AP2_CTHS_2001

Question 1

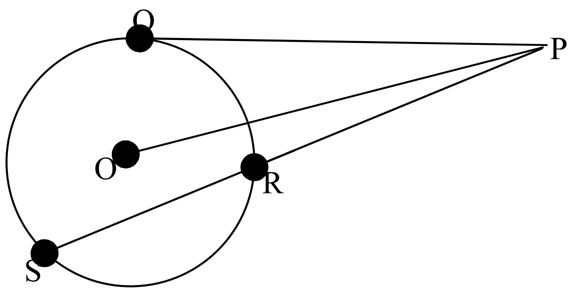
a) For the parabola $y=(x+2)^2 - 5$, state the vertex and axis of symmetry. 2

b) The line l has an x-intercept of -6 and y-intercept of 5. 3
The line, k has an x-intercept of 3 and y-intercept of 4.
Find the acute angle between the lines giving your answer to nearest minute.

c) (i) On the same axes, sketch the curves of $y = x^2$ and $y= |x|$ 2

(ii) Hence, or otherwise, solve $x^2 < |x|$ 1

d)



O is the centre of a circle of radius 6cm, PQ is a tangent and PRS is a secant. PO = 10cm, PR=6cm. Find SR.

Question 2

a) The roots of the quadratic equation $x^2 +4x+2 = 0$ are α and β.

(i) Find $(\alpha-2)(\beta-2)$. 2

(ii) Find $\alpha^{2+}\beta^2$ 2

(iii) Hence, show that the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = -5$ 2

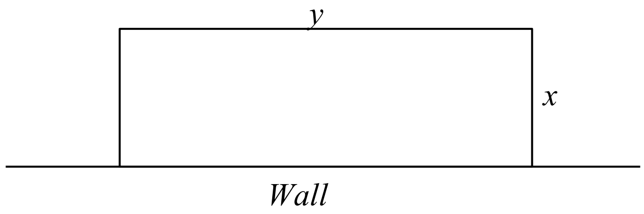
b) If A is acute, B is obtuse and $\cos A= \frac{5}{13}$ and $\sin B = \frac{3}{5}$, 2
find cosecAtanB.

c) Find the equation of the tangent to the curve $y = 2x\sqrt{x+1}$ 4
at the point where x = 3.

Question 3

a) A is the point (5,0) and O is the origin. Given that the point 4
B(x,y) lies on the line $y = 1 - 3x$ and that OB is perpendicular to AB, find the coordinates of B.

b) Farmer Joe decided to make a rectangular chicken run using an existing wall as one side.



He has 18m of wire netting to use.

(i) Show that $y = 18 - 2x$ 1

(ii) Show that the area, A m^2, of the run is given by 1
$A = 18x - 2x^2$

(iii) Find the maximum possible area of the run. 3
(Use quadratic theory only)

c) For what values of k, does the quadratic equation 3
$kx^2 + 4x + k + 3 = 0$, have real roots?

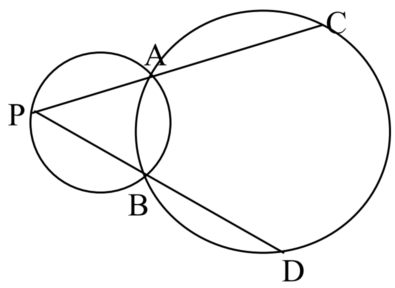
Question 4

a) $y = x^2 + bx + c$

(i) Find $\frac{dy}{dx}$ 1

(ii) The line $x+7y - 15 = 0$ is a normal to the curve 4
$y=x^2 +bx+c$ at the point (1,2). Find b and c.

