Sydney Grammar School
Mathematics Department
Yearly Examinations 2008

+ Solutions

~~Y11 Task~~

Y11 2U Yearly
2008

Syd Gram 2008 Y11 2U Q8,5'

FORM V

MATHEMATICS

Examination date

Tuesday 2nd September 2008

Time allowed

2 hours

Instructions

All eight questions may be attempted.
All eight questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

Write your name, class and master clearly on each booklet.
Hand in the eight questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

5P: KWM 5Q: JMR 5R: LYL

Checklist

Folded A3 booklets: 8 per boy. A total of 500 booklets should be sufficient.
Candidature: 50 boys.

Examiner

KWM

QUESTION ONE (15 marks) Use a separate writing booklet.

(a) Simplify $\sqrt{18} - \sqrt{8}$.

(b) Use your calculator to find the value of $\cos 56°43'$ correct to three decimal places.

(c) Simplify $\log_a a^2$.

(d) Solve the equation $x^2 - 2x + 1 = 0$.

(e) Find the gradient of the interval joining the points $P(-2, 1)$ and $Q(5, -2)$.

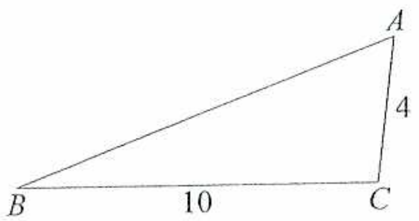
(f) Factorise $a^2 - ab + 2a - 2b$.

(g) Find the value of p given that 5, p, 21 is an arithmetic sequence.

(h) Find $\dfrac{dy}{dx}$ given $y = 5x^4 - 3x + 7$.

(i) Simplify $\sqrt{49x^2y^6}$.

(j)



In the triangle above $AC = 4$ cm, $BC = 10$ cm and $\sin B = \frac{1}{5}$. Find $\angle A$ given that it is acute.

(k) Write down the exact value of $\cos 30°$.

(l) Factorise $p^3 - 8$.

QUESTION TWO (15 marks) Use a separate writing booklet.

(a) (i) Sketch the parabola $y = (x-1)(x+3)$, showing all intercepts with the axes and the vertex.

(ii) Hence, or otherwise, solve the inequation $x^2 + 2x - 3 < 0$.

(b) Write down the domain of the function $g(x) = \sqrt{x}$.

(c) Write down a quadratic equation that has roots 2 and -3.

(d) Solve:

(i) $|x+1| = 3$

(ii) $|x+1| < 3$

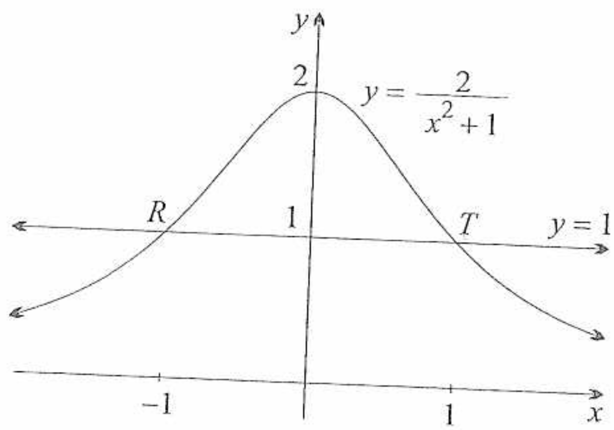
(e) Solve $\tan\theta = -1$, for $0^\circ \leq \theta \leq 360^\circ$.

(f) Given $2^x = 60$.

(i) Rewrite this exponential statement in logarithmic form.

(ii) Hence solve $2^x = 60$, writing your solution correct to two decimal places.

(g)



The diagram shows the curve $y = \dfrac{2}{x^2+1}$ and the line $y = 1$ intersecting the curve at points $R(-1,1)$ and $T(1,1)$.

(i) Write down the range of $y = \dfrac{2}{x^2+1}$.

(ii) Explain why $y = \dfrac{2}{x^2+1}$ is an even function.

