SYDNEY GRAMMAR SCHOOL

2013 Annual Examination

FORM V
MATHEMATICS 2 UNIT

Wednesday 28th August 2013

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your name, class and master on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your name and master on this question paper and submit it with your answers.

5P: MLS 5Q: GMC 5R: BR

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 35 boys

Examiner
MLS

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

What is 71·06784 to three significant figures?

(A) 71·068

(B) 71·067

(C) 71·0

(D) 71·1

QUESTION TWO

What is the gradient of the interval joining the points $P(5,-3)$ and $Q(1,7)$?

(A) $-\dfrac{2}{5}$

(B) $\dfrac{3}{2}$

(C) $-\dfrac{5}{2}$

(D) $\dfrac{2}{3}$

QUESTION THREE

What are the solutions of the quadratic equation $2x^2 + 3x - 2 = 0$?

(A) $x = \dfrac{1}{2}$ and $x = -2$

(B) $x = \dfrac{1}{2}$ and $x = 2$

(C) $x = -1$ and $x = 2$

(D) $x = 1$ and $x = -2$

Exam continues next page ...

QUESTION FOUR

What is the perpendicular distance from the point $(1, -2)$ to the line $2x - y + 1 = 0$?

(A) $\dfrac{5}{\sqrt{5}}$

(B) $\dfrac{4}{\sqrt{5}}$

(C) $\dfrac{5}{\sqrt{3}}$

(D) $\dfrac{4}{\sqrt{3}}$

QUESTION FIVE

The quadratic equation $x^2 + 5x - 1 = 0$ has roots α and β.
What is the value of $\alpha + \beta$?

(A) -1

(B) 1

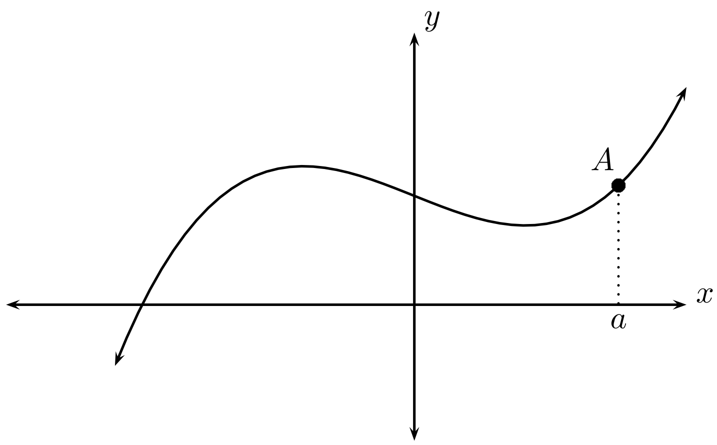
(C) -5

(D) 5

Exam continues overleaf ...

QUESTION SIX

The diagram shows the graph of $y = f(x)$.



Which of the following statements is true?

(A) The gradient at A is positive.

(B) The gradient at A is negative.

(C) $f'(a) = 0$

(D) $f(a) = 0$

QUESTION SEVEN

If the discriminant of a quadratic equation is 0, which of the following types of roots will the equation have?

(A) Real, rational and distinct roots

(B) Equal real roots

(C) No real roots

(D) Real, irrational and distinct roots

QUESTION EIGHT

Given the sequence $5, 8, 11, 14, \ldots$, which of the following statements is correct?

(A) The eighth term is 29.

(B) The ninth term is 29.

(C) The sequence has a limiting sum.

(D) The common ratio is 3.

Exam continues next page ...

QUESTION NINE

Which of the following is the derivative of $6x^3 - 7x + 3$?

(A) $18x^2 - 7$

(B) $18x^2 - 7x$

(C) $6x^2 - 7$

(D) $18x^2 - 7x + 3$

QUESTION TEN

Which of the following is the correct statement of the quotient rule used to differentiate $f(x) = \dfrac{u}{v}$?