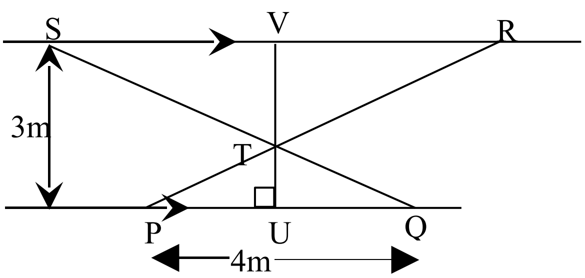
b)



Copy this diagram on to your answer sheet. In the diagram PAC and PBC are straight lines. Prove that DC is parallel to the tangent P. 3

c) Solve $2cos^2 2\theta - cos2\theta - 1 = 0$ *for* $0 \leq \theta \leq 360$.

Question 5



In the diagram, PQ and SR are parallel railings which are 3 Metres apart. The points P and Q are fixed 4m apart on the Lower railing. Two crossbars PR and QS intersect at T as shown in the diagram. The line perpendicular to PQ intersects PQ at U and SR at V. the length of UT is y metres.

(i) Show that ΔSRT is similar to ΔQPT. 3

(ii) Hence explain why $\frac{SR}{PQ} = \frac{VT}{UT}$ 1

(iii) Show that SR = $\frac{12}{y} - 4$ 1

(iv) Hence, show that the total area, A, of ΔQPT and ΔRTS is $A = 4y - 12 + \frac{18}{y}$ 2

b) If $a^x = 7$ and $a^{x+3} = 56$, find the values of a^{x-3} 2

c) A and B have coordinates (–1, 7) and (5, –2) respectively, P divides the interval AB in the ratio *k*: 1

(i) Write down the coordinates of P in terms of *k*. 1

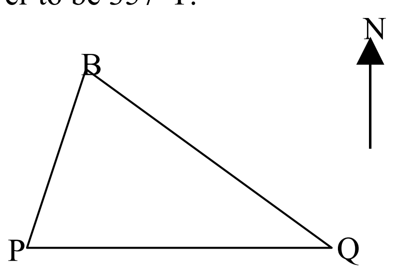
(ii) Hence find AP : PB when P lies on the line $5x - 4y - 1 = 0$. 2

Question 6

a)(i) Determine whether $f(x) = x^2 - 2x + 2$ is positive definite, negative definite or neither. 2

(ii) Hence, solve $\frac{x^2 - 2x + 2}{3 - x}$ 2

b) Two observers P and Q are 150m apart. The observer at P finds the bearing of Q to be 087°T and a distant tower AB, to have a bearing of 006°T. The observer at Q finds the bearing of the tower to be 357°T.



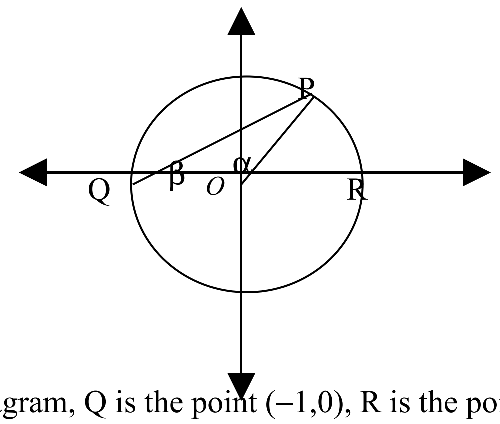
(i) Copy the diagram onto your answer sheet and add any given information. 1

(ii) Show that BQ = $\frac{150 \sin 81^\circ}{\sin 9^\circ}$ 2

(iii) The observer at Q finds the angle of elevation of the top of the tower to be 7°. Find the height of the tower correct to the nearest metre. 2

c) The line $3x + 4y + 7 = 0$ is a tangent to a circle with centre (2,1). Find the equation of the circle. 3

Question 7



In the diagram, Q is the point (−1,0), R is the point (1,0), and P is another point on the circle with centre O and radius 1 unit. Let $\angle POR = \alpha$ and $\angle PQR = \beta$, and let $\tan\beta = m$.

