Name: ______________________________

Teacher: ____________________________

Class: ______________________________

FAER EST SUAE QUISQUE FORTUNAE

FORT STREET HIGH SCHOOL

2011

PRELIMINARY SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: FINAL PRELIMINARY EXAMINATION

Mathematics Extension I

TIME ALLOWED: 1½ HOURS

Outcomes Assessed	Questions	Marks
Demonstrates the ability to manipulate and simplify expressions to solve problems involving inequalities, division of an interval into a given ratio and circle geometry.	1	12
Solves problems involving polynomials, angle between two lines, quadratics and graphs.	2	10
Uses appropriate techniques to solve problems involving radians, trigonometric rules and graphs.	3	11
Uses appropriate techniques to solve problems involving trigonometric identities and working in three dimensions.	4	14
Uses appropriate techniques to solve problems involving trigonometric equations.	5	13
Solves problems using differentiation techniques.	6	12

Question	1	2	3	4	5	6	Total	%
Marks	/12	/10	/11	/14	/13	12	/72	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started in a new booklet

Marks

QUESTION ONE (12 marks)

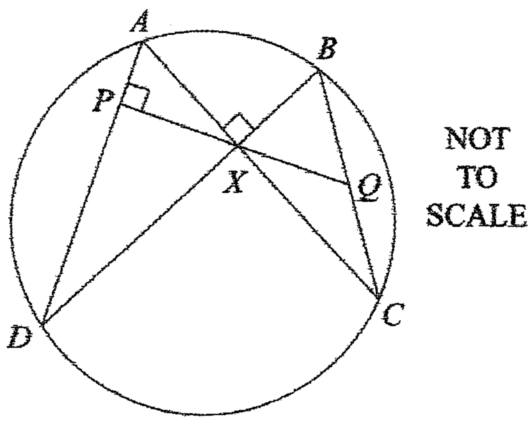
a) Factorise $8x^3 + 27$ **2**

b) Solve the inequality **3**

$$\frac{x+3}{2x} > 1$$

c) The interval AB, where A is (4,5) and B is (19,-5), is divided internally in the ratio 2:3 by the point P (x,y). Find the values of x and y. **2**

d)



The diagram show points A, B, C and D on a circle. The lines AC and BD are perpendicular and intersect at X The perpendicular to AD through X meets AD at P and BC at Q.

Copy or trace this diagram into your writing booklet.

(i) Prove that angle QXB =angle QBX. **3**

(ii) Prove that Q bisects BC. **2**

Marks

QUESTION TWO (10 marks)

a) The polynomial $P(x) = x^2 + ax + b$ has a zero at x=2. When p(x) is divided by x+1, the remainder is 18. Find the values of a and b. **3**

b) The cubic polynomial $P(x) = x^3 + rx^2 + sx + t$, where r, s and t are real numbers has three real zeros, 1, α and -α.

(i) Find the value of r. **1**

(ii) Find the value of s + t. **2**

c) The acute angle between the lines $y = 3x + 5$ and $y = mx + 4$ is 45^0. Find two possible values of m. **2**

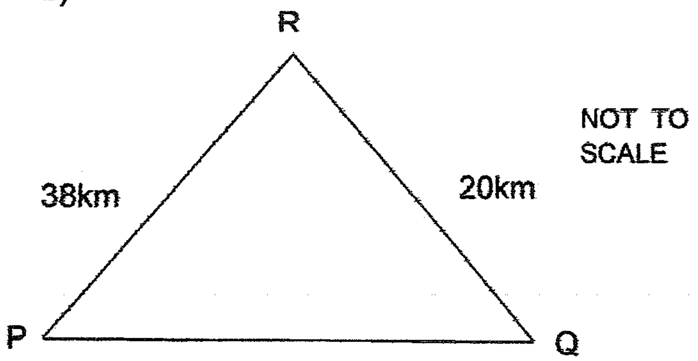
d) For what values of k does $x^2 - kx + 4 = 0$ have no real roots? **2**

Marks

QUESTION THREE (11 marks)

a) Find the exact area of an isosceles triangle with equal sides 8cm about an included angle of 45^0. **2**

b)



In the diagram, the point Q is due east of P.
The point R is 38km from P and 20km from Q.
The bearing of R from Q is 325^0.