(A) $f'(x) = \dfrac{u\dfrac{dv}{dx} - v\dfrac{du}{dx}}{v^2}$

(B) $f'(x) = \dfrac{u\dfrac{dv}{dx} + v\dfrac{du}{dx}}{u^2}$

(C) $f'(x) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

(D) $f'(x) = \dfrac{u\dfrac{dv}{dx} - v\dfrac{du}{dx}}{u^2}$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Solve $x^2 = 7x$. **2**

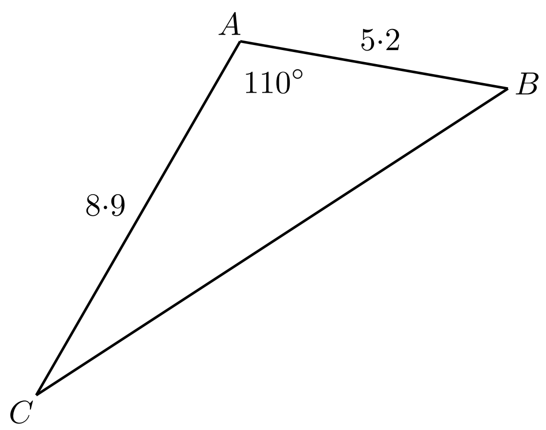
(b) Differentiate $y = x^5 + 2x + 1$. **2**

(c) Solve $|x - 5| = 3$. **2**

(d) State the domain and range of the function $y = \sqrt{25 - x^2}$. **3**

(e) Find all values of θ, where $0° \leq \theta \leq 360°$, that satisfy the equation $\cos\theta - \frac{2}{5} = 0$. Answer to the nearest degree. **2**

(f)



In the diagram, ABC is a triangle where $AB = 5{\cdot}2$ metres, $AC = 8{\cdot}9$ metres and $\angle BAC = 110°$.

(i) Find the length of BC to the nearest metre. **2**

(ii) Calculate the area of triangle ABC to the nearest square metre. **2**

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Differentiate the following:

(i) $y = x^3 - 7x^2 + 3x - 5$ **2**

(ii) $y = \dfrac{3}{x}$ **2**

(iii) $y = \dfrac{3x^4 - 2x^3}{x^2}$ **2**

(iv) $y = (3x + 7)^3$ **2**

(v) $y = \dfrac{x}{(x-1)^3}$ **2**

(b) A geometric series has a first term of 8 and a common ratio of $\frac{1}{2}$. Calculate the sum of the first 5 terms. **2**

(c) (i) Write down the discriminant of $3x^2 + 2x + k$. **1**

(ii) For what values of k does the equation $3x^2 + 2x + k = 0$ have real roots? **2**

Exam continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let α and β be the roots of the equation $3x^2 - 5x - 2 = 0$. Without solving the equation find:

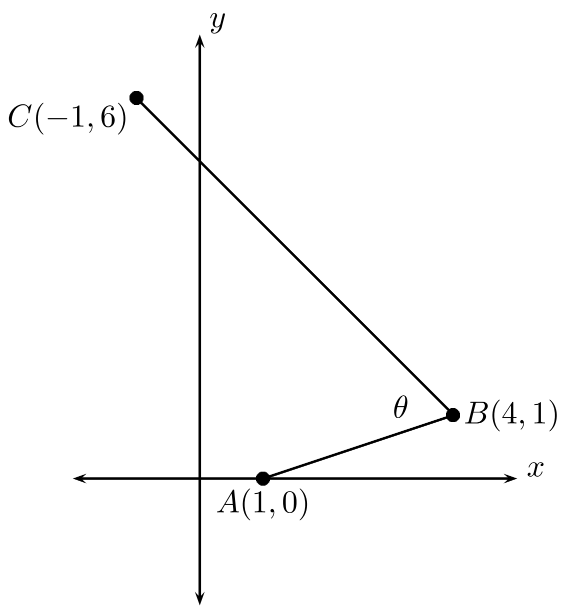
(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **2**

(iv) $\alpha^2 + \beta^2$ **2**

(b)



The diagram shows points $A(1,0), B(4,1)$ and $C(-1,6)$ in the Cartesian plane.
Angle ABC is θ.
Copy this diagram onto your answer sheet.

