BAULKHAM HILLS HIGH SCHOOL

YEAR 11

YEARLY EXAMINATION

2006

MATHEMATICS

EXTENSION 1

Time Allowed - Two hours
(Plus five minutes reading time)

DIRECTIONS TO CANDIDATES

* Approved calculators may be used.
* There are 7 questions of equal value. Start each question on a new page.
* Show all necessary working. Marks may be deducted otherwise.

Begin each question on a new page

Question 1

a) Solve $16^{4-x}=\frac{1}{8^x}$. **3**

b) Solve $\frac{x-6}{x}\geq 3$ **3**

c) Find the exact value of the gradient of the tangent to $y=\frac{1}{\sqrt{2x-4}}$ at the point where $x=3$. **3**

d) Find the coordinates of the point P which divides the interval joining points A(4,6) and B(13,5) externally in the ratio 4:1. **2**

e) Evaluate $\sum_{k=3}^{6}(-1)^k k$ **1**

Question 2

a) Find the acute angle between the lines $y=2x-1$ and $3x-2y+1=0$. Answer correct to the nearest degree **3**

b) Solve $\sin 2\theta=-\sqrt{3}\cos\theta$ for $0°\leq x\leq 360°$ **3**

c) Solve $x^4+2x^2-8=0$ **3**

d) Find, from first principles, the derivative of $y=x^2-2x$ **3**

Question 3

a) The line $3x+4y-k=0$ is a tangent to the circle $x^2+y^2=16$. Find the possible values of k. 3

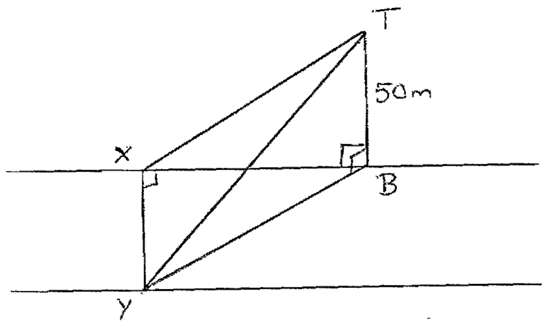
b) (i) On the same axes, sketch the graphs of $y=|x|$ and $y=(x-2)^2$ 2

(ii) Hence solve $|x|<(x-2)^2$ 3

c) The curve $y=x^3+ax^2+bx+c$ has a point of inflexion at (1,2) and a stationary point when $x=4$. Find the value of a, b and c. 4

Question 4

a)



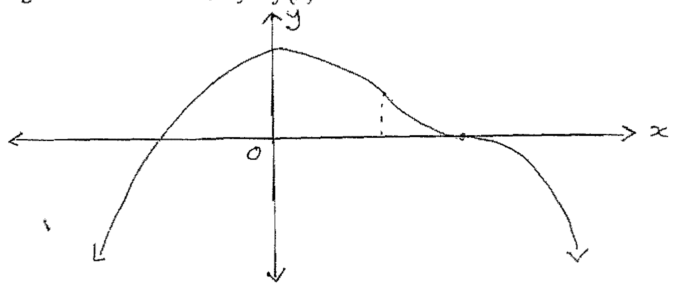
From a point X on a canal bank, a surveyor measures the angle of elevation of a tower 50m high and on the same bank to be $56°$. From a point Y on the other bank and exactly opposite X, another surveyor finds the angle of elevation of the tower is $39°$. Assuming that X, Y and the foot of the tower are all on the same level, show that the breadth of the canal is given by

$$b=50\sqrt{\cot^2 39°-\cot^2 56°}$$

and evaluate this expression, giving your answer correct to 3 significant figures. 3

b) Write the expansion for cos (A-B). Hence find the exact value for cos $15°$. 3

c) The diagram shows a function $y=f(x)$.



Copy the diagram neatly and sketch the curve $y=f'(x)$ on the same axes. 2

d) If α and β are the roots of $2x^2-5x-4=0$, form a quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. 4

Question 5

a) Consider the function $f(x)=\dfrac{x}{4-x^2}$

(i) State the domain of the function 1

(ii) Show that the function is an odd function 2

(iii) Show that the function is increasing throughout its domain. 3

(iv) Evaluate $\lim\limits_{x\to\infty} f(x)$ 1

(v) Sketch the graph of $y=f(x)$ showing all important features 2

b) Prove by Mathematical Induction that $1+4+16+64+\ldots..+4^{n-1}=\frac{1}{3}(4^n-1)$ for all positive integers n. 3

