Sydney Grammar School
Mathematics Department
Yearly Examinations 2007

+ Solutions

FORM V

MATHEMATICS

Y11 2U Yearly
2007

Syd Gram 2007 Y11 2U Q&S

Examination date

Thursday 18th October 2007

Time allowed

2 hours

Instructions

All nine questions may be attempted.
All nine questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Start each question on a new booklet.
Approved calculators and templates may be used.

Collection

Write your name, class and master clearly on each booklet.
Hand in the nine questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

5P: TCW 5Q: RCF 5R: DNW

Checklist

SGS booklets: 9 per boy. A total of 500 booklets should be sufficient.
Candidature: 45 boys.

Examiner

TCW

QUESTION ONE (12 marks) Use a separate writing booklet.

(a) Simplify:

(i) $\left(2\sqrt{3}\right)^2$

(ii) $8^{-\frac{1}{3}}$

(b) Solve:

(i) $|x| = 3$

(ii) $x(x+10) = 0$

(c) Evaluate:

(i) $\log_6 1$

(ii) $\log_6 180 - \log_6 5$

(d) (i) Find the exact value of $\sin 225°$.

(ii) Find the acute α, correct to the nearest minute, given that $\cot \alpha = \frac{1}{3}$.

(e) Factorise:

(i) $x^2 - 100$

(ii) $x^3 - 1000$

QUESTION TWO (12 marks) Use a separate writing booklet.

(a) Rationalise the denominator then simplify: $\dfrac{10}{\sqrt{5} - \sqrt{3}}$.

(b) (i) Write $y = x^2 + 4x + 7$ in the form $y = (x-h)^2 + k$ by completing the square.

(ii) Hence or otherwise write down the coordinates of the vertex of the parabola $y = x^2 + 4x + 7$.

(c) Differentiate:

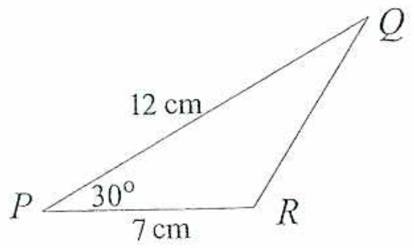
(i) $y = 5x^4$

(ii) $y = 2\sqrt{x}$

(iii) $y = \dfrac{6x^2 - 14x^5}{2x}$

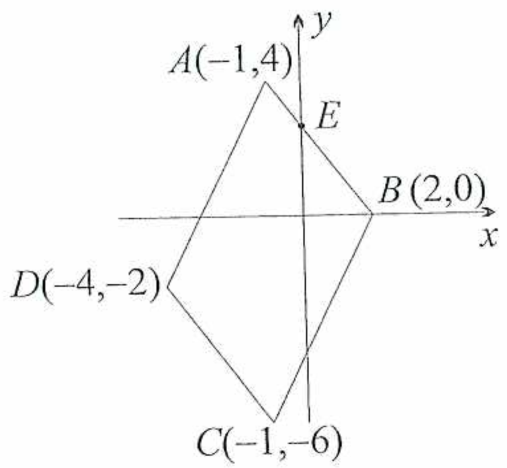
Exam continues next page ...

(d)



Find the area of $\triangle PQR$ in the diagram above.

QUESTION THREE (12 marks) Use a separate writing booklet.



The diagram above shows the parallelogram $ABCD$ with vertices $A(-1, 4)$, $B(2, 0)$, $C(-1, -6)$ and $D(-4, -2)$. The point E is the y-intercept of the line AB.

(a) Show that $AB \parallel DC$.

(b) Show that the equation of the line AB is $4x + 3y - 8 = 0$.

(c) Find the coordinates of E.

(d) Find the coordinates of the midpoint M of AB.

(e) Find the equation of the circle with centre A and radius AB.

(f) Show that the perpendicular distance from D to AB is 6 units.

(g) Hence find the area of parallelogram $ABCD$.
Note: The area of a parallelogram is given by:
Area = Base × Perpendicular Height.

QUESTION FOUR (12 marks) Use a separate writing booklet.

(a) Use the quadratic formula to solve $2x^2 - 6x - 3 = 0$.

(b) For what values of c is the quadratic $2x^2 - 3x + c$ positive definite?

(c) For what values of m does the equation $2x^2 - mx + 2 = 0$ have two distinct real roots?