(iii) Use the diagram to solve $\dfrac{2}{x^2+1} \leq 1$.

QUESTION THREE (15 marks) Use a separate writing booklet.

(a) Find $\dfrac{dy}{dx}$ for each of the following functions:

(i) $y = \sqrt{x}$

(ii) $y = \dfrac{1}{2x^2}$

(iii) $y = \dfrac{4x - 3x^2}{x}$

(b) Let $f(x) = x^2 + 3x - 4$.

(i) Find:

(α) $f(1)$

(β) $f'(x)$

(γ) $f'(1)$

(ii) Hence find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 1$.

(c) For some real number t, the first three terms of an arithmetic sequence are $2t$, $5t - 1$ and $6t + 2$.

(i) Find the value of t.

(ii) Hence find the fourth term of the sequence.

(d) Evaluate $\displaystyle\lim_{h \to 0} \frac{3x^2h - h^2}{h}$.

(e) Use the quotient rule to find $\dfrac{dy}{dx}$, given $y = \dfrac{2x}{x^2 + 1}$.

Exam continues next page ...

QUESTION FOUR (15 marks) Use a separate writing booklet.

(a) Solve $3x^2 - 2x - 2 = 0$ using the quadratic formula, writing your solutions in simplest exact form.

(b) (i) Find the discriminant of the quadratic equation $2x^2 - 3x + 1 = 0$.

(ii) Select the letter of the statement that correctly describes the nature of the roots of this quadratic:

A. real, rational and equal

B. unreal

C. real, rational and unequal

D. unreal, irrational and unequal

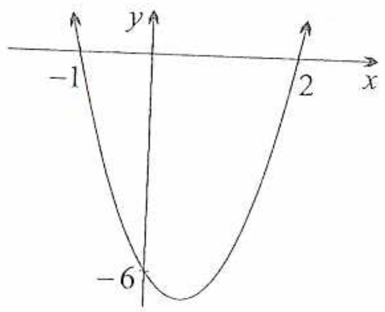
(c) Let α and β be the roots of the quadratic $x^2 - 5x + 10 = 0$. Find the value of:

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(iii) $\alpha^2 + \beta^2$

(d)



The diagram above shows a parabola that has x-intercepts -1 and 2, and a y-intercept of -6. Find the equation of the parabola.

(e) (i) Express the quadratic function $y = x^2 - 4x + 1$ in the form $y = (x - h)^2 + k$.

(ii) Hence, or otherwise, write down the minimum value of $x^2 - 4x + 1$.

(f) Factorise $10x^2 - 13xy - 9y^2$.

Exam continues overleaf ...

QUESTION FIVE (15 marks) Use a separate writing booklet.

(a) Consider the sequence 10, 13, 16, ...

(i) Show that the sequence is arithmetic and hence find the common difference.

(ii) Find T_{26}, the twenty-sixth term of the sequence.

(iii) Find the sum of the first twenty-six terms of the sequence.

(b) The third term of an arithmetic sequence is 18 and the eighth term is 48. Find the first term, the common difference and hence write down the first three terms of the sequence.

(c) (i) Find the tenth term of the geometric sequence 3, 6, 12, ...

(ii) Calculate the sum of the first 10 terms of the series $3 + 6 + 12 + \cdots$.

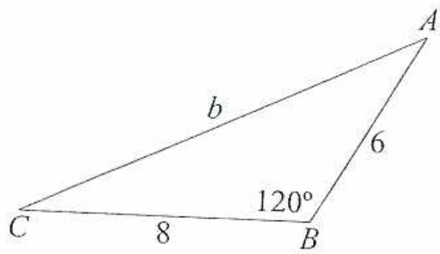
(iii) How many terms of the series $3 + 6 + 12 + \ldots$ need to be added to give a sum greater than 30 000?

(d) A pendulum is set swinging. It covers a distance of 27 centimetres on its first swing, 18 centimetres on its second swing, 12 centimetres on its third swing and so on. Find the total distance it swings through before coming to rest.

Exam continues next page ...

QUESTION SIX (15 marks) Use a separate writing booklet.