(i) Explain why ΔOPQ is isosceles, and hence deduce that $\alpha = \beta$ 2

(ii) Find the equation of the line PQ. 1

(iii) Show that the coordinates of P and Q are solutions of the equation 2

$$(1+m^2)x^2 + 2m^2x + m^2 - 1 = 0.$$

(iv) Using this equation, find the coordinates of P in terms of m. 3

(v) Hence deduce that 2

$$\tan 2\beta = \frac{2\tan\beta}{1 - \tan^2\beta}$$

(vi) Hence, using part(v) only, show that 2

$\tan 15° = 2 - \sqrt{3}$

Question 1

a) $y=(x+2)^2-5$

Vertex= $(-2,-5)$

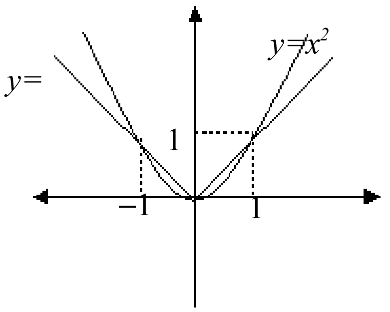
Axis of symmetry $x=-2$

b) $m_l=\dfrac{5}{6}$ $\qquad m_k=\dfrac{-4}{3}$

$$\tan\theta=\left|\frac{m_k-m_l}{1+m_km_l}\right| = \left|\frac{\frac{-4}{3}-\frac{5}{6}}{1+\frac{-4}{3}\times\frac{5}{6}}\right| = 19\tfrac{1}{2}$$

$\angle\theta = 87°4'$

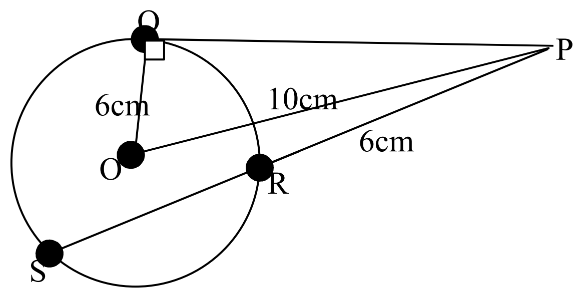
c)(i)



(ii) For $x^2<|x|$,

$-1<x<0$ or $0<x<1$

d)



$\angle POQ = 90°$ (tangent is perpendicular to radius)

$10^2 = PQ^2 + 6^2$ (Pythag. Theorem)

$\therefore$ PQ = 8cm

$PQ^2 = SP \times PR$ (Square of tangent = product of intercept of secants)

$8^2 = (SR+6)\times 6$

$SR = 4\dfrac{2}{3}$

Question 2

a) $x^2+4x+2=0$

$\alpha+\beta=-4$ and $\alpha\beta=2$

(1) $(\alpha-2)(\beta-2) = \alpha\beta-2(\alpha+\beta)+4$

$= 2-2\times -4+4 = 14$

(ii) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$

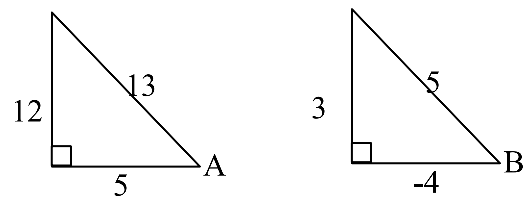
$= (-4)^2-2\times 2 = 12$

(iii) $\dfrac{1}{\alpha^3}+\dfrac{1}{\beta^3} = \dfrac{\alpha^3+\beta^3}{\alpha^3\beta^3}$

$= \dfrac{(\alpha+\beta)^3-(\alpha^2+\beta^2-\alpha\beta)}{\alpha^3\beta^3}$

$= \dfrac{(-4)^3-(12-2)}{2^3} = -5$

b)