(d) If α and β are the zeroes of $-x^2 - 4x + 10$, find:

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(iii) $(\alpha + 2)(\beta + 2)$

(iv) $\alpha^2 + \alpha\beta + \beta^2$

QUESTION FIVE (12 marks) Use a separate writing booklet.

(a) A point $P(x, y)$ moves so that it is always 3 units from the line $x = 1$. Find the equations of the locus of P.

(b) Consider the parabola with equation $x^2 = -12y$.

(i) Write down the coordinates of the vertex.

(ii) Write down the coordinates of the focus.

(iii) Write down the equation of the directrix.

(c) An arithmetic sequence has first term 15 and common difference 25.

(i) Write out the first three terms of the sequence.

(ii) Find the sum of the first 41 terms.

(d) In a certain athletics season, each time Rory runs an 800 metre race he lowers his time by $2\frac{1}{4}$ seconds. In his first 800 metre race of the season his time is 2 minutes 30 seconds.

(i) Rory's successive race times in seconds form an arithmetic sequence. Show that the nth term in the sequence is $T_n = \dfrac{609 - 9n}{4}$.

(ii) How many races will he have to run throughout the season to lower his time below the magic 1 minute 50 second barrier?

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet.

(a) Sketch the graph of $y = \log_5 x$, clearly showing the asymptote, the x-intercept and one other point.

(b) Find the equation of the parabola with vertex $(-2, 1)$ and focus $(-2, 3)$.

(c) (i) Differentiate $y = (x+3)^{10}$.

(ii) Use the quotient rule to find the derivative of $y = \dfrac{x^2}{x-2}$.

(iii) (α) Use the product rule to show that $\dfrac{d}{dx}\left(5x(x-4)^4\right) = 5(5x-4)(x-4)^3$.

(β) Hence find the x-coordinates of the points on the curve $y = 5x(x-4)^4$ where the tangents to the curve are horizontal.

QUESTION SEVEN (12 marks) Use a separate writing booklet.

(a) In $\triangle TUV$, $\angle T = 120^\circ$, $TU = 10$ cm and $TV = 15$ cm.

(i) Draw a diagram representing $\triangle TUV$, clearly labelling the vertices and marking the given side lengths and $\angle T$.

(ii) Show that the exact side length of UV is $5\sqrt{19}$ cm.

(b) Evaluate: $\displaystyle\sum_{n=1}^{20} 3 \times 2^{n-1}$.

(c) The fourth term in a geometric sequence is 4 and the eleventh term is -512. Find the eighth term.

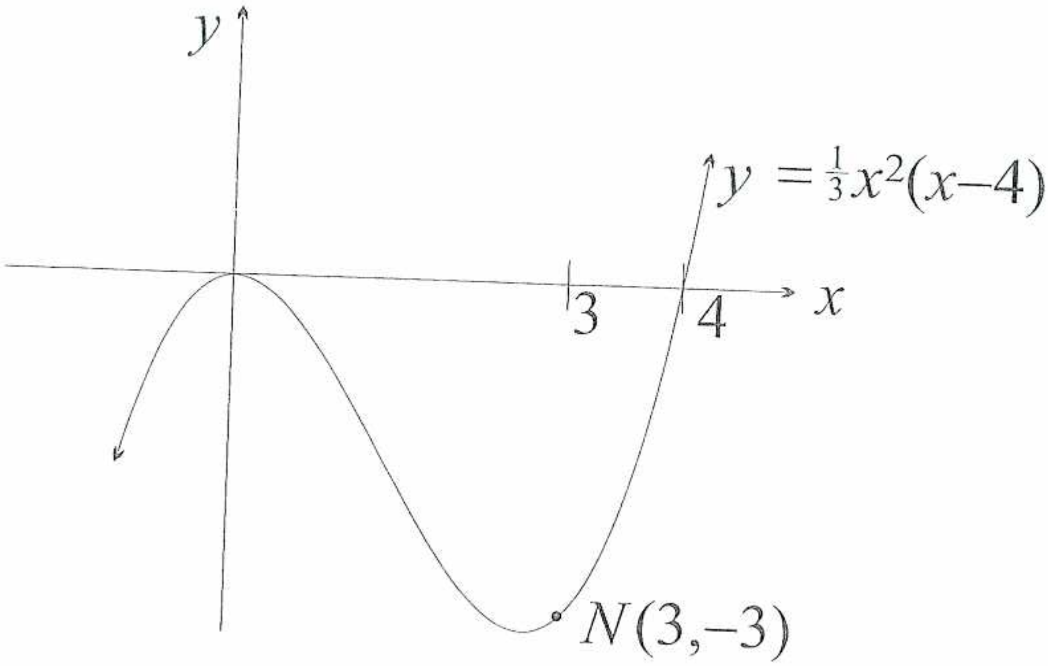
(d) Without finding the point of intersection, find the equation of the line through the point $(1, -3)$ that also passes through the intersection of the two lines $4x - 3y + 2 = 0$ and $x - 2y - 3 = 0$. Give your answer in simplest general form.

