BAULKHAM HILLS HIGH SCHOOL

YEARLY EXAMINATIONS

YEAR 11

2009

MATHEMATICS

EXTENSION 1

Time Allowed: Two hours
(plus 5 mins reading time)

Instructions:

- Write in the answer booklets provided
- Do not write on this question paper
- Show all working
- Use black or blue pens only
- Start a new page for each question
- Write your name and your teacher's name at the top of each page
- Board approved calculators may be used

Marks

Question 1 (12 marks) Start a new page.

a) Find the coordinates of the point which divides the interval joining (-3,1) and (5,6) externally in the ratio 3 : 4 **2**

b) Solve $\dfrac{4}{x-2} \le 1$ **3**

c) A curve has parametric equations $x = t + 2,\ y = 4t^2$
Find the Cartesian equation. **2**

d) Evaluate $\lim\limits_{x \to \infty} \dfrac{x^2 - 4x + 2}{2x^2 + 3}$ **1**

e) i) Express $\cos x - \sin x$ in the form $R\cos(x + \alpha)$ **2**

ii) Hence solve $\cos x - \sin x = 1$ for $0^o \le x \le 360^o$ **2**

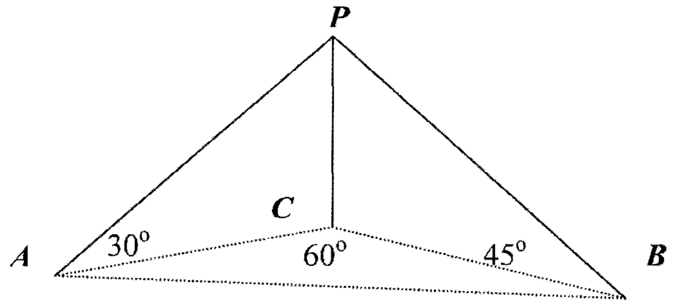
Marks

Question 2 (12 marks) Start a new page.

a) Solve the equation $2\sin^2\theta = \sin 2\theta$ for $0^0 \le \theta \le 360^0$ **3**

b) Prove the identity $\dfrac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A} = 2 - \sin 2A$ **2**

c) The angle of elevation of the top of a vertical cliff P from boat A is 30° and from boat B is 45°
The two boats subtend an angle 60° at the base of the cliff and are 1 km apart.



Find the height of the cliff. (Answer to the nearest metre). **4**

d) For what values of 'b' is the line $y = 3x + b$, a tangent to $y = x^3$? **3**

Marks

Question 3 (12 marks) Start a new page.

a) Find the acute angle between lines $2x - 3y + 4 = 0$ and $4x + y - 6 = 0$ to the nearest minute. **3**

b) Consider the curve $y = \dfrac{x - 1}{x^2 - 9}$

i) Find the intercepts with the coordinate axes **2**

ii) Locate stationary points (if any) **3**

iii) Write the equations of any asymptotes **2**

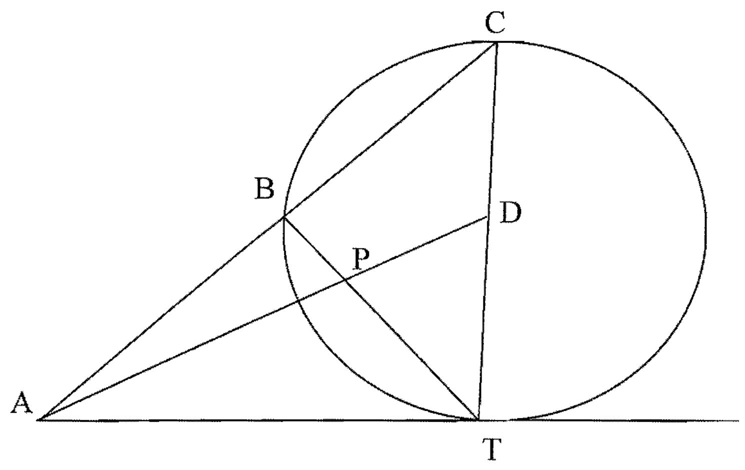
iv) Draw a graph of the function and use it to deduce the number of points of inflexion **2**