(a)



The diagram above shows $\triangle ABC$ with $a = 8$ cm, $c = 6$ cm and $\angle ABC = 120°$. Calculate the exact length b, leaving your answer in simplest surd form.

(b) (i) Find the gradient of the line $3x - 4y + 1 = 0$.

(ii) Find in general form, the equation of the line that passes through the point $(3, 1)$ and is perpendicular to the line $3x - 4y + 1 = 0$.

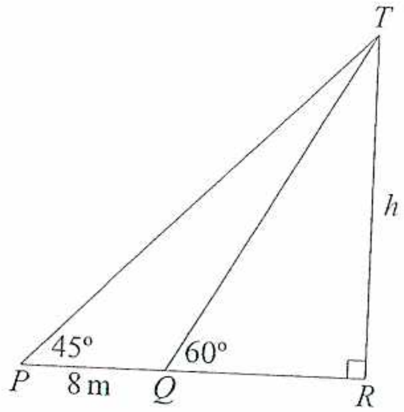
(iii) Calculate the perpendicular distance from the point $(3, 1)$ to the line $3x - 4y + 1 = 0$.

(c) Simplify:

(i) $\log_3 81$

(ii) $\log_{10} 5 + 3\log_{10} 2 - \log_{10} 4$

(d)



In the diagram above two points P and Q are 8 metres apart and on the same level as the foot R of a vertical tower. The angles of elevation to the top T of the tower of height h are $45°$ and $60°$ respectively. Let h metres be the height of the tower. Show that $h = (12 + 4\sqrt{3})$ metres.

Exam continues overleaf ...

QUESTION SEVEN (15 marks) Use a separate writing booklet.

(a) Find the equation of the line that passes through the point $K(2,1)$ and also through the point of intersection of the lines $l_1 : 2x - y + 1 = 0$ and $l_2 : x + y - 2 = 0$.

(b) Find the equation of the tangent to the curve $y = \sqrt{x-1}$ at the point $Q(2,1)$. Express the equation in general form.

(c) Given that $a + \dfrac{a}{4} + \dfrac{a}{16} + \cdots = \dfrac{128}{3}$, find the value of a.

(d) (i) Given $f(x) = 3x(2x-1)^3$, find $f'(x)$ in fully factored form.

(ii) Hence find the x coordinates of the points where the curve $y = 3x(2x-1)^3$ has a gradient of zero.

(e) Solve $\tan^2\theta - \tan\theta - 2 = 0$, for $0^\circ \le \theta \le 360^\circ$. Where necessary leave your answer correct to the nearest minute.

Exam continues next page …

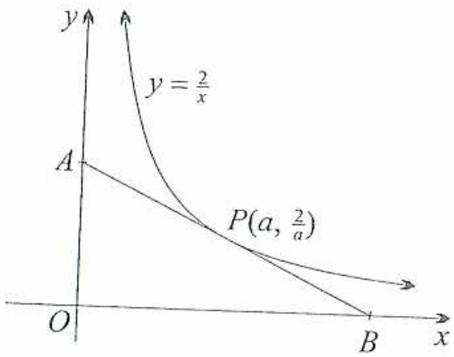
QUESTION EIGHT (15 marks) Use a separate writing booklet.

(a) (i) Find the discriminant of the quadratic equation

$$kx^2 - 4kx - (k-5) = 0.$$

(ii) Hence find the values of k for which the quadratic equation $kx^2 - 4kx - (k-5) = 0$ has no real solutions.

(b)



The diagram above shows part of the hyperbola $y = \dfrac{2}{x}$ and the tangent to the hyperbola drawn at an arbitrary point $P(a, \frac{2}{a})$. The tangent intersects the y-axis and x-axis at A and B respectively.

(i) Find the gradient of the curve at the point $P(a, \frac{2}{a})$.

(ii) Hence show that the equation of the tangent to the curve $y = \dfrac{2}{x}$ at the point $P(a, \frac{2}{a})$ is

$$2x + a^2y - 4a = 0.$$

(iii) Show that the area of $\triangle AOB$ is constant.