Question 6

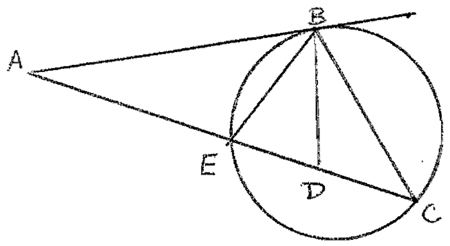
a) (i) Express $\sin x+\sqrt{3}\cos x$ in the form $R\sin(x+\alpha)$, where R>0 and $0° < x < 90°$. 2

(ii) Hence or otherwise solve the equation $\sin x+\sqrt{3}\cos x=\sqrt{2}$ for $0° \le x \le 360°$ 2

b) Find the values of k for which the equation $2x^2-4x+k=0$ has real roots 3

c) Simplify $\dfrac{3^{n+1}+3^{n-1}}{3^{n-2}+3^{n}}$ 2

d) AB is a tangent and AC is a secant of the circle.



BD bisects $\angle$CBE.
Let $\angle CBD = x$.
Prove that ΔABD is isosceles. 3

Question 7

a) For which values of x is the curve $y=x^3-12x+16$ decreasing **and** concave down? 3

b) Simplify $\frac{1}{2}(\cot\frac{\theta}{2}-\tan\frac{\theta}{2})$, expressing any trigonometric ratios in terms of θ 2

b) Anna wins $20 000 in a lottery and decides to invest it in an account paying 10% p.a. compound interest, being compounded annually. She decides to withdraw $2500 each year immediately after the annual interest has been credited to her account.

(i) After Anna makes her first withdrawal, how much remains in the account? 1

(ii) Let $ A_n =value of the investment after the n^{th} withdrawal. Show that $A_n=25000-5000(1.1)^n$ 4

(iii) How many withdrawals of $2500 will Anna be able to make? 2

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Q1

a) $(2^4)^{4-x} = (2^{-3})^x$ 1

$2^{16-4x} = 2^{-3x}$

$16 - 4x = -3x$ 1

$16 = x$ 1

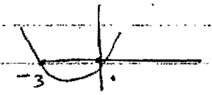
b) $\frac{x-6}{x} \geq 3$ $\quad \times x^2 \quad \times x^2$ $\quad x \neq 0$

$x^2 - 6x \geq 3x^2$ 1

$0 \geq 2x^2 + 6x$

$2x^2 + 6x \leq 0$

$2x(x+3) \leq 0$ 1



$-3 \leq x < 0$ 1

c) $y = (2x-4)^{-1/2}$

$\frac{dy}{dx} = \underbrace{-\frac{1}{2}(2x-4)^{-3/2}}_{1} \cdot \underbrace{2}_{1}$

$= -(2x-4)^{-3/2}$

$= \frac{-1}{(\sqrt{2x-4})^3}$

$= \frac{-1}{(\sqrt{6-4})^3}$ When $x=3$

$= \frac{-1}{2\sqrt{2}}$ 1

d) $A(4,6)$ $\quad B(13,5)$

$4 : -1$

$\left(\frac{4(13)-1(4)}{4-1}, \frac{4(5)-1(6)}{4-1}\right)$

$= \left(\frac{48}{3}, \frac{14}{3}\right)$

$= \left(\underbrace{16}_{1}, \underbrace{14/3}_{1}\right)$

e) $-3+4-5+6$

$= 2$ 1

1 d) Internal division $\left(\frac{56}{3}, \frac{26}{3}\right)$ = 1 mark only

12 marks each question

TOTAL /84

Q2.

a) $y = 2x - 1$, $m_1 = 2$

$3x - 2y + 1 = 0$, $m_2 = \frac{3}{2}$ 1

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2 - 3/2}{1 + 2(\frac{3}{2})}\right|$ 1