Marks

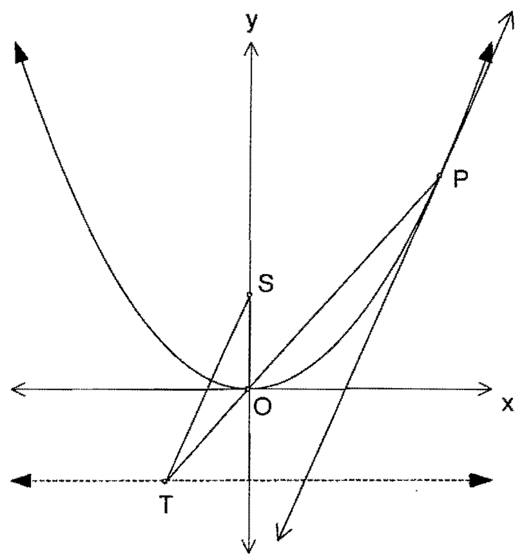
Question 4 (11 marks) Start a new page.

a) AT is a tangent to the circle at T. ABC is a secant and AD bisects $\angle CAT$

Prove that $TP = TD$ **3**



b) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$



i) Derive the equation of the tangent at P **2**

ii) The line from P to the vertex is produced to cut the directrix at T.
If S is the focus, prove that TS is parallel to the tangent at P. **3**

c) Differentiate from first principles, $f(x) = \dfrac{1}{x}$ **3**

Marks

Question 5 (12 marks) Start a new page.

a) Use mathematical induction to prove that

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots\ldots\ldots+\frac{1}{(2N-1)(2N+1)}=\frac{N}{2N+1}$$

for positive integers $N \geq 1$ **4**

b) If α, β are the roots of $2x^2 - 5x + 8 = 0$
write a quadratic equation with roots α^2, β^2 **3**

c) AB is a chord of length 16 units of the circle $x^2 + y^2 = 100$ and P is the mid-point of AB. Determine the locus of P. **2**

d) State the domain of the function $f(x) = \sqrt{9 - x^2} + \dfrac{1}{x+3}$ **3**

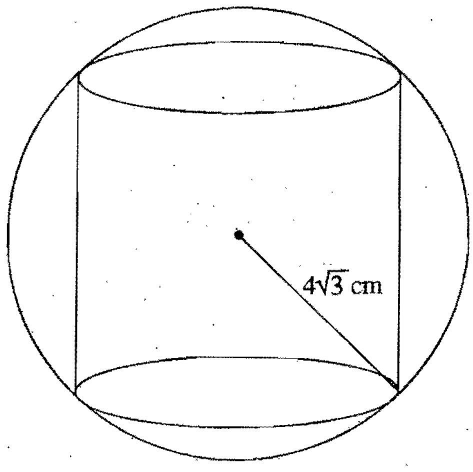
Marks

Question 6 (12 marks) Start a new page.

a) Prove that $\tan\theta = \sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$

hence show the exact value of $\tan 22\frac{1}{2}^0 = \sqrt{2} - 1$ **4**

b) Find the height of the cylinder with maximum volume which fits exactly into a sphere of radius $4\sqrt{3}$ cm **4**



c) By making the substitution $t = \tan\dfrac{\theta}{2}$ or otherwise

show that $\operatorname{cosec}\theta + \cot\theta = \cot\dfrac{\theta}{2}$ **2**

d) Solve for x: $2.2^x - 5 = \dfrac{12}{2^x}$ **2**

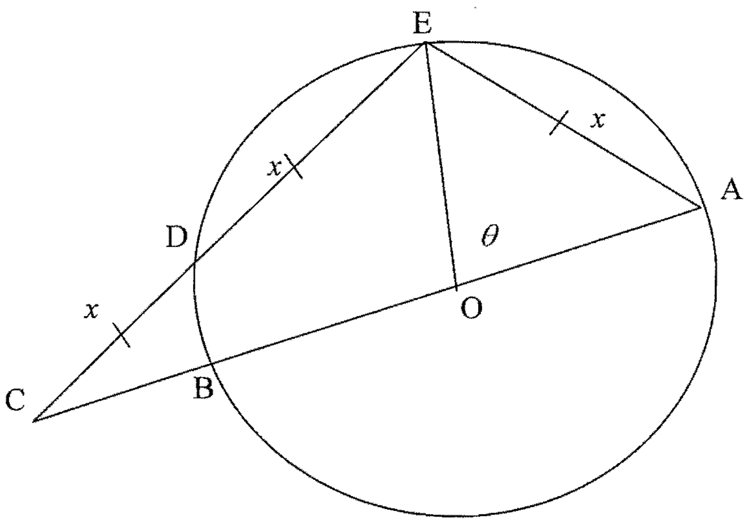
Marks

Question 7 (12 marks) Start a new page.