QUESTION EIGHT (12 marks) Use a separate writing booklet.

(a) Solve $\cos\theta = 1$ for $0° \leq \theta \leq 360°$.

(b) Solve $2\log_{10} x = \log_{10}(6 - x)$.

(c)



The diagram above shows the graph of $y = \frac{1}{3}x^2(x-4)$. N is a point on the curve with coordinates $(3, -3)$.

(i) Show that the tangent to $y = \frac{1}{3}x^2(x-4)$ at N has equation $y = x - 6$.

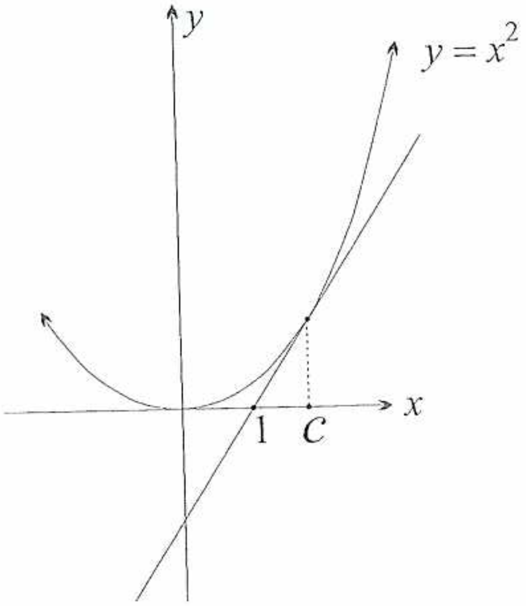
(ii) Show that the normal to $y = \frac{1}{3}x^2(x-4)$ at N has equation $x + y = 0$.

(iii) The points W and X are the points where the tangent and normal at N meet the x-axis. Find the area of $\triangle WXN$.

QUESTION NINE (12 marks) Use a separate writing booklet.

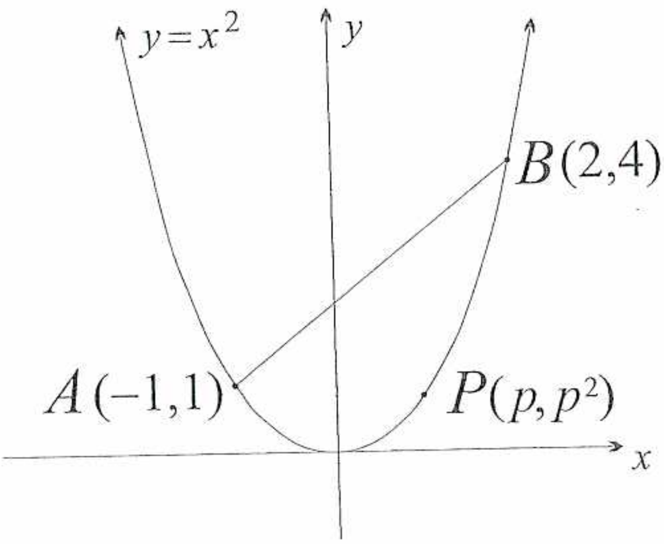
(a) Find the first three terms of the geometric sequence that has a common ratio of $\dfrac{1}{t}$ and a limiting sum of $\dfrac{1}{1-t}$.

(b)



The diagram above shows a graph of the parabola $y = x^2$, and the tangent to the parabola at the point where $x = c$, for $c \neq 0$. For what value of c does the tangent intersect the x-axis at $x = 1$? Clearly show your working.

(c)



In the diagram above $A(-1,1)$ and $B(2,4)$ are points on the parabola $y = x^2$. The point $P(p, p^2)$ is a variable point on the parabola that lies below the line AB.