$= \left|\frac{1/2}{4}\right|$

$= 1/8$

$\theta = \tan^{-1}(1/8) \doteqdot 7^\circ$ 1

b) $2\sin\theta\cos\theta + \sqrt{3}\cos\theta = 0$ 1

$\cos\theta(2\sin\theta + \sqrt{3}) = 0$

$\cos\theta = 0$ or $\sin\theta = \frac{-\sqrt{3}}{2}$

$\theta = \underbrace{90^\circ, 270^\circ}_{1}, \underbrace{240^\circ, 300^\circ}_{1}$

c) Let $u = x^2$

$u^2 + 2u - 8 = 0$ 1

$(u+4)(u-2) = 0$ 1

$u = x^2 = -4$ No soln $\qquad u = x^2 = 2$

$x = \pm\sqrt{2}$ 1

d) $f(x+h) = (x+h)^2 - 2(x+h)$

$= x^2 + 2xh + h^2 - 2x - 2h$ 1

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ 1

$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 2)}{\cancel{h}}$ 1

$= 2x - 2$

Q3

a) ⊥d of line from centre (0,0)

$$= \left| \frac{3(0)+4(0)-k}{\sqrt{3^2+4^2}} \right|$$

$$= \left| \frac{-k}{5} \right|$$

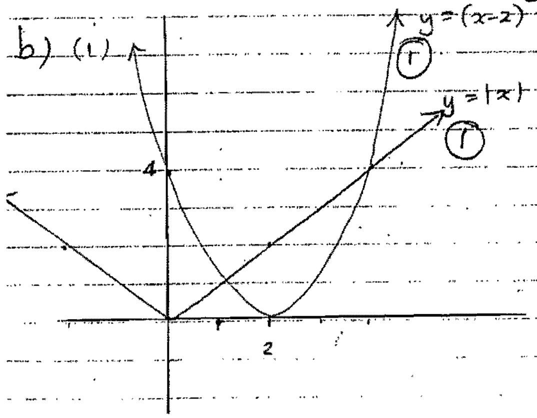
If line is a tangent, then this distance = radius

$$\left| \frac{-k}{5} \right| = 4$$

$$\frac{k}{5} = 4 \quad \text{or} \quad \frac{-k}{5} = 4$$

$$k = 20 \quad \text{or} \quad -20$$

b) (i)



(ii) Graphs intersect when

$$x = (x-2)^2$$
$$x = x^2 - 4x + 4$$
$$0 = x^2 - 5x + 4$$
$$0 = (x-4)(x-1)$$
$$x = 4, \quad x = 1 \qquad (2)$$

Soln to inequality is

$$x < 1, \quad x > 4 \qquad (1)$$

c) $y = x^3 + ax^2 + bx + c$

$$\left.\begin{aligned} y' &= 3x^2 + 2ax + b \\ y'' &= 6x + 2a \end{aligned}\right] (1) \text{ both}$$

~~Passes through (1,2):~~

~~$2 = 1 + a + b + c$~~

~~$a + b + c = 1$~~

Inflexion at (1,2) — Set $y''=0$

$$6(1) + 2a = 0$$
$$2a = -6$$
$$a = -3 \qquad (1)$$

Stat pt when $x = 4$ — Set $y' = 0$

$$3(4)^2 - 6(4) + b = 0$$
$$48 - 24 + b = 0$$
$$b = -24 \qquad (1)$$

$$y = x^3 - 3x^2 - 24x + c$$

Passes through (1,2)

$$2 = 1 - 3 - 24 + c$$
$$c = 28 \qquad (1)$$

Q4

a) $\tan 56° = \dfrac{50}{XB}$

$$XB = \frac{50}{\tan 56°} = 50 \cot 56°$$

$$\tan 39° = \frac{50}{YB} \qquad (1)$$

$$YB = \frac{50}{\tan 39°} = 50 \cot 39°$$

$$XY^2 = YB^2 - BX^2 \quad \text{(Pythag)}$$

$$XY^2 = 50^2 \cot^2 39° - 50^2 \cot^2 56°$$

$$XY^2 = 50^2 (\cot^2 39° - \cot^2 56°) \qquad (1)$$

$$b = XY = 50\sqrt{\cot^2 39° - \cot^2 56°}$$

$$= 51.7\text{ m} \qquad (1)$$

b) $\cos(A-B)$

$$= \cos A \cos B + \sin A \sin B \qquad (1)$$

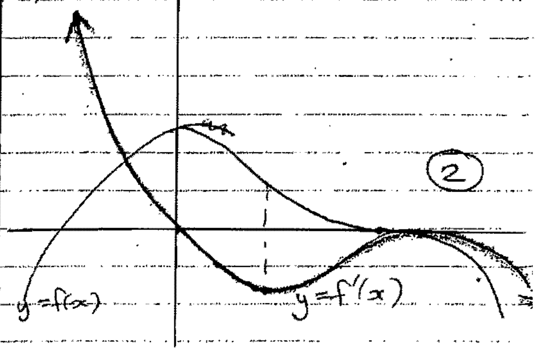
$\cos 15°$

$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \qquad (1)$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \qquad (1)$$

c)