(i) Show that the equation of line AC is $y = 3 - 3x$. **2**

(ii) Show that the gradient of AB is $\frac{1}{3}$. **1**

(iii) Show that AB and AC are perpendicular. Mark the right angle on your diagram. **2**

(iv) Find the lengths of AB and AC. **2**

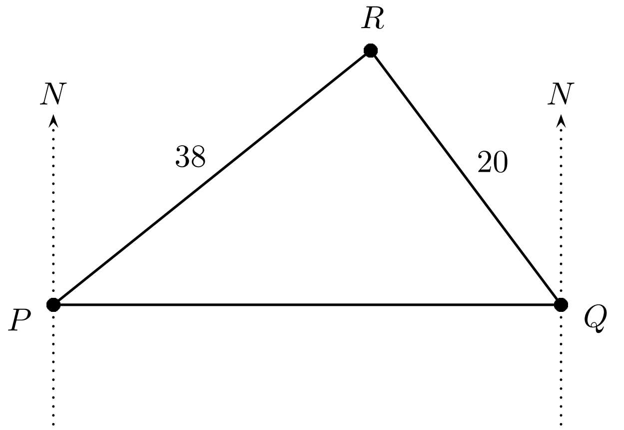
(v) Find the area of triangle ABC. **2**

Exam continues next page ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Use your calculator and the change of base formula to find the value of $\log_5 70$ to two decimal places. **2**

(b)



In the diagram, the point Q is due east of P. The point R is 38 km from P and 20 km from Q. The bearing of R from Q is 325°.

(i) What is the size of $\angle PQR$? **1**

(ii) What is the bearing of R from P? Give your answer to the nearest degree. **3**

(c) (i) Find the gradient of the tangent to the curve $y = x^2 - 3x$ at the point $P(1, -2)$. **2**

(ii) Find the equation of the tangent to the curve at P. **1**

(d) Determine algebraically whether the function $f(x) = \dfrac{x^3 - 3x}{2x^2 + 1}$ is even, odd or neither. **2**

(e) The tenth term of an arithmetic sequence is 29 and the fifteenth term is 44.

(i) Show that the common difference is 3. **1**

(ii) Find the first term. **1**

(iii) Find the sum of the first 75 terms. **2**

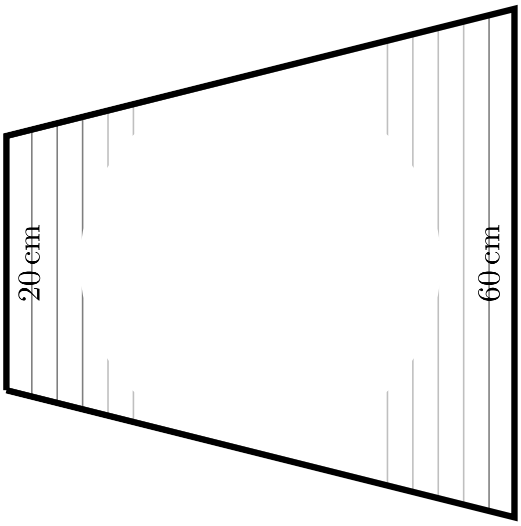
QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\sum_{n=2}^{4} n^2$. **1**

(b) Simplify $\sin^3 x + \sin x \cos^2 x + \sin x$. **2**

(c) Solve $\log_2(3x - 4) = 5$. **2**

(d)



William builds a stringed musical instrument. The diagram above shows this instrument with a few of its strings drawn. The difference between the lengths of adjacent strings is constant, so that the lengths of the strings are the terms of an arithmetic sequence.
The shortest string is 20 cm long and the longest string is is 60 cm long. The sum of the lengths of the strings is 840 cm.