Cosec A tan B = $\dfrac{13}{12}\times\dfrac{3}{-4} = \dfrac{-13}{16}$

c) $y = 2x\sqrt{x+1}$

$\dfrac{dy}{dx} = 2x(x+1)^{\frac{1}{2}}$

$= 2(x+1)^{\frac{1}{2}} + 2x\times\dfrac{1}{2}(x+1)^{-\frac{1}{2}}$

$= 2\sqrt{x+1}+\dfrac{x}{\sqrt{x+1}}$

coordinates of tangents at x = 3

$m_t = 2\sqrt{4}+\dfrac{3}{\sqrt{3+1}} = \dfrac{11}{2}$

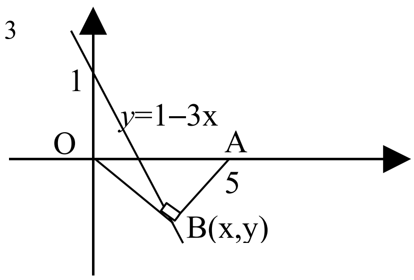
when x = 3, $y = 2\times 3+\sqrt{4} = 12$

Equation of tangent

$y-12 = \dfrac{11}{2}(x-3)$

$11x-2y-9=0$

Question 3



$m_{OB} = \dfrac{y}{x}$ $\qquad m_{AB} = \dfrac{y-0}{x-5}$

as OB $\perp$AB

$\dfrac{y}{x}\times\dfrac{y}{x-5} = -1$

$y^2 = -x^2 + 5x$ ……………….①

But B(x, y) lies on y = 1 – 3x

$\therefore\ (1-3x)^2 = -x^2 + 5x$

$1 - 6x + 9x^2 = -x^2 + 5x$

$10x^2 - 11x + 1 = 0$

$(10x - 1)(x - 1) = 0$

$x = \dfrac{1}{10}$ or 1

$\therefore B(\dfrac{1}{10}, \dfrac{7}{10})$ or $(1, -2)$

b)(i) perimeter = 2x + y

ie. 2x + y = 18

hence, y = 18 – 2x

(ii) Area A = xy

= x(18 – 2x)

$A = 18x - 2x^2$

(iii) As this is a quadratic function, $a = -2 < 0$ concave down.

Hence, the maximum value will exist.

Axis of symmetry $x = \dfrac{-b}{2a} = \dfrac{-(-18)}{2(-2)} = \dfrac{9}{2}$

$x = \dfrac{9}{2}$

$\therefore A = \dfrac{9}{2}(18 - 2\times\dfrac{9}{2}) = 40½$

$\therefore$ Maximum area = 40 ½ m^2

c) $kx^2 + 4x + k + 3 = 0$

For real roots $\Delta \geq 0$

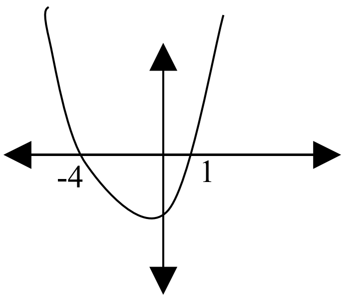
ie. $4^2 - 4\times k(k+3) \geq 0$

$16 - 4k^2 - 12k \leq 0$

$4k^2 + 12k - 16 \geq 0$

$k^2 + 3k - 4 \geq 0$

$(k+4)(k-1) \geq 0$



$-4 \leq x \leq 1$

Question 4

a)(i) $\dfrac{dy}{dx} = 2x + b$

(ii) x + 7y – 15 = 0

$y = -\dfrac{1}{7}x + \dfrac{15}{7}$

$m_{line} = -\dfrac{1}{7} = m_{normal}$

$m_{tangent} = 7$

Gradient of tangent at (1,2) = 2× 1 + b = 7

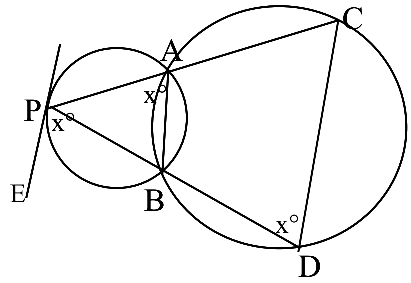
$\therefore$ b = 5

(1,2) is on the curve.