Last portion may lie above or below $y=f(x)$

d) $\left.\begin{aligned} \alpha + \beta &= 5/2 \\ \alpha\beta &= -2 \end{aligned}\right]$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} \qquad (1)$$

$$= \frac{\frac{25}{4} + 4}{-2}$$

$$= -41/8 \qquad (1)$$

$$\frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1 \qquad (1)$$

Equation is

$$x^2 + \frac{41}{8}x + 1 = 0 \qquad (1)$$

Q5.

a) (i) $x \neq 2, -2$ (1)

(ii) $f(-x) = \dfrac{-x}{4-(-x)^2}$ (1)

$= \dfrac{-x}{4-x^2}$

$= -\left(\dfrac{x}{4-x^2}\right)$ (1)

$= -f(x)$ ∴ Fn is odd.

(iii) $f'(x) = \dfrac{(4-x^2) \cdot 1 - x \cdot -2x}{(4-x^2)^2}$ (1)

$= \dfrac{4-x^2+2x^2}{(4-x^2)^2}$

$= \dfrac{x^2+4}{(4-x^2)^2}$ (1)

Numerator + denominator are both always positive (for ~~f(x)~~ all x in domain) (1)

∴ $f'(x) > 0$ for all x in domain

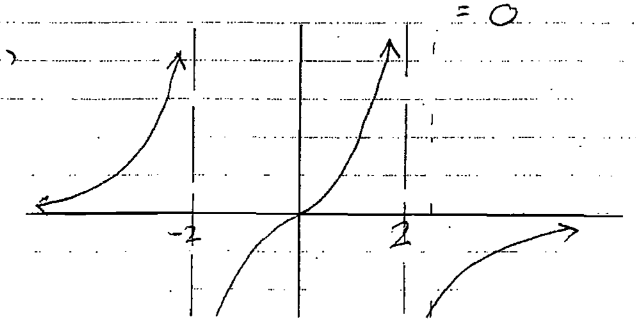
ie $f(x)$ is increasing

(iv) $\lim\limits_{x \to \infty} \dfrac{x/x^2}{4/x^2 - 1} = \lim\limits_{x \to \infty} \dfrac{1/x}{4/x^2 - 1}$

$= \dfrac{0}{0-1}$

$= 0$ (1)

(v)



(1) for [curve sketch]

(1) for [curve sketch]

b) To prove $1 + 4 + 16 + 64 + \dots + 4^{n-1} = \frac{1}{3}(4^n - 1)$ for all pos. integers n

If $n = 1$: LHS $= 4^{1-1} = 4^0 = 1$

RHS $= \frac{1}{3}(4^1 - 1) = \frac{1}{3}(3) = 1$ (1)

∴ Statement is true for $n = 1$

Assume true for $n = k$

ie Assume $1 + 4 + 16 + 64 + \dots + 4^{k-1} = \frac{1}{3}(4^k - 1)$

Need to prove true for $n = k+1$

ie Need to prove:

$1 + 4 + 16 + 64 + \dots + 4^{k-1} + 4^k = \frac{1}{3}(4^{k+1} - 1)$ (1)

LHS $= \frac{1}{3}(4^k - 1) + 4^k$ by assumption

$= \frac{1}{3}(4^k) - \frac{1}{3} + 4^k$

$= \frac{4}{3}(4^k) - \frac{1}{3}$

$= \frac{1}{3}(4(4^k) - 1)$

$= \frac{1}{3}(4^{k+1} - 1)$ (1)

$=$ RHS

∴ If statement is true for $n = k$, then it is also true for $n = k+1$

Now statement is true for $n = 1$

∴ Also true for $n = 1+1 = 2$

" " " $n = 2+1 = 3$ etc

By induction, statement is true for all pos. integers n.

[−1 if conclusion is incomplete or incorrect]

Q6.