a) Find the vertex, focus, directrix of parabola with equation

$$(x+4)^2 = -16(y-3)$$ **3**

b) AB is the diameter of a circle of radius 1 unit with centre O.
$CD = DE = AE = x$ and $BC = 1$ $\angle AOE = \theta$



Prove $\cos\theta = \dfrac{1}{4}$ **4**

c) If $5\cos A + 3 = 0$ and $\tan A > 0$ find the exact value of $\cos(A + 90^\circ)$ **2**

d) The normals at P and Q on $x^2 = 4ay$ intersect at R whose coordinates are

$$x = a\left(t - \frac{1}{t}\right) \quad y = a\left(t^2 + 1 + \frac{1}{t^2}\right)$$

Find the equation in Cartesian form of the locus of R. **2**

e) Evaluate $\sum_{r=1}^{4} r^r$ **1**

End of Paper

YR11 YEARLY EXT 1 SEPT 2009. SOLUTIONS for Q2: NOTE CHANGE 2 C to MARKS

Q1 a) $(-3,1)$ and $(5,6)$

$-3 : 4$

$\left(\frac{-3\times5+4\times-3}{-3+4}, \frac{-3\times6+4\times1}{-3+4}\right)$

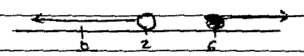
$= (-27, -14)$ (1 each) [2]

b) $\frac{4}{x-2} \le 1$

$x \neq 2$

$4 = x-2$

$x = 6$ (1) [3]



Test $x=3$ → false

$\therefore x<0$ or $x \ge 6$ (1) (1)

c) $x = t+2$; $y = 4t^2$

$t = x-2$ — ① (1) [2]

Sub ① → y.

$y = 4(x-2)^2$ (1)

d) $\lim_{x\to\infty} \frac{x^2-4x+2}{2x^2+3} \begin{matrix}\div x^2\\ \div x^2\end{matrix}$

$= \lim_{x\to\infty} \frac{1-\frac{4}{x}+\frac{2}{x^2}}{2+\frac{3}{x^2}}$

$= \frac{1}{2}$ [1]

e) i) $\cos x - \sin x = R\cos(x+\alpha)$

$R\cos(x+\alpha) =$

$= 2\cos x\cos\alpha - R\sin x\sin\alpha$

$\therefore R\cos\alpha =$

$R\sin\alpha = 1$

$\tan\alpha = 1$

$\alpha = 45°$, $R=\sqrt{2}$. (1) (1) [2]

$\therefore \cos x - \sin x = \sqrt{2}\cos(x+45°)$

ii) $\cos x - \sin x = 1$

$\sqrt{2}\cos(x+45°) = 1$

$\cos(x+45°) = \frac{1}{\sqrt{2}}$

$x+45° = 45°, 315°$ (1) [2]

$x = 0, 270°, 360°$ (1)

Q2 a) $2\sin^2\theta = \sin 2\theta$

$2\sin^2\theta = 2\sin\theta\cos\theta$

$2\sin^2\theta - 2\sin\theta\cos\theta = 0$

$2\sin\theta(\sin\theta - \cos\theta) = 0$ (1)

$\therefore \sin\theta = 0$ | $\sin\theta = \cos\theta$

$\theta = 0, 180°, 360°$ (1) | $\tan\theta = 1$

$\theta = 45°$ or $225°$ (1) [3]

$\therefore \theta = 0, 45°, 180°, 225°, 360°$

b) LHS $= \frac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A}$

$= \frac{2(\sin^3 A + \cos^3 A)}{\sin A + \cos A}$ (1)

$= \frac{2(\sin A + \cos A)(\sin^2 A - \sin A\cos A + \ldots)}{(\sin A + \cos A)}$

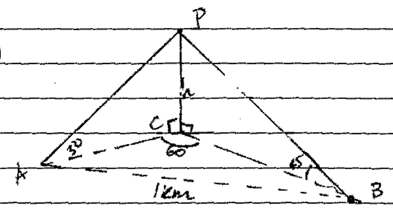
$= 2(1 - \sin A\cos A)$ (1)

$= 2 - 2\sin A\cos A$ [2]

$= 2 - \sin 2A$

$=$ RHS

c)