So, $2 = 1^2 + 5\times1 + c$

$\therefore c = -4$

b)



Let ∠EPB = x°

1. ∠EPB = ∠PAB = x° (∠ bet. Tangent and chord = ∠ in alternate segment)
2. ∠PAB = ∠BDC = x° (Ext. ∠ of a cyclic quadrilateral = opp. Interior angle

$\therefore$ ∠EPB = ∠BDC = x° (Alternate angles)

$\therefore$ EP ∥ CD

ie. tangent at P is parallel to CD.

c) $2cos^2 2\theta - cos2\theta - 1 = 0$ *for* $0 \leq \theta \leq 360$.

$2cos^2 2\theta - cos2\theta - 1 = 0$

$(2\cos2\theta+1)(\cos2\theta-1) = 0$

$\therefore$ Cos2θ = – ½ or Cos2θ = 1

∠2θ = 120°, 240°, 480°, 600° or 0°, 360°, 720°

$\therefore$ θ = 60°, 120°, 240°, 300° or 0°, 180°, 360°

Question 5

a) (i) In ΔSRT and QPt,

1. ∠STR = ∠PTQ (Vert. Opp. Angles)
2. ∠VST = ∠TQP (alternate angles equal as SR ∥ PQ)

$\therefore$ ΔSRT ||| ΔQPT (equiangular)

(ii) $\dfrac{SR}{PQ} = \dfrac{VT}{UT}$ (corresponding sides of similar triangles are proportional. $\therefore$ heights are proportional)

(iii) $\dfrac{SR}{4} = \dfrac{3-y}{y} = \dfrac{3}{y} - 1$

$\therefore SR = \dfrac{12}{y} - 4$

(iv) Area = ΔPTQ + ΔRTS

$= ½ \times 4\times y + ½\,(\dfrac{12}{y} - 4)(3-y)$

$= 2y + (\dfrac{6}{y} - 2)(3-y)$

$= 4y - 12 + \dfrac{18}{y}$

b) $a^x = 7$ $\qquad a^{x+3} = 56$

$\therefore \dfrac{a^{x+3}}{a^x} = \dfrac{56}{7} = 8$

ie. $a^3 = 8 \Rightarrow a = 2$

$\therefore a^{x-3} = a^x \times a^{-3}$

$= 7\times 8^{-1} = \dfrac{7}{8}$

c)(i) $(-1, 2)$ $\quad k : 1 \quad$ $(5, -2)$

$$P = \left(\frac{5k-1}{k+1}, \frac{-2k+7}{k+1}\right)$$

(ii) As P lies on line $5x-4y-1 = 0$,

$$5\left(\frac{5k-1}{k+1}\right) - 4\times\left(\frac{-2k+7}{k+1}\right) - 1 = 0$$

$$25k - 5 + 8k - 28 - k - 1 = 0$$

$$32k - 34 = 0$$

$$\therefore k = \frac{34}{32} = \frac{17}{16}$$

$$\therefore AP : PB = \frac{17}{16} : 1 = 17 : 16$$

Question 6

a)(i) $f(x) = x^2 - 2x + 2$

$\Delta = (-2)^2 - 4\times1\times2 = -4$

$\therefore$ $\Delta < 0$ and as $a = 1 > 0$, $f(x)$ is a positive definite.

(ii) $\dfrac{x^2 - 2x + 2}{3 - x} \le 0$

as $3 - x \ne 0$. ie. $x \ne 3$

Multiplying by $(3-x)^2$ on both sides,

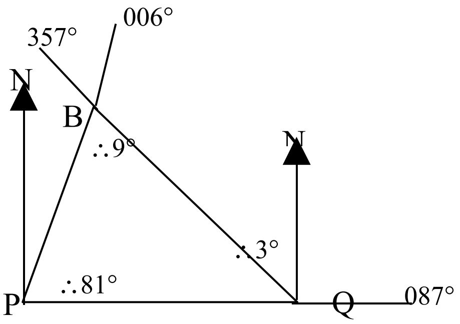
$$(3-x)(x^2 - 2x + 2) \le 0$$

But, using part(ii) $x^2 - 2x + 2 > 0 \quad \forall\ x.$

$\therefore 3 - x < 0$

$\Rightarrow x > 3$

b)(i)