(i) Show that line AB has equation $x - y + 2 = 0$.

(ii) Find the maximum possible area of $\triangle APB$.

END OF EXAMINATION

V MATHEMATICS (2 UNIT) - ANNUAL EXAMINATION 2007

9 × 12 marks = 108 marks TOTAL

Q1

(a) (i) $(2\sqrt{3})^2 = 12$ ✓

(ii) $8^{-\frac{1}{3}} = \frac{1}{2}$ ✓

(b) (i) $|x| = 3$

$x = 3$ or -3 ✓

(ii) $x(x+10) = 0$

$x = 0$ or -10 ✓

(c) (i) $\log_6 1 = 0$ ✓

(ii) $\log_6 180 - \log_6 5 = \log_6 \frac{180}{5}$ ✓

$= \log_6 36$

$= 2$ ✓

(d) (i) $\sin 225° = -\sin 45°$

$= -\frac{1}{\sqrt{2}}$ ✓

(ii) $\cot \alpha = \frac{1}{3}$

$\tan \alpha = 3$ ✓

[no penalty for rounding] $\alpha \doteqdot 71°34'$ ✓ (nearest minute)

(e) (i) $x^2 - 100 = (x-10)(x+10)$ ✓

(ii) $x^3 - 1000 = x^3 - 10^3$

$= (x-10)(x^2+10x+100)$ ✓

/12

Q2

(a) $\frac{10}{\sqrt{5}-\sqrt{3}} = \frac{10}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ ✓

$= \frac{10(\sqrt{5}+\sqrt{3})}{2}$

$= 5(\sqrt{5}+\sqrt{3})$ ✓

[OR $5\sqrt{5} + 5\sqrt{3}$]

(b) (i) $y = x^2 + 4x + 7$

$y = x^2 + 4x + 4 + 3$

$y = (x+2)^2 + 3$ ✓✓

(ii) vertex $= (-2, 3)$ ✓

(c) (i) $y = 5x^4$

$y' = 20x^3$ ✓

(ii) $y = 2x^{\frac{1}{2}}$ ✓

$y' = 2 \times \frac{1}{2} x^{-\frac{1}{2}}$

$= x^{-\frac{1}{2}}$ ✓

$= \frac{1}{\sqrt{x}}$

(iii) $y = \frac{6x^2}{2x} - \frac{14x^5}{2x}$

$y = 3x - 7x^4$ ✓

$y' = 3 - 28x^3$ ✓

(d) Area $= \frac{1}{2} \times 7 \times 12 \times \sin 30°$ ✓

$= 21 \text{ cm}^2$ ✓

[-1 for incorrect or no units]

/12

Q3 $A(-1,4)$ $B(2,0)$ $C(-1,-6)$ $D(-4,-2)$

(a) $m_{AB} = \frac{0-4}{2+1} = -\frac{4}{3}$ ✓ $m_{DC} = \frac{-2+6}{-4+1} = -\frac{4}{3}$ ✓

$m_{AB} = m_{DC}$, so $AB \parallel DC$.

(b) AB: $y - 0 = -\frac{4}{3}(x-2)$ ✓ ✓

$3y = -4x + 8$

$4x + 3y - 8 = 0$

(c) when $x = 0$, $3y = 8$

$y = \frac{8}{3}$ ✓

$E = (0, 2\frac{2}{3})$

(d) $M_{AB} = \left(\frac{2-1}{2}, \frac{0+4}{2}\right)$

$= \left(\frac{1}{2}, 2\right)$ ✓

(e) centre $A(-1,4)$

$AB^2 = 3^2 + 4^2$ ✓

$AB = 5$

circle : $(x+1)^2 + (y-4)^2 = 25$ ✓ ✓

(f) Perp. distance D to AB

$= \frac{|4(-4) + 3(-2) - 8|}{\sqrt{4^2+3^2}}$ ✓

$= \frac{|-30|}{5}$ ✓

$= 6$ units

(g) Area $= 6 \times AB$

$= 6 \times 5$

$= 30$ units^2 ✓

Q4

(a) $2x^2 - 6x - 3 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(2)(-3)}}{4}$ ✓