(i) What is the size of angle PQR? **1**

(ii) What is the bearing of R from P? **3**

c) An arc of length 5 units subtends an angle of θ at the centre of a circle of radius 3 units. Find the value of θ to the nearest degree. **2**

d) (i) On the same set of axes, sketch the graphs of $y = cos2x$ and $y = \frac{x+1}{2}$ for $-\pi \leq x \leq \pi$. **2**

(ii) Use your graph to determine how many solutions there are to the equation $2cos2x = x + 1$ for $-\pi \leq x \leq \pi$. **1**

Marks

QUESTION FOUR (14 marks)

a) Simply fully $cos5xcos3x + sin5xsin3x.$ **2**

b) Given that θ is acute and sin 2θ = $\frac{5}{13}$, find tan θ. **4**

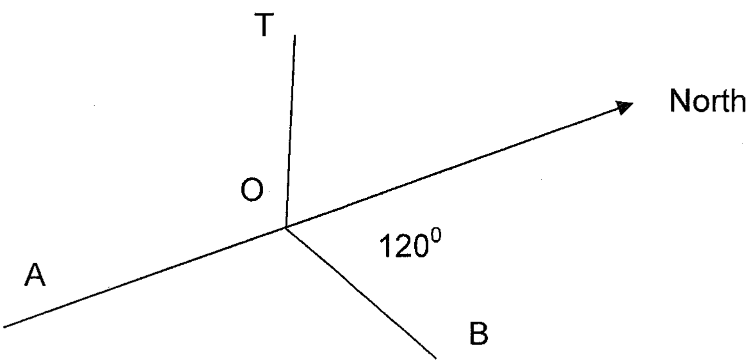
c) (i) Prove that $tan^2\theta = \frac{1-cos2\theta}{1+cos2\theta}$

provided that cos 2θ ≠ -1 **2**

(ii) Hence show the exact value of **2**

$tan\frac{\pi}{8}$ is $\sqrt{2}$ - 1

d) From a point A due south of a tower, the angle of elevation of the top of the tower T, is 23^0. From another point B, on a bearing of 120^0 from the tower, the angle of elevation of T is 32^0. The distance AB is 200 metres.



(i) Copy or trace the diagram, adding the given information to your diagram. **1**

(ii) Hence find the height of the tower to the nearest metre. **3**

Marks

QUESTION FIVE (13 marks)

a) (i) Express sinx and cosx in terms of t, **2**

where t = tan $\frac{x}{2}$.

(ii) Use the *t* formula to solve $sinx + cosx = -1$ **4**

for $0 \leq x \leq 2\pi$.

b) (i) Express 3sinx + 4cosx in the form Asin(x + α) **2**

where $0 \leq \alpha \leq \frac{\pi}{2}$.

(ii) Hence or otherwise, solve 3sinx + 4cosx = 5 **2**

for $0 \leq \alpha \leq 2\pi$. Give your answer or answers, in radians correct to two decimal places.

c) Find the general solution for $\sqrt{2}cos\ 2\theta + 1 = 0$ **3**

Marks

QUESTION SIX (12 marks)

a) Find the derivative of each of the following functions:

(i) $y = \frac{1}{x} + \frac{1}{x^2}$ (Answer with positive indices) **2**

(ii) $y = \sqrt{x}$ (Answer in surd form) **2**

(iii) $y = (3x - 1)(3x^2 + 1)$ **2**

(iv) $y = (3x^2 - 2x - 1)^4$ **1**

(v) $y = \frac{4x^2}{x^2+5}$ **2**

b) (i) Write down the rule for differentiation by first principles. **1**

(ii) Hence show by first principles that the derivative of

$3x^2 - 4x$ is $6x - 4$. **2**

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QUESTION ONE

a) $8x^3 + 27 = (2x)^3 + 3^3 = (2x+3)(4x^2 - 6x + 9)$

b) $\dfrac{x+3}{2x} > 1$

$4x^2\left(\dfrac{x+3}{2x}\right) > 4x^2$

$2x(x+3) > 4x^2$

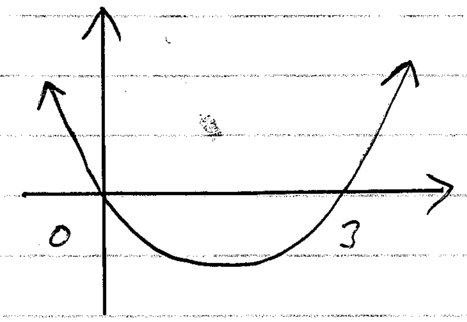
$2x^2 + 6x > 4x^2$

$0 > 2x^2 - 6x$

$2x^2 - 6x < 0$

$2x(x-3) < 0$

zeros $x = 0$ $x = 3$



$\therefore 0 < x < 3$ from graph

c) $A(4, 5)$ $B(19, -5)$ $P(x, y)$

$x = \dfrac{mx_2 + nx_1}{m+n}$, $y = \dfrac{my_2 + ny_1}{m+n}$ $m:n = 2:3$

$x = \dfrac{2 \times 19 + 3 \times 4}{2+3}$, $y = \dfrac{2 \times -5 + 3 \times 5}{2+3}$

$= \dfrac{50}{5}$ $= \dfrac{5}{5}$

$\therefore x = 10$ $y = 1$

d) (i) $\angle PAX = \angle QBX$ ($\angle$s in the same segment on CD) ✓
$\angle PXA = 90^\circ - \angle PAX$ (angle sum $\triangle APX$)
$\angle QXC = 90^\circ - \angle PAX$ (vert opposite $\angle$s) ✓
$\angle QXB = 90^\circ - \angle QXC$ (angles on st line)
$= 90^\circ - (90^\circ - \angle PAX)$
$= \angle PAX$ ✓
$\therefore \angle QXB = \angle QBX$

(ii) $\angle QXB = \angle QBX$ (from (i))
$\therefore \triangle BXQ$ is isoceles
$QB = QX$ (sides opposite equal $\angle$s of $\triangle BXQ$) 1
$\angle QXC = 90^\circ - \angle QXB$ ($AC \perp BD$)
Also $\angle BCX = 90^\circ - \angle QBX$ (angle sum $\triangle XBC$)
$= 90^\circ - \angle QXB$ (from (i)) 1
$\therefore \angle QXC = \angle BCX$
$\therefore \triangle XQC$ is isoceles
$\therefore QX = QC$ (sides opposite equal $\angle$s of $\triangle QXC$)
$\therefore QB = QC$ (both equal to QX)
$\therefore Q$ bisects BC

QUESTION TWO

(a) $P(x) = x^2 + ax + b.$

Zero at $x=2$ $\therefore$ $P(2) = (2)^2 + a(2) + b = 0$

$4 + 2a + b = 0$ (1)

$P(-1) = 18$, by remainder theorem ✓

$(-1)^2 + a(-1) + b = 18$

$1 - a + b = 18$ (2)

From (1) $2a + b = -4$ (3)

From (2) $-a + b = 17$ (4)

(3) − (4) $3a = -21$

$a = -7$ ✓

Sub $a = -7$ in (1) $4 + 2 \times -7 + b = 0$

$-10 + b = 0$

$b = 10$ ✓

$\therefore a = -7 \quad b = 10$

(b) (i) $P(x) = x^3 + rx^2 + sx + t$

The sum of zeros $= -\frac{b}{a}$

$1 + \alpha + (-\alpha) = -\frac{r}{1}$

$r = -1$ ✓

(ii) The product of zeros $= -\frac{d}{a}$

$1 \times \alpha \times (-\alpha) = -\frac{t}{1}$

$\alpha^2 = t$ ✓

The sum of zeros 2 at a time $= \frac{c}{a}$

$1\alpha + \alpha(-\alpha) + (\alpha)1 = \frac{s}{1}$

$\alpha - \alpha^2 + \alpha = s$

$s = -\alpha^2$

$s + t = -\alpha^2 + \alpha^2 = 0$ ✓

(c) $y = 3x + 5$ has $m_1 = 3$

$y = mx + 4$ has $m_2 = m$

If θ is the angle between the lines

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \text{and} \quad \theta = 45^\circ$$