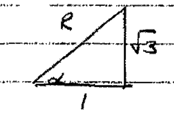
a) (i) $R\sin(x+\alpha)$

$= R\sin x\cos\alpha + R\cos x\sin\alpha$

$= \sin x + \sqrt{3}\cos x$

$R\cos\alpha = 1$ $\qquad$ $R\sin\alpha = \sqrt{3}$

$\cos\alpha = 1/R$ $\qquad$ $\sin\alpha = \frac{\sqrt{3}}{R}$



$R = 2$

$\underline{R = 2}$ (1) , $\alpha = \tan^{-1}\sqrt{3} = \underline{60^\circ}$ (1)

$\therefore \sin x + \sqrt{3}\cos x = 2\sin(x+60^\circ)$

(ii) $2\sin(x+60^\circ) = \sqrt{2}$

$\sin(x+60^\circ) = \frac{\sqrt{2}}{2}$ (or $\frac{1}{\sqrt{2}}$)

$\boxed{\begin{array}{l} 0^\circ \le x \le 360^\circ \\ 60^\circ \le x+60^\circ \le 420^\circ \end{array}}$

$x+60^\circ = \cancel{45^\circ}, 135^\circ, 405^\circ$

$x = 75^\circ$ (1), 345° (1).

(Deduct 1 if -15° is left as a soln)

b) Real roots : ($\Delta \ge 0$) — (1)

$\Delta = (-4)^2 - 4.2.k$

$= 16 - 8k \ge 0$ — (1)

$-8k \ge -16$

$k \le 2$ — (1)

c) $\frac{3^{\cancel{1}}(3^1 + 3^{-1})}{3^{\cancel{2}}(3^{-2}+1)}$ (1) $= \frac{3\frac{1}{3}}{1\frac{1}{9}}$

$= 3$ (1)

d) If $\angle CBD = x$, then $\angle DBE = x$ (BD bisects $\angle CBE$)

$\angle ABE = \angle BCD$ ($\angle$ between tangent + chord $= \angle$ in alt. segment) (1)

$\angle BDE = x + \angle DCB$ (ext $\angle$ of ΔDBC = sum of int. opp $\angle$s) (1)

$\angle ABD = x + \angle ABE$ (addition of adj. $\angle$s)

but $\angle ABE = \angle DCB$ (proven above)

$\therefore \angle EDB = \angle ABD$ (addition of equal $\angle$s)

$\therefore \Delta ADB$ is isosceles (2 equal $\angle$s) (1)

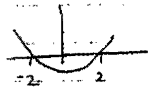
Q7.

a) $y = x^3 - 12x + 16$

$y' = 3x^2 - 12$

$y'' = 6x$] (1)

$y' < 0$ for $3(x-2)(x+2) < 0$



$-2 < x < 2$ (1)

$y'' < 0$ for $x < 0$

$y' < 0$ and $y'' < 0$ for $\underline{-2 < x < 0}$ (1)

decreasing $\qquad$ conc. down

~~b)~~ c) (i) $\$20\,000(1.1) - 2500$ (1)

b) $\frac{1}{2}\left(\frac{1}{t} - t\right) = \frac{1}{2}\cdot\frac{1-t^2}{t} = \frac{1-t^2}{2t} = \cot\theta$

$[t = \tan\frac{\theta}{2}]$

(ii) $A_1 = 20\,000(1.1) - 2500$

$A_2 = A_1 \times 1.1 - 2500$

$= 20\,000(1.1)^2 - 2500(1.1) - 2500$

$= 20\,000(1.1)^2 - 2500(1.1 + 1)$ (1)

develop sequence

$A_3 = A_2 \times 1.1 - 2500$

$= 20\,000(1.1)^3 - 2500(1.1^2 + 1.1) - 2500$

$= 20\,000(1.1)^3 - 2500(1.1^2 + 1.1 + 1)$

$$A_n = 20\,000(1.1)^n - 2500\underbrace{(1.1^{n-1} + 1.1^{n-2} + \dots + 1.1 + 1)}_{\text{geom. seq. } a=1 \ r=1.1 \ n \text{ terms}} \quad (1)$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \cancel{\frac{1(1.1^n - 1)}{1.1 - 1}}$$

$$= \frac{1.1^n - 1}{0.1} \quad (1)$$

$$= 10(1.1^n - 1)$$

$A_n = 20\,000(1.1)^n - 2500 \times 10(1.1^n - 1)$

$= 20\,000(1.1)^n - 25\,000(1.1^n - 1)$ (1)

$= 20\,000(1.1)^n - 25\,000(1.1)^n + 25\,000$

$= 25\,000 - 5000(1.1)^n$

iii) She can make withdrawals until $A_n = 0$

If $A_n = 0$, then $25\,000 = 5000(1.1)^n$

$(1.1)^n = 5$ (1)

By trial + error (or logarithms), 16 withdrawals can be made. (1)

[$(1.1)^{16} = 4.6$ ✓ $(1.1)^{17} = 5.1$ – too high]