{2}$ ✓

$k^2 = x^2 + 9 - 3x = x^2 - 3x + 9$

(ii) $x^2 - 3x + 9 - k^2 = 0$

$\Delta = (-3)^2 - 4\times 1\times (9-k^2)$

$= 9 - 36 + 4k^2$ ✓

$= 4k^2 - 27$

For real roots, $4k^2 - 27 \geq 0$

$k^2 \geq \frac{27}{4}$

$k \geq \frac{\sqrt{27}}{2}$ $(k > 0)$

$k \geq \frac{3\sqrt{3}}{2}$ ✓

(c) (i) $T_n = a + (n-1)d$ ✓

$T_{n+1} = a + nd$

(ii) $S_n = \frac{n}{2}[2a + (n-1)d]$ (1) ✓

(Sum of first n terms)

For sum of next n terms :

first term $= a + nd$

Sum $= \frac{n}{2}[2(a+nd) + (n-1)d]$

$= \frac{n}{2}[2a + 2nd + nd - d]$ ✓

$= \frac{n}{2}(2a + 3nd - d)$ (2)

Now, $S_n = 2 \times \text{Sum}$ ✓

$\frac{n}{2}(2a + nd - d) = n(2a + 3nd - d)$

$2a + nd - d = 2(2a + 3nd - d)$

$2a + nd - d = 4a + 6nd - 2d$

$d - 5nd = 2a$

$d(1 - 5n) = 2a$ ✓

$d = \frac{2a}{1 - 5n}$