$\tan 30° = \frac{h}{AC}$ $\tan 45° = \frac{h}{CB}$

$AC = \frac{h}{\tan 30}$ $CB = \frac{h}{\tan 45}$

$AC = \sqrt{3}h$ $CB = h$ (1)

In $\triangle ABC$

$1^2 = (h\sqrt{3})^2 + h^2 - 2(h\sqrt{3})h\cos 60$ (1)

$1 = 4h^2 - \sqrt{3}h^2$

$h^2 = \frac{1}{4-\sqrt{3}}$ (1) [4]

$h = 664.7$ (1) (1)

Q2 d) $y = x^3$

$\frac{dy}{dx} = 3x^2$ — ①

$y = 3x+b$ — ③

$m = 3$. — ②

①=② $3x^2 = 3$

$x = \pm 1$ (1) [3]

when $x=+1$ | when $x=-1$

$y = 1^3 = 1$ | $y = (-1)^3 = -1$

③→ $1 = 3(1)+b$ | ③→ $-1 = 3(-1)+b$

$b = -2$ (1) | $b = 2$ (1)

$\therefore$ when $b = \pm 2$

Q3 a) $2x - 3y + 4 = 0$ — ①

$4x + y - 6 = 0$ — ②

① → $3y = 2x+4$

$m_1 = \frac{2}{3}$

② → $y = -4x+6$ (1)

$m_2 = -4$

$\tan\theta = \left|\frac{\frac{2}{3}-(-4)}{1+\frac{2}{3}\times-4}\right|$

$\tan\theta = \frac{14}{5}$ (1) [3]

$\theta = 70°21'$ (1)

b) $y = \frac{x-1}{(x+3)(x-3)}$

i) y int ; $x=0$. (1)

$y = \frac{1}{9}$

x int ; $y=0$

$0 = \frac{x-1}{x^2-9}$ (1)

$x = 1$

$\therefore$ x int $= 1$, y int $= \frac{1}{9}$

ii) Stat pt when $\frac{dy}{dx} = 0$.

$u = x-1$ $v = x^2-9$

$u' = 1$ $v' = 2x$. (1)

$\frac{dy}{dx} = \frac{x^2-9-2x(x-1)}{(x^2-9)^2}$

$0 = x^2-9-2x^2+2x$

$0 = x^2-2x+9$ (1)

$x = \frac{2 \pm \sqrt{2^2-4\times9}}{2}$

Since $\Delta < 0$

$\therefore$ no stationary points. (1)

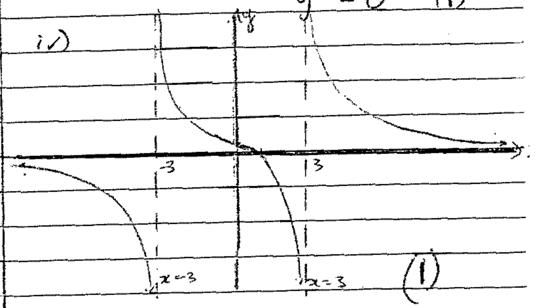
iii) $x^2-9 \neq 0$

$\therefore x \neq \pm 3$.

asymptotes at $x = \pm 3$. (1)

$y = 0$ (1)

iv)



(1)

$x=-2$; $y = \frac{3}{5}$

$x=2$; $y = -\frac{1}{5}$

$x=4$; $y = \frac{3}{7}$

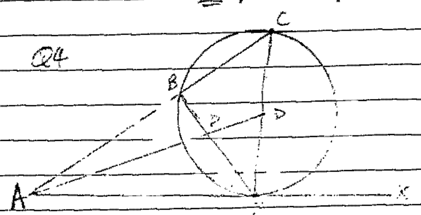
$x \to \infty$; $y \to 0^+$

$x \to -\infty$; $y \to 0^-$

(1)

$\therefore$ it has 1 point of inflexion.

Q4



$\angle BAP = \angle TAP = x$ (GIVEN)

$\angle ATP = \angle BCT = y°$ (ANGLE between tangent chord = $\angle$ in ALT SEGMENT) (1)

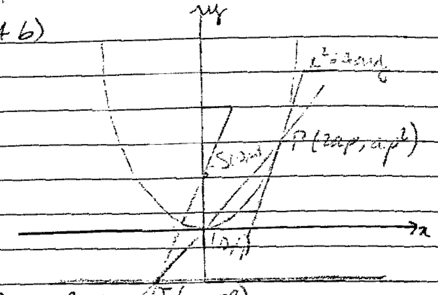
$\angle DPT = x+y$ (EXT $\angle$ of $\triangle$ = SUM of INT OPP $\angle$S) (1)