(i) Find the number of strings. **2**

(ii) Find the difference in length between adjacent strings. **2**

(e) Solve $\cot x = 2$, for $-180° \le x \le 180°$, giving your solutions correct to the nearest degree. **3**

(f) (i) Show that $\cos\theta \tan\theta = \sin\theta$. **1**

(ii) Hence solve $8 \sin\theta \cos\theta \tan\theta = -\operatorname{cosec}\theta$, for $0° \le \theta \le 360°$. **2**

Exam continues next page ...

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Sketch $y = |x - 2| - 4$, showing clearly any intercepts with the axes. **2**

(b) (i) Show that for all values of m, the line $y = mx - 3m^2$ is a tangent to the parabola $x^2 = 12y$. **2**

(ii) Find the values of m for which this line passes through the point $(-5, 2)$. **2**

(iii) Hence determine the equations of the two tangents to the parabola $x^2 = 12y$ from the point $(-5, 2)$. **1**

(c) Solve $\sin(x + 60^\circ) = \dfrac{1}{\sqrt{2}}$, for $0^\circ \le x \le 360^\circ$. **3**

(d) Consider the geometric series

$$1 - \tan^2\theta + \tan^4\theta - \tan^6\theta + \cdots.$$

(i) Assuming that the limiting sum exists, show that the limiting sum is $\cos^2\theta$. **2**

(ii) For what values of θ, given $0^\circ \le \theta < 90^\circ$, does the limiting sum exist? **3**

End of Section II

END OF EXAMINATION

Tear-off pages follow ...

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2013
Annual Examination
FORM V
MATHEMATICS 2 UNIT
Wednesday 28th August 2013

- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

NAME: ....

CLASS: ....MASTER: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

Solutions

Form V 2U 2013 Yearly.

1. $71.06784 \simeq 71.1$ — D

2. $m = \frac{7--3}{1-5} = \frac{10}{-4} = -\frac{5}{2}$ — C

3. $(2x^2+3x-2) = 0$
   $(2x-1)(x+2) = 0$
   $x = \frac{1}{2}$ or -2 — A

4. $d = \left|\frac{2+2+1}{\sqrt{4+1}}\right|$ — A
   $= \frac{5}{\sqrt{5}}$

5. $\alpha+\beta = \frac{-b}{a} = -5$ — C

6. A

7. B

8. AP $a=5,\ d=3$
   $T_8 = 5+21 = 26$
   $T_9 = 5+24 = 29$. — B

9. $18x^2-7$ — A

10. C

11.

a) $x^2 = 7x$
$x^2 - 7x = 0$ ✓
$x(x-7) = 0$
$x = 0$ or 7 ✓ (one ✓ for $x=7$ only).

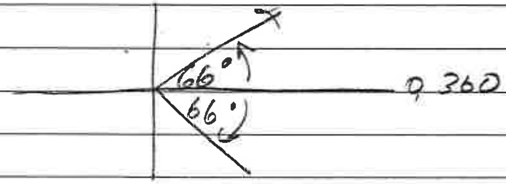
(b) $y = x^5+2x+1$
$\frac{dy}{dx} = 5x^4+2$ ✓✓

(c) $|x-5| = 3$.
$x-5 = 3$ or $x-5 = -3$
$x = 8$ ✓ or $x = 2$ ✓

(d) Domain: $25-x^2 \geq 0$ ✓
$-5 \leq x \leq 5$ ✓
Range: $0 \leq y \leq 5$ ✓

(e) $\cos\theta - \frac{2}{5} = 0$
$\cos\theta = \frac{2}{5}$

related angle is $66°$ ✓



$\theta = 66°$ or $294°$ ✓

(f)

(i) $BC^2 = 8.9^2 + 5.2^2 - 2 \times 8.9 \times 5.2 \times \cos 110^0$

$= 137.9073845$

$BC = 12$ m

(ii) area $ABC = \frac{1}{2} \times 8.9 \times 5.2 \times \sin 110^0$

$= 22$ m^2

12.