(c) (i) Find an expression for the nth term T_n of the sequence $1, 3, 5, \ldots$.

(ii) Show that the nth partial sum of the series $1 + 3 + 5 + \cdots$ is given by $S_n = n^2$.

(iii) Show that $\displaystyle\sum_{k=1}^{n} \left(2^k - (2k-1)\right) = 2^{n+1} - n^2 - 2$.

END OF EXAMINATION

FORM V MATHEMATICS SOLUTIONS YEARLY 2008.

QUESTION 1

a) $\sqrt{18} - \sqrt{8} = 3\sqrt{2} - 2\sqrt{2}$
$= \sqrt{2}$ ✓

b) $\cos 56°43' \doteqdot 0.549$ ✓

c) $\log_a a^2 = 2\log_a a$
$= 2$ ✓

(d) $x^2 - 2x + 1 = 0$
$(x-1)^2 = 0$
$x = 1$ ✓

(e) $P(-2,1) \quad Q(5,-2)$
$m_{PQ} = \frac{1--2}{-2-5}$ ✓
$= -\frac{3}{7}$ ✓

(f) $a^2 - ab + 2a - 2b$
$= a(a-b) + 2(a-b)$ ✓
$= (a+2)(a-b)$ ✓

(g) $21 - p = p - 5$
$21 = 2p - 5$
$2p = 26$
$p = 13$ ✓

(h) $y = 5x^4 - 3x + 7$
$\frac{dy}{dx} = 20x^3 - 3$ ✓

(i)
$\sqrt{49x^2y^6} = 7xy^3$ ✓

(j) $\frac{\sin A}{a} = \frac{\sin B}{b}$
$\sin A = \frac{\frac{1}{5} \times 10}{4}$ ✓
$\sin A = \frac{1}{2}$
$A = 30°$ ✓

(k) $\cos 30° = \frac{\sqrt{3}}{2}$ ✓

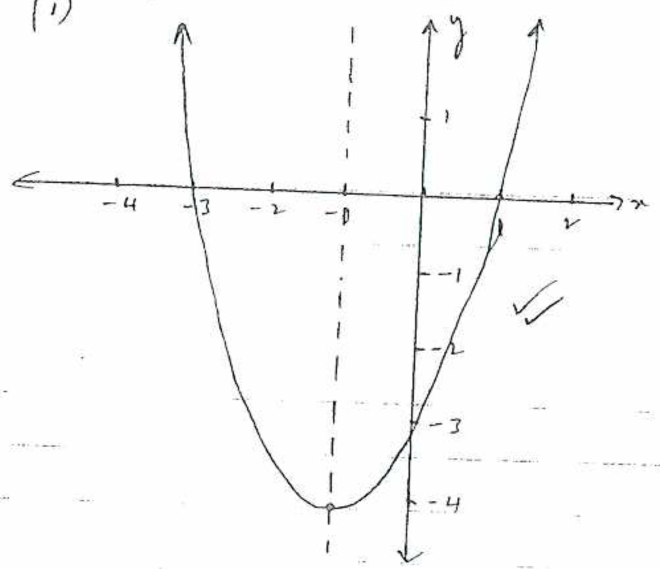
(l) $p^3 - 8 = (p-2)(p^2 + 2p + 4)$ ✓

15

QUESTION 2

(a) $y = (x-1)(x+3)$

(i)



(ii) $x^2 + 2x - 3 < 0$

$-3 < x < 1$ ✓

(b) Domain $x \geq 0$. ✓

(c) $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$x^2 + x - 6 = 0$ ✓

(d) (i) $|x+1| = 3$

$x+1 = 3$ or $x+1 = -3$

$x = 2$ ✓ or $x = -4$ ✓

(ii) $x+1 < 3$ or $x+1 > -3$

$x < 2$ $\quad$ $x > -4$

$-4 < x < 2$ ✓

(e) $\tan\theta = -1$

$\theta = 135°$ ✓ or $\theta = 315°$ ✓

(f) $2^x = 60$

(i) $x = \log_2 60$ ✓

(ii) $x = \dfrac{\log_{10} 60}{\log_{10} 2}$

$x \doteqdot 5.91$ ✓

(g) (i) Range: $0 < y \leq 2$ ✓

(ii) The curve is symmetrical about the y-axis. ✓

(iii) $x \leq -1$ or $x \geq 1$ ✓

15

QUESTION 3

(a)