$x = \frac{3 \pm \sqrt{15}}{2}$ ✓

(b) $2x^2 - 3x + c$

positive definite : $a > 0$, $\Delta < 0$

$a = 2$ and $(-3)^2 - 4(2)(c) < 0$ ✓

$9 - 8c < 0$

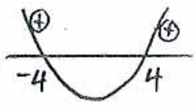
$c > 1\frac{1}{8}$ ✓

(c) $2x^2 - mx + 2 = 0$

two distinct, real roots : $\Delta > 0$ ✓

$m^2 - 4(2)(2) > 0$

$(m-4)(m+4) > 0$



$m < -4$ or $m > 4$ ✓

(d) $-x^2 - 4x + 10$

(i) $\alpha + \beta = -\frac{b}{a}$

$= -4$ ✓

(ii) $\alpha\beta = \frac{c}{a}$

$= -10$ ✓

(iii) $(\alpha+2)(\beta+2) = \alpha\beta + 2\alpha + 2\beta + 4$

$= \alpha\beta + 2(\alpha+\beta) + 4$ ✓

$= -10 + 2(-4) + 4$

$= -14$ ✓

(iv) $\alpha^2 + \alpha\beta + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2 - \alpha\beta$

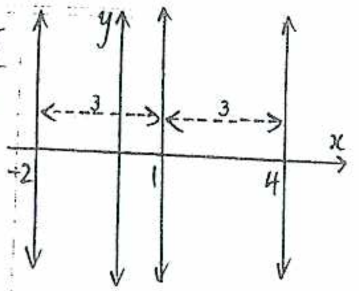
$= (\alpha+\beta)^2 - \alpha\beta$ ✓

$= 16 + 10$

$= 26$ ✓

Q5

(a)



locus of P :

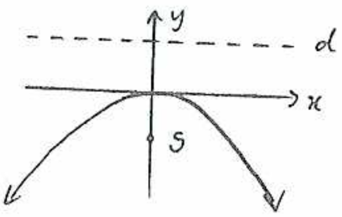
$x = 4$ and $x = -2$ ✓ ✓

(b)

$x^2 = -12y$

$4a = 12$

$a = 3$



(i) vertex $= (0,0)$ ✓

(ii) focus $= (0,-3)$ ✓

(iii) directrix : $y = 3$ ✓

(c) (i) $15, 40, 65$ ✓

(ii) $S_{41} = \frac{41}{2}(2a + 40d)$ ✓

$= 41(15 + 20 \times 25)$

$= 21\,115$ ✓

(d) (i) AP : $150, 147\frac{3}{4}, 145\frac{1}{2}, \ldots$ ✓

$a = 150$, $d = -2\frac{1}{4}$

$T_n = a + (n-1)d$

$T_n = 150 - \frac{9}{4}(n-1)$ ✓

$T_n = \frac{609 - 9n}{4}$

(ii) 1 min 50 s = 110 s

$110 = \frac{609 - 9n}{4}$ ✓

$440 = 609 - 9n$

$9n = 169$

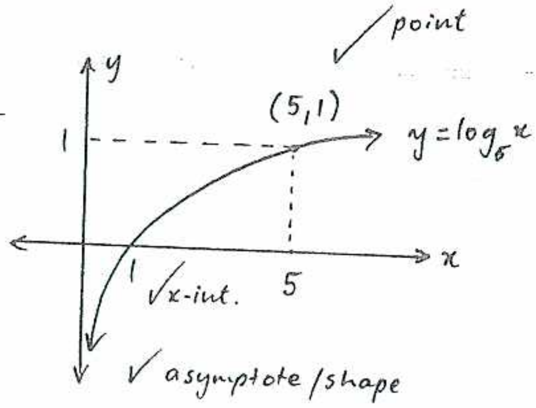
$n = 18\frac{7}{9}$

So Rory will have to run 19 races. ✓

12

Q6

(a)