(a) (i) $y = x^3 - 7x^2 + 3x - 5$

$\frac{dy}{dx} = 3x^2 - 14x + 3$

(ii) $y = \frac{3}{x}$

$= 3x^{-1}$

$\frac{dy}{dx} = -3x^{-2}$

$= \frac{-3}{x^2}$

(iii) $y = \frac{3x^4 - 2x^3}{x^2}$

$= 3x^2 - 2x$

$\frac{dy}{dx} = 6x - 2$

iv) $y = (3x+7)^3$

$\frac{dy}{dx} = 3(3x+7)^2 \times 3$

$= 9(3x+7)^2$

v) $y = \frac{x}{(x-1)^3}$ $u = x$ $v = (x-1)^3$

$u' = 1$ $v' = 3(x-1)^2$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(x-1)^3 - 3x(x-1)^2}{(x-1)^6}$

$= \frac{(x-1)^2[x-1-3x]}{(x-1)^6}$ (no need to simplify).

$= \frac{-2x-1}{(x-1)^4}$

(b) $a = 8, \quad r = \frac{1}{2}$.

$$S_5 = \frac{a(1-r^n)}{1-r}$$

$$= \frac{8\left(1-\left(\frac{1}{2}\right)^5\right)}{1-\frac{1}{2}}$$

$$= 16\left(1 - \frac{1}{2^5}\right)$$

$$= 16 - \frac{2^4}{2^5}$$

$$= 15\frac{1}{2}$$

(c) (i) $\Delta = b^2 - 4ac$

$$= 4 - 4 \times 3 \times k$$

$$= 4 - 12k$$

(ii) For real roots $\Delta \geq 0$.

$$4 - 12k \geq 0$$

$$4 \geq 12k$$

$$\frac{1}{3} \geq k$$

ie $k \leq \frac{1}{3}$

Q13.

a) (i) $\alpha + \beta = \frac{-b}{a}$

$$= \frac{5}{3}.$$

(ii) $\alpha\beta = \frac{c}{a}$

$$= -\frac{2}{3}$$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$$= \frac{5}{3} \div \left(-\frac{2}{3}\right)$$

$$= \frac{5}{3} \times \frac{-3}{2}$$

$$= -\frac{5}{2}$$

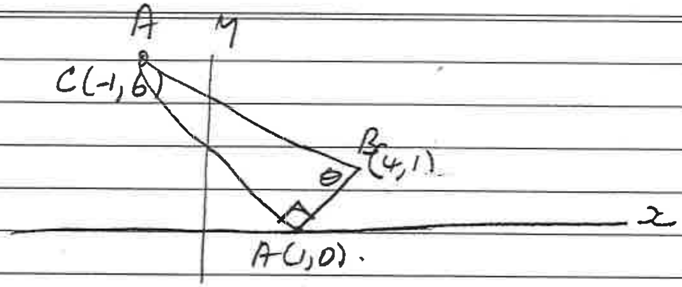
(iv) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$= \left(\frac{5}{3}\right)^2 - 2 \times \left(-\frac{2}{3}\right)$$

$$= \frac{25}{9} + \frac{4}{3}$$

$$= \frac{37}{9}$$

b).



(i) $m_{AC} = \frac{6-0}{-1-1} = -3$ ✓

$y - y_1 = m(x - x_1)$

$y - 0 = -3(x-1)$ ✓

$y = 1 - 3x$

(ii) $M_{AB} = \frac{1-0}{4-1}$ ✓

$= \frac{1}{3}$

(iii) $M_{AC} \times M_{AB} = -3 \times \frac{1}{3}$

$= -1$ ✓

$\therefore AC \perp AB$

✓ for marking on diagram.