(i) $y = \sqrt{x}$

$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$ ✓

(ii) $y = \frac{1}{2}x^{-2}$

$\frac{dy}{dx} = -x^{-3}$

$= -\frac{1}{x^3}$ ✓

(iii) $y = \frac{4x - 3x^2}{x}$

$y = 4 - 3x$ ✓

$\frac{dy}{dx} = -3$ ✓

(b) $f(x) = x^2 + 3x - 4$

(i) $f(1) = 1 + 3 - 4$

$= 0$ ✓

(ii) $f'(x) = 2x + 3$ ✓

(iii) $f'(1) = 5$ ✓

(v) $y - y_1 = m(x - x_1)$

$(1, 0)$: $y - 0 = 5(x - 1)$ ✓

$y = 5x - 5$

$5x - y - 5 = 0$ ✓

(c) $5t - 1 - 2t = 6t + 2 - (5t - 1)$

$3t - 1 = t + 3$

$2t = 4$

$t = 2$ ✓

4, 9, 14.

(d) $\lim_{h \to 0} \frac{3x^2h - h^2}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 - h)}{\cancel{h}}$ ✓

$= \lim_{h \to 0} 3x^2 - h$

$= 3x^2$ ✓

(e) $f(x) = \frac{2x}{x^2 + 1}$

$f'(x) = \frac{(x^2+1)2 - 2x \times 2x}{(x^2+1)^2}$ ✓

$= \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2}$

$= \frac{2 - 2x^2}{(x^2+1)^2}$ ✓

15

QUESTION 4

(a) $3x^2 - 2x - 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{4+24}}{6} \checkmark$$

$$x = \frac{2 \pm 2\sqrt{7}}{6}$$

$$x = \frac{1+\sqrt{7}}{3} \checkmark \quad \text{or} \quad x = \frac{1-\sqrt{7}}{3} \checkmark$$

(b) (i) $2x^2 - 3x + 1 = 0$

$\Delta = b^2 - 4ac$

$\Delta = 9 - 4 \times 2 \times 1$

$\Delta = 1$ ✓

(ii) C ✓

(c) $x^2 - 5x + 10 = 0$

(i) $\alpha + \beta = \frac{-b}{a}$

$= 5$ ✓

(ii) $\alpha\beta = \frac{c}{a}$

$= 10$ ✓

(iii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$ ✓

$= 25 - 20$

$= 5$ ✓

(d) $y = a(x+1)(x-2)$ ✓

when $x = 0$, $y = -6$.

$-6 = a \times 1 \times -2$

$a = 3$.