(b) $V(-2,1)$ $S(-2,3)$

$(x-h)^2 = 4a(y-k)$

$(x+2)^2 = 8(y-1)$ ✓✓

(c) (i) $y = (x+3)^{10}$

$y' = 10(x+3)^9$ ✓

(ii) $y = \frac{x^2}{x-2}$

$y' = \frac{2x(x-2) - 1(x^2)}{(x-2)^2}$ ✓

$= \frac{2x^2 - 4x - x^2}{(x-2)^2}$

$= \frac{x^2 - 4x}{(x-2)^2}$ ✓

$= \frac{x(x-4)}{(x-2)^2}$

(iii) (α) Let $y = 5x(x-4)^4$

$\frac{dy}{dx} = 5(x-4)^4 + 5x \times 4(x-4)^3$ ✓ ✓

$= 5(x-4)^3(x - 4 + 4x)$ ✓

$= 5(5x-4)(x-4)^3$

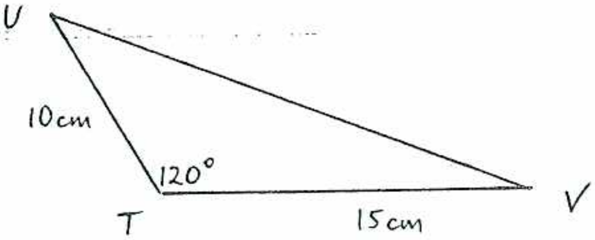
(β) horizontal tangent : $y' = 0$

$x = 4$ or $\frac{4}{5}$ ✓

12

Q7

(a) (i)



(ii) By the cos rule,

$$UV^2 = 10^2 + 15^2 - 2 \times 10 \times 15 \cos 120^\circ$$
$$UV^2 = 325 - 300 \times -\tfrac{1}{2}$$
$$UV = \sqrt{475}$$
$$UV = \sqrt{25 \times 19}$$
$$UV = 5\sqrt{19}$$

(b)
$$\sum_{n=1}^{20} 3 \times 2^{n-1} = 3(2^0) + 3(2^1) + 3(2^2) + \dots + 3(2^{19})$$
$$= \underbrace{3 + 6 + 12 + \dots + 3(2^{19})}_{}$$

GP: $a = 3$, $r = 2$, $n = 20$

$$= \frac{3(2^{20} - 1)}{2 - 1}$$
$$= 3\ 145\ 725$$

(c)
$$\frac{T_{11}}{T_4} = \frac{-512}{4}$$
$$\frac{ar^{10}}{ar^3} = -128$$
$$r^7 = -128$$
$$r = -2$$

(or equations)

$$ar^3 = 4$$
$$a(-2)^3 = 4$$
$$a = -\frac{1}{2}$$

$$T_8 = ar^7$$
$$= \left(-\tfrac{1}{2}\right) \times (-2)^7$$
$$= 64$$

(d) line through the intersection: $4x - 3y + 2 + k(x - 2y - 3) = 0$

substitute $(1, -3)$: $4 + 9 + 2 + k(1 + 6 - 3) = 0$

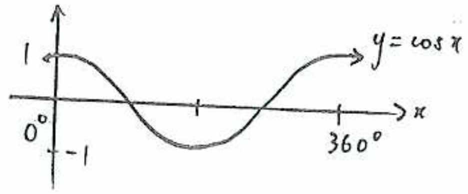
$$4k = -15$$
$$k = -\frac{15}{4}$$

equation: $4x - 3y + 2 - \frac{15}{4}(x - 2y - 3) = 0$

$$16x - 12y + 8 - 15x + 30y + 45 = 0$$
$$x + 18y + 53 = 0$$

12

Q8 (a) $\cos\theta = 1, \quad 0° \leq \theta \leq 360°$



$\theta = 0°$ or $360°$ ✓✓

(b)

$2\log_{10} x = \log_{10}(6-x)$

$\log_{10} x^2 = \log_{10}(6-x)$

$x^2 = 6-x, \quad x > 0$ and $6-x > 0$ ✓

$x^2+x-6 = 0, \quad 0 < x < 6$

$(x+3)(x-2) = 0, \quad 0 < x < 6$ ✓

$x = 2$ or $-3, \quad 0 < x < 6$

so $x = 2$ is the only solution. ✓

[two solutions = 2/3]

(c) (i) $y = \frac{1}{3}x^3 - \frac{4}{3}x^2$

$\frac{dy}{dx} = x^2 - \frac{8}{3}x$ ✓

when $x=3$, $\frac{dy}{dx} = 9 - \frac{8}{3}\times 3$

$= 1$ ✓

tangent at $N(3,-3)$:

$y+3 = 1(x-3)$ ✓

$y = x-6$ ——— ①

(ii) normal at $N(3,-3)$:

$y+3 = -1(x-3)$ ✓

$y+3 = -x+3$ ✓

$x+y = 0$ ——— ②

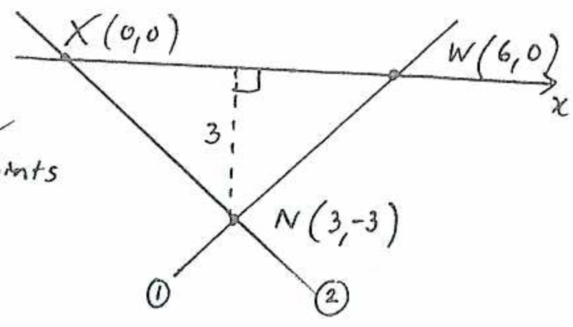
(iii) ① : when $y=0$, $x=6$

$W = (6,0)$

② : when $y=0$, $x=0$

$X = (0,0)$

✓ points



Area of $\Delta WXN = \frac{1}{2}\times 6\times 3$

$= 9$ units^2 ✓ area

Q9 (a)

GP : $r = \frac{1}{t}$ $S_\infty = \frac{1}{1-t}$

$$\frac{a}{1-\frac{1}{t}} = \frac{1}{1-t}$$

$$\frac{at}{t-1} = \frac{1}{1-t}$$

$$at(1-t) = t-1$$

$$a = \frac{\cancel{t-1}}{-t(\cancel{t-1})}$$

$$a = -\frac{1}{t}$$

GP : $-\frac{1}{t}, -\frac{1}{t^2}, -\frac{1}{t^3}, \ldots$

(b)

$$y = x^2$$
$$y' = 2x$$

when $x = c$, $y' = 2c$ and $y = c^2$

tangent at $x = c$:

$$y - c^2 = 2c(x-c)$$
$$y = 2cx - c^2$$

for x-intercept $(1,0)$:

$$0 = 2c - c^2$$
$$c(2-c) = 0$$
$$c = 0 \text{ or } 2$$

note : when $c=0$ the tangent is the x-axis, but $c \neq 0$.

solution : when $c = 2$ the tangent intersects the x-axis at $x = 1$.

(c) (i) $m_{AB} = \frac{4-1}{2+1} = 1$

line AB: $y-1 = 1(x+1)$

$y - 1 = x + 1$

$x - y + 2 = 0$

OR $x - y + 2 = 0$

sub $A(-1,1)$:

LHS $= -1-1+2$

$= 0$

$=$ RHS

sub $B(2,4)$:

LHS $= 2-4+2$

$= 0$

$=$ RHS

So A, B lie on the line.

AB is the line $x-y+2=0$

(ii) Perpendicular distance from

$$P \text{ to } AB = \frac{|p - p^2 + 2|}{\sqrt{1^2+1^2}}$$

$$= \frac{|-p^2+p+2|}{\sqrt{2}}$$

Area of $\Delta APB = \frac{1}{2} \times AB \times \text{height}$

$$= \frac{1}{2} \times \sqrt{9+9} \times \frac{|-p^2+p+2|}{\sqrt{2}}$$

$$= \frac{1}{2} \times 3\cancel{\sqrt{2}} \times \frac{|-p^2+p+2|}{\cancel{\sqrt{2}}}$$

$$= \frac{3}{2}|-p^2+p+2| \quad \text{where } -1<p<2$$

The area is maximised when $-p^2+p+2$, $-1<p<2$ is maximised:

ie when $p = -\frac{b}{2a}$

$p = -\frac{1}{-2}$

$p = \frac{1}{2}$

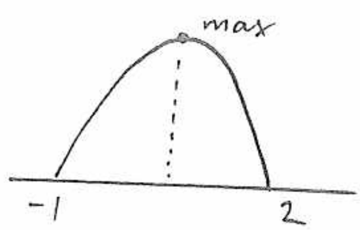
Maximum area of $\Delta APB = \frac{3}{2}\left|-\left(\frac{1}{2}\right)^2 + \frac{1}{2} + 2\right|$

$= \frac{3}{2} \times \frac{9}{4}$

$= 3\frac{3}{8}$ units^2

OR area $= \frac{3}{2}(-p^2+p+2)$ since $-1<p<2$

$= \frac{3}{2}(p+1)(2-p)$



$p = \frac{2-1}{2}$

$p = \frac{1}{2}$

Max. area $= \frac{3}{2}\left(\frac{1}{2}+1\right)\left(2-\frac{1}{2}\right)$

$= 3\frac{3}{8}$ units^2

12