(iv) $AB = \sqrt{(4-1)^2 + (1-0)^2}$

$= \sqrt{9+1}$

$= \sqrt{10}$ ✓

$AC = \sqrt{(1+1)^2 + (6)^2}$

$= \sqrt{4+36}$

$= \sqrt{40}$ ✓

$= 2\sqrt{10}$

v. Area $\triangle ABC = \frac{1}{2}bh$

$= \frac{1}{2} \times \sqrt{10} \times 2\sqrt{10}$ ✓

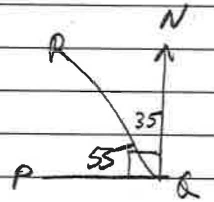
$= 10 \ u^2$ ✓

Q14.

a) $\log_5 70 = \dfrac{\log_{10} 20}{\log_{10} 5}$

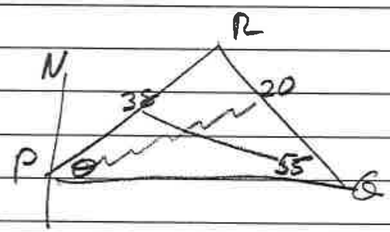
$= 2.64$

(b)
(i)



$\angle PQR$ is $55°$

(ii)



$\dfrac{\sin\theta}{20} = \dfrac{\sin 55}{38}$

$\sin\theta = \dfrac{\sin 55}{38} \times 20$

$= 0.431132$

$\theta = 25.5°$ or $26°$

Bear is $90° - 25.5° = 065°$ (take $064°$ or $64°$)

C,
(i) $y = x^2 - 3x$
$\dfrac{dy}{dx} = 2x - 3$

at $(1, -2)$, $m = 2 - 3 = -1$

ii) Tangent at P is
$y - y_1 = m(x - x_1)$
$y + 2 = -1(x - 1)$
$y + 2 = -x + 1$
$y = -x - 1$
or $x + y + 1 = 0$

(d) $f(x) = \dfrac{x^3 - 3x}{2x^2 + 1}$

$f(-x) = \dfrac{(-x)^3 - 3(-x)}{2(-x)^2 + 1}$

$= \dfrac{-x^3 + 3x}{2x^2 + 1}$

$= -\dfrac{(x^3 - 3x)}{2x^2 + 1}$

$= -f(x)$

So odd

e. AP

(i) $T_{10} = a + 9d = 29$ ①
$T_{15} = a + 14d = 44$ ②
② − ① $5d = 15$
$d = 3$. or any sensible method.

(ii) $a + 9d = 29$
$a + 27 = 29$
$a = 2$

(iii) $S_{75} = \frac{n}{2}(2a + (n-1)d)$
$= \frac{75}{2}(4 + 74 \times 3)$
$= 8475$

Q15.

(a) $\sum_{n=2}^{4} n^2 = 2^2 + 3^2 + 4^2$
$= 4 + 9 + 16$
$= 29$

(b) $\sin^3 x + \sin x \cos^2 x + \sin x$
$= \sin^3 x + \sin x(1 - \sin^2 x) + \sin x$
$= \sin^3 x + \sin x - \sin^3 x + \sin x$
$= 2\sin x$

(c) $\log_2(3x-4) = 5$
$3x - 4 = 2^5$
$= 32$
$3x = 36$
$x = 12$.

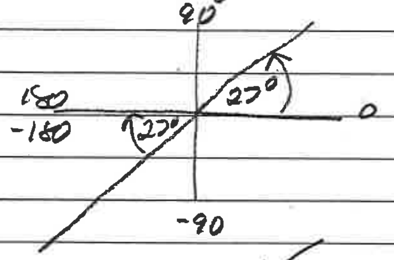
(d)
(i) $S_n = \frac{n}{2}(a + l)$
$840 = \frac{n}{2}(20 + 60)$
$840 = 40n$
$n = \frac{840}{40}$
$= 21$.