The parabola has equat^n

$y = 3(x+1)(x-2)$ ✓

(e)(i) $y = x^2 - 4x + 1$

$y = (x-2)^2 + 1 - 4$

$y = (x-2)^2 - 3$ ✓

(ii) The minimum value of $x^2 - 4x + 1$ is -3. ✓

(f) $10x^2 - 13xy - 9y^2$

$$\left.\begin{aligned} ab &= -90 \\ a+b &= -13 \end{aligned}\right\} -18, 5$$

$$= \frac{(10x - 18y)(10x + 5y)}{10}$$

$$= (5x - 9y)(2x + y) \checkmark\checkmark$$

15

QUESTION 5

(a) $10, 13, 16, \ldots$

(i) $13-10 = 16-13 = 3$

$d = 3$ ✓

(ii) $t_n = a + (n-1)d$

$t_{26} = 10 + 25 \times 3$

$t_{26} = 85$ ✓

(iii) $S_n = \frac{n}{2}(a+l)$

$S_{26} = 13(10+85)$

$S_{26} = 1{,}235$ ✓

(b) $\begin{cases} ① \; a + 2d = 18 \\ ② \; a + 7d = 48 \end{cases}$ ✓

② − ①: $5d = 30$

$d = 6$ ✓

① $a + 12 = 18$

$a = 6$ ✓

$6, 12, 18, \ldots$ ✓

(c) (i) $3, 6, 12, \ldots$

$a = 3 \qquad r = 2$ ✓

$t_n = ar^{n-1}$

$t_{10} = 3 \times 2^9$

$t_{10} = 1536$ ✓

(ii) $S_n = \frac{a(r^n - 1)}{r-1}$

$S_{10} = \frac{3(2^{10}-1)}{1}$

$S_{10} = 3{,}069$ ✓

(iii) $S_n = \frac{3(2^n - 1)}{1} > 30\,000$

$2^n - 1 > 10\,000$

$2^n > 10\,001$ ✓

by calculator:

$2^{14} > 10{,}001$

14 terms need to be added.

or

$n \log 2 > \log 10\,001$

$n > \frac{\log 10001}{\log 2}$

$n > 13.29$

$\therefore n = 14$. 14 terms. ✓

(d) $27, 18, 12, \ldots$

$r = \frac{18}{27} = \frac{2}{3} \qquad \frac{12}{18} = \frac{2}{3}$ ✓

$-1 < \frac{2}{3} < 1$, $\therefore S_\infty$ exists

$S_\infty = \frac{a}{1-r}$

$= \frac{27}{1-\frac{2}{3}}$ ✓

$= \frac{27}{\frac{1}{3}}$

$= 81$

Total distance 81 cm. ✓

15

QUESTION 6

(a)

$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 64 + 36 - 2 \times 8 \times 6 \times \cos 120°$ ✓

$b^2 = 64 + 36 - 2 \times 8 \times 6 \times -\frac{1}{2}$

$b^2 = 100 + 48$

$b = \sqrt{148}$ ✓

$b = 2\sqrt{37}$ cm.

(b) (i) $3x - 4y + 1 = 0$

(i)

$4y = 3x + 1$

$y = \frac{3}{4}x + 1$

gradient $= \frac{3}{4}$. ✓

(ii) $(3,1)$ $y - y_1 = m(x - x_1)$

$\perp m = -\frac{4}{3}$ ✓ $y - 1 = -\frac{4}{3}(x - 3)$ ✓

$3y - 3 = -4x + 12$

$4x + 3y - 15 = 0$ ✓

(iii) $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

$(3,1)$ $3x - 4y + 1 = 0$

$d = \left|\frac{9 - 4 + 1}{\sqrt{9 + 16}}\right|$ ✓

$d = 1\frac{1}{5}$ units. ✓

(c) (i) $\log_3 81 = 4$ ✓

(ii) $\log_{10} 5 + \log_{10} 2^3 - \log_{10} 4$

$= \log_{10}\left(\frac{5 \times 8}{4}\right)$ ✓

$= \log_{10} 10$

$= 1$ ✓

(d) $QR = h - 8$ (isosceles $\triangle TPR$) ✓

$\tan 60° = \frac{h}{h-8}$ ✓

$\frac{h}{h-8} = \sqrt{3}$

$h = \sqrt{3}h - 8\sqrt{3}$

$8\sqrt{3} = \sqrt{3}h - h$

$h(\sqrt{3} - 1) = 8\sqrt{3}$

$h = \frac{8\sqrt{3}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$ ✓

$h = \frac{24 + 8\sqrt{3}}{2}$

$\therefore h = 12 + 4\sqrt{3}$ m ✓

15

QUESTION 7

(a)