(ii) $\angle QPB = 87° - 6° = 81°$

$\angle BQN = 360 - 357 = 3°$

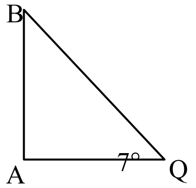
$\angle BQP = 357 - (87+180) = 90°$

$\therefore \angle PBQ = 9°$

$$\therefore \frac{BQ}{\sin 81} = \frac{150}{\sin 9}$$

$\therefore BQ = 947.06m$

(iii)



$$\text{Tan } 7° = \frac{AB}{BQ} = \frac{947.06}{BQ}$$

$\therefore BQ = 947.06\times\tan 7° = 116m$

c) Perpendicular distance $= \left|\dfrac{3\times2 + 4\times1 + 7}{\sqrt{3^2+4}}\right| = \dfrac{17}{5}$

Cenre (2, 1)

$\therefore$ Equation of the circle $= (x-2)^2 + (y-1)^2 = \left(\dfrac{17}{5}\right)^2$

Question 7

a) $OP = OQ$ (equal radii)

$\therefore \Delta OPQ$ is isosceles

$\angle OPQ = \angle PQO = \beta$ (base angles equal as Δ is isosceles)

So, $\alpha = \beta + \beta$ (ext.$\angle$ = sum of opp. Int. $\angle$s)

$\alpha = 2\beta$ (angle at the centre = twice $\angle$ at the circumference of arc PR)

b) As PQ has slope $m_{PQ} = \tan\beta$, $\quad Q = (-1,0)$

$\therefore$ equation is $y - 0 = m(x+1)$

$y = mx + m$

c) the x-value of P and Q are the x-values of the simultaneous equations.

$x^2 + y^2 = 1$ ………………(1)

$y = mx + m$ ………………(2)

So, $x^2 + (mx+m)^2 = 1$

$x^2 + m^2x^2 + 2m^2x + m^2 = 1$

$\Rightarrow (1+m^2)x^2 + 2m^2x + m^2 - 1 = 0$

d) the x-values of P is x_P.

But x-value of Q is $x_Q = -1$

Solving (c) $x = -1$ or $\dfrac{1-m^2}{1+m^2}$

Using $\alpha\beta = \dfrac{c}{a}$, $\quad -1\times x_p = \dfrac{m^2-1}{1+m^2}$

$$\Rightarrow x_p = \frac{1-m^2}{1+m^2}$$

$$\therefore y = m\left(\frac{1-m^2}{1+m^2}\right) + m$$

$$= \frac{2m}{1+m^2}$$

$$\therefore P = \left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$$

e) As $\alpha = 2\beta$

$\therefore \tan\alpha = \tan 2\beta$

ie. $m_{OP} = \tan\alpha$

So, $\tan 2\beta = m_{OP}$

$$= \frac{2m}{1+m^2} \div \frac{1-m^2}{1+m^2}$$

$$= \frac{2m}{1-m^2}$$

$$\therefore \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta}$$

f) Let $\beta = 15°$

So, $\tan 30 = \dfrac{2\tan 15^\circ}{1 - \tan^2 15^\circ}$

ie. $\dfrac{1}{\sqrt{3}} = \dfrac{2\tan 15^\circ}{1 - \tan^2 15^\circ}$

$$\tan^2 15^\circ + 2\sqrt{3}\tan 15^\circ - 1 = 0$$

Solving the quadratic equation,

$\tan 15° = \pm(2 - \sqrt{3})$

But $\tan 15° > 0$, $\therefore \tan 15° = 2 - \sqrt{3}$