$\angle PDT = x+y$ (AS ABOVE) (1)

$\angle DPT = \angle PDT$ [3]

$TP = TD$ (base $\angle$s of ISOS $\triangle$s)

Q4 b)



i) $x^2=4ay$

$y=\frac{x^2}{4a}$

$\frac{dy}{dx}=\frac{x}{2a}$

at $(2ap, ap^2)$

$\frac{dy}{dx}=\frac{2ap}{2a}=p$. (1)

eqn of tangent [2]

$y-ap^2=p(x-2ap)$

$y=px-ap^2$ (1)

ii) $M_{OP}=\frac{ap^2-0}{2ap-0}$

$=p/2$.

Eqn OP $y-0=\frac{p}{2}(x-0)$

$y=\frac{p}{2}x$. (1)

at T: $y=-a$

$-a=\frac{p}{2}x$

$x=-\frac{2a}{p}$

$T\left(-\frac{2a}{p},-a\right)$ (1)

$M_{ST}=\frac{a--a}{0--2a/p}$ [3]

$=\frac{2a}{2a/p}$

$=p$. (1)

$\therefore M_{ST}=M_{tangent}$

ST // tangent.

c) $f(x)=\frac{1}{x}$.

$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$.

$=\lim_{h\to0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$.

$=\lim_{h\to0}\frac{x-x-h}{x(x+h)}\times\frac{1}{h}$.

$=\lim_{h\to0}\frac{-1}{x^2+xh}$ [3]

$=\frac{-1}{x^2}$

Q5 a)

$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\dots+\frac{1}{(2N-1)(2N+1)}=\frac{N}{2N+1}$

Proof:

Test for $n=1$ (1)

$LHS=\frac{1}{1\times3}=\frac{1}{3}$

$RHS=\frac{1}{2\times1+1}=\frac{1}{3}$.

$LHS=RHS$

$\therefore$ True for $n=1$

Assume true for $n=k$

ie $\frac{1}{1\times3}+\frac{1}{3\times5}+\dots\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$

Investigate for $n=k+1$ [4]

ie try to prove

$\frac{1}{1\times3}+\frac{1}{3\times5}+\dots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$

$=\frac{k+1}{2k+3}$

$LHS=\frac{1}{1\times3}+\frac{1}{3\times5}+\dots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$

$=\frac{k\times(2k+3)}{(2k+1)(2k+3)}\frac{1}{(2k+1)(2k+3)}$ (1)

$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$=\frac{(k+1)(2k+1)}{(2k+1)(2k+3)}$

$=\frac{k+1}{2k+3}$ (1)

$=RHS$

$\therefore$ Since it is true for $n=1$, $n=1+1=2$, $n=k+1$ ie all positive integers of n (1)

Q5 b) $2x^2-5x+8=0$

$\alpha+\beta=+\frac{5}{2}$ $\alpha\beta=4$

$\alpha^2+\beta^2=\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta$

$=(\alpha+\beta)^2-2\alpha\beta$

$=\left(\frac{5}{2}\right)^2-2\times4$

$=-\frac{11}{4}$. (1)

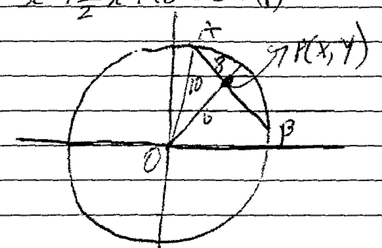
$\alpha^2\beta^2=(\alpha\beta)^2$ [3]

$=4^2$

$=16$. (1)

$\therefore x^2+\frac{11}{2}x+16=0$. (1)

c)



BP = PA (P is the midpt of AB)

BA ⊥ PO (line thru centre that bisects the chord meets at 90°)

AO = 10 (radius)

$PO=\sqrt{10^2-8^2}$ [2]

PO = 6 (equal chords are equi-distant from the centre)

$\therefore$ locus of P is

$x^2+y^2=36$ (1)

d) for $\sqrt{9-x^2}$ D: $-3\le x\le3$ (1)

for $\frac{1}{x+3}$ D: $x\ne-3$, all real x (1)

$\therefore$ for $f(x)=\sqrt{9-x^2}+\frac{1}{x+3}$ [3]

D: $-3<x\le3$ (1)

Q6 a) $\tan\theta=\sqrt{\frac{1-\cos2\theta}{1+\cos2\theta}}$ (1)

RHS $=\sqrt{\frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta-1)}}$

$=\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}$

$=\tan\theta$ (1)

$=LHS$.