(ii) find d.
$T_{21} = 60 = a + (n-1)d$
$60 = 20 + 20d$
$20d = 40$
$d = 2$ cm

e. $\cot x = 2$

$\tan x = \frac{1}{2}$ ✓

related angle is 26.57°



✓ ✓

$x = 27^\circ$ or -153°

(f) (i) $LHS = \cos\theta \tan\theta$

$= \cos\theta \dfrac{\sin\theta}{\cos\theta}$ ✓

$= \sin\theta$

$= RHS$

(ii) $8\sin\theta\cos\theta\tan\theta = -\operatorname{cosec}\theta$

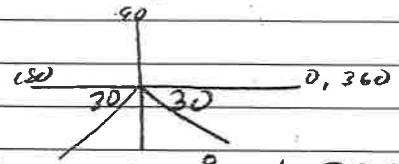
$8\sin^2\theta = -\dfrac{1}{\sin\theta}$

$-8\sin^3\theta = -1$

$\sin^3\theta = -\frac{1}{8}$

$\sin\theta = -\frac{1}{2}$ ✓

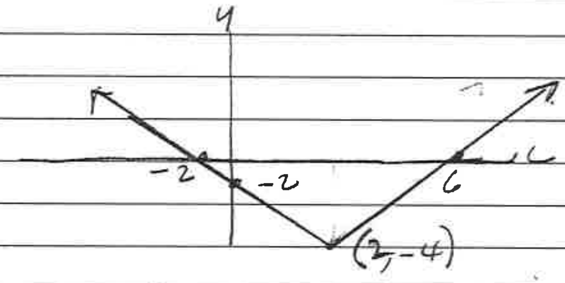
related angle is 30°



$\theta = 210^\circ$ or 330° ✓

Q16.

a).



✓ for shape

✓ for all x, y intercepts

b)(i) At point of intersection

$mx - 3m^2 = \frac{1}{12}x^2$ ✓

$12mx - 36m^2 = x^2$

$x^2 - 12mx + 36m^2 = 0$

$\Delta = b^2 - 4ac$

$= 144m^2 - 4 \times 36m^2$ ✓

$= 0$

So there is 1 point of intersection

So the line touches the parabola

(ii)

At $(-5, 2)$, $y = mx - 3m^2$

$2 = -5m - 3m^2$

$3m^2 + 5m + 2 = 0$ ✓

$(3m+2)(m+1) = 0$

$m = -\frac{2}{3}$ or -1. ✓

$\frac{4}{4}$

(iii) $m = -1$, tangent is $y = -x - 3$ ✓

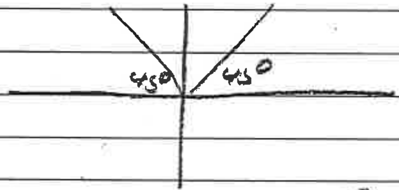
$m = -\frac{2}{3}$, tangent is $y = -\frac{2}{3}x - \frac{4}{3}$ ✓

c) $\sin(x+60^\circ) = \frac{1}{\sqrt{2}}$ $\qquad 0^\circ \le x \le 360^\circ$

let $u = x + 60^\circ$, $\qquad 60^\circ \le u \le 420^\circ$

$\sin u = \frac{1}{\sqrt{2}}$

related angle is 45° ✓



$u = 180 - 45^\circ,\ 360 + 45$

$= 135^\circ$ or 405°.

$x + 60^\circ = 135^\circ$ or 405°

$x = 75^\circ$ or 345°

✓ ✓

d) $1 - \tan^2\theta + \tan^4\theta - \tan^6\theta \ldots$

$a = 1 \qquad r = -\tan^2\theta$

(i)

limiting sum $= \frac{a}{1-r}$

$= \frac{1}{1+\tan^2\theta}$ ✓

$= \frac{1}{\sec^2\theta}$ ✓

$= \cos^2\theta$.

ii) For limiting sum $|r| < 1$

So we want $-1 < -\tan^2\theta < 1$ ✓

or $-1 < \tan^2\theta < 1$

But $\tan^2\theta \ge 0$

so $0 < \tan^2\theta < 1$

ie $0 < \tan\theta < 1$, ✓ $0 \le \theta < 90^\circ$

so $0 < \theta < 45^\circ$ ✓