$$(2x-y+1)+k(x+y-2)=0$$

$$(4-1+1)+k(2+1-2)=0 \checkmark$$

$$4+k=0$$

$$k=-4 \checkmark$$

$$2x-y+1-4(x+y-2)=0 \checkmark$$

$$2x-y+1-4x-4y+8=0$$

$$-2x-5y+9=0$$

$$2x+5y-9=0 \checkmark$$

(b) $y=(x-1)^{1/2}$

$$\frac{dy}{dx}=\frac{1}{2}(x-1)^{-1/2} \checkmark$$

gradient at $Q(2,1)$

$$m=\frac{1}{2}(2-1)^{-1/2}$$

$$m=\frac{1}{2} \checkmark$$

$$y-y_1=m(x-x_1)$$

$$y-1=\frac{1}{2}(x-2)$$

$$2y-2=x-2$$

$$x-2y=0 \checkmark$$

(c) a, $r=\frac{1}{4}$

$$S_\infty: \quad \frac{a}{1-\frac{1}{4}}=\frac{128}{3} \checkmark$$

$$\frac{a}{\frac{3}{4}}=\frac{128}{3}$$

$$\frac{4a}{3}=\frac{128}{3}$$

$$4a=128$$

$$a=32 \checkmark$$

(d)

(i) $f(x)=3x(2x-1)^3$

$$f'(x)=3(2x-1)^3+x(2x-1)^2\times 2 \checkmark$$

$$f'(x)=3(2x-1)^2\{2x-1+6x\}$$

$$f'(x)=3(2x-1)^2(8x-1) \checkmark$$

(ii) $3(2x-1)^2(8x-1)=0$

$x=\frac{1}{2} \checkmark$ or $x=\frac{1}{8} \checkmark$

(e) $\tan^2\theta-\tan\theta-2=0 \checkmark$

$$(\tan\theta+1)(\tan\theta-2)=0$$

$\tan\theta=-1$ or $\tan\theta=2$

$\theta=135^\circ$ or $\theta=63^\circ 26'$

$\theta=315^\circ \checkmark$ or $\theta=243^\circ 26'$.

15

QUESTION 8

(a) $Kx^2 - 4Kx - (K-5) = 0$

$\Delta = b^2 - 4ac$

$\Delta = 16K^2 + 4K(K-5)$ ✓

$\Delta = 16K^2 + 4K^2 - 20K$

$\Delta = 20K^2 - 20K$ ✓

(ii) no real solutions $\Delta < 0$

$20K^2 - 20K < 0$

$K^2 - K < 0$

$K(K-1) < 0$ ✓ 

$0 < K < 1$ ✓

(b) $y = 2x^{-1}$

$\frac{dy}{dx} = -2x^{-2}$

$= \frac{-2}{x^2}$ ✓

i) at $x = a$, gradient $= -\frac{2}{a^2}$ ✓

$P(a, \frac{2}{a})$

(ii) $y - y_1 = m(x - x_1)$

$y - \frac{2}{a} = -\frac{2}{a^2}(x - a)$

$a^2y - 2a = -2x + 2a$

$\underline{2x + a^2y - 4a = 0}$ ✓

(iii) B: $2x - 4a = 0$

$2x = 4a$

$x = 2a.$

B. $(2a, 0)$ ✓

A: $a^2y - 4a = 0$

$a^2y = 4a$

$y = \frac{4}{a}$

A $(0, \frac{4}{a})$ ✓

Area $\Delta AOB = \frac{1}{2}bh$

$= \frac{1}{2} \times 2a \times \frac{4}{a}$

$= 4$ sq units ✓

(c)

(i) $1 + 3 + 5 \ldots$

$a = 1 \qquad d = 2$ ✓

$t_n = a + (n-1)d$

$t_n = 1 + (n-1)2$

$\underline{t_n = 2n - 1}$ ✓

(ii) $S_n = \frac{n}{2}(a + l)$

$= \frac{n}{2}(1 + 2n - 1)$ ✓

$= \frac{n}{2}(2n)$

$= n^2$

(iii) $\sum_{k=1}^{n} 2^k - \sum_{k=1}^{n} (2k-1)$

$(2 + 2^2 + \ldots) - (1 + 3 + 5 + \ldots$

GP ✓ AP, from (ii) $= n^2$

$a = 2 \quad r = 2$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_n = \frac{2(2^n - 1)}{2 - 1}$

$= 2^{n+1} - 2$

15

$\therefore 2^{n+1} - 2 - n^2$

$= 2^{n+1} - n^2 - 2.$ ✓