$\therefore\tan22\frac{1}{2}°=\sqrt{\frac{1-\cos(22\frac{1}{2}\times2)}{1+\cos(22\frac{1}{2}\times2)}}$

$=\sqrt{\frac{1-\frac{1}{\sqrt2}\times\sqrt2}{1+\frac{1}{\sqrt2}\times\sqrt2}}$ (1)

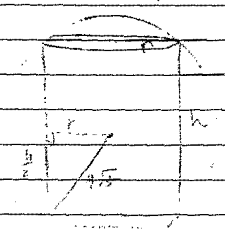
$=\sqrt{\frac{(\sqrt2-1)(\sqrt2-1)}{(\sqrt2+1)(\sqrt2-1)}}$ [4]

$=\sqrt{(\sqrt2-1)^2}$ (1)

$=\sqrt2-1$

$=RHS$.

b)



$V_{cylinder}=\pi r^2h$

$r^2+\left(\frac{h}{2}\right)^2=(4\sqrt3)^2$

$r^2=48-\frac{h^2}{4}$ (1)

$V=\pi\left(48-\frac{h^2}{4}\right)h$

$V=48\pi h-\frac{h^3\pi}{4}$

$\frac{dV}{dh}=48\pi-\frac{3h^2\pi}{4}$ (1)

$\frac{d^2V}{dh^2}=-\frac{3\pi h}{2}$

when $\frac{dV}{dh}=0$

$\frac{3\pi h^2}{4}=48\pi$

$h=8$ $h\ge0$

when $h=8$ (1)

$\frac{d^2V}{dh^2}=-3\pi\times8<0$ (1)

$\therefore$ max height at $h=8$

Q6 c) $\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}$

if $t = \tan\frac{\theta}{2}$

$LHS = \frac{1+t^2}{2t} + \frac{1-t^2}{2t}$ (1)

$= \frac{2}{2t}$ [2]

$= \frac{1}{\tan\frac{\theta}{2}}$ (1)

$= \cot\frac{\theta}{2}$ = RHS.

d) $2.2^x - 5 = \frac{12}{2^x}$

Let $2^x = a$

$2a^2 - 5a = 12$

$2a^2 - 5a - 12 = 0$

$(2a+3)(a-4) = 0$ (1) [2]

$2^x = \frac{-3}{2}$ or $2^x = 4$

not possible $x = 2$ (1)

Q7 a) $(x+4)^2 = -16(y-3)$

vertex $= (-4, 3)$ (1)

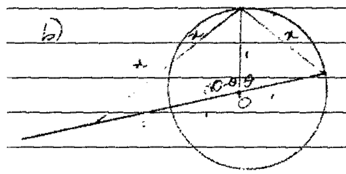
$-16 = 4a$

$a = -4$ [3]

focus $= (-4, -1)$ (1)

Directrix $y = 7$ (1)

b)



$\cos\theta = \frac{1^2+1^2-x^2}{2\times1\times1}$

$\cos\theta = \frac{2-x^2}{2}$ —①

$\cos(180-\theta) = \frac{1^2+2^2-(2x)^2}{2\times2\times1}$

$-\cos\theta = \frac{5-4x^2}{4}$ —② (1)

$\frac{-2+x^2}{2} = \frac{5-4x^2}{4}$

$-4+2x^2 = 5-4x^2$

$x^2 = \frac{9}{6}$ $\therefore x^2 = 3/2$ (1)

① → $\cos\theta = \frac{2-9/6}{2} = \frac{1}{4}$ (1)

7c) $\cos A = -3/5$

A in QUAD 3

$\cos(A+90°) = \cos A\cos 90 - \sin A\sin 90$ (1)

$= - -4/5 \cdot 1$

[2] $= 4/5$. (1)

7d) $\left(t-\frac{1}{t}\right)^2 = t^2 - 2 + \frac{1}{t^2}$ (1)

$\left(x/a\right)^2 = \frac{y}{a} - 3$ (1)

$x^2/a^2 = \frac{y}{a} - 3$ [2]

$x^2 = ay - 3a^2$.

7e) $\sum_{r=1}^{4} r^r = 1^1 + 2^2 + 3^3 + 4^4$

$= 1 + 4 + 27 + 256$

$= 288.$ (1)