$$\tan 45^\circ = \left|\frac{3 - m}{1 + 3m}\right|$$

$$\frac{3-m}{1+3m} = 1 \qquad \text{or} \qquad \frac{3-m}{1+3m} = -1$$

$$3 - m = 1 + 3m. \qquad\qquad 3 - m = -1 - 3m$$

$$2 = 4m \qquad\qquad 2m = -4$$

$$m = \tfrac{1}{2} \qquad\qquad m = -2$$

$\therefore\ m = \frac{1}{2}$ or -2

(d) $x^2 - kx + 4 = 0$

$$\Delta = b^2 - 4ac$$

$$= (k)^2 - 4 \times 1 \times 4$$

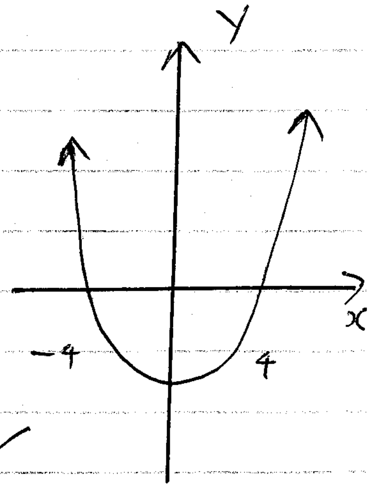
$$= k^2 - 16$$

For no real roots $\Delta < 0$

$$k^2 - 16 < 0$$

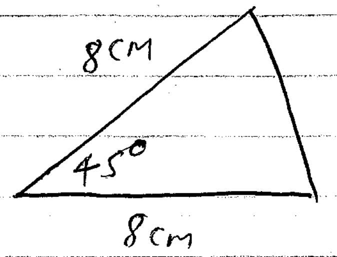
$$(k - 4)(k + 4) < 0$$

$$-4 < k < 4 \text{ (from graph)}$$



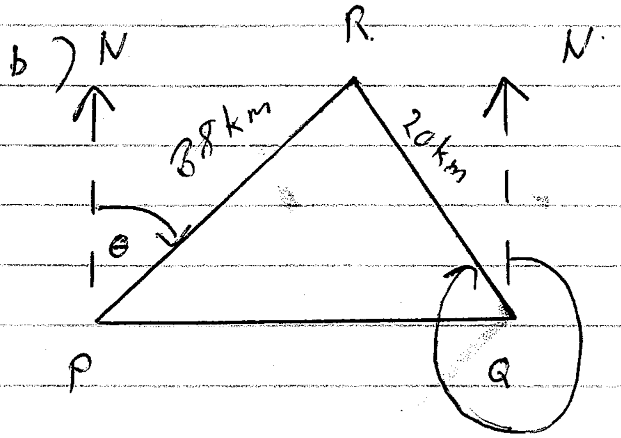
QUESTION THREE

a)



$$\text{Area} = \frac{1}{2} \times 8 \times 8 \times \sin 45^\circ$$
$$= \frac{1}{2} \times 8 \times 8 \times \frac{1}{\sqrt{2}} \checkmark$$
$$= \frac{32}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= 16\sqrt{2}\ \text{cm}^2. \checkmark$$

b)



(i) $\angle PQR = 325^\circ - 270^\circ$
$= 55^\circ$ ✓

(ii) Let bearing equal θ as shown.

$$\frac{\sin \angle RPQ}{20} = \frac{\sin 55^\circ}{38} \checkmark$$
$$\sin \angle RPQ = \frac{20 \sin 55^\circ}{38}$$
$$\sin \angle RPQ = 0.431133654\ldots$$
$$\angle RPQ = 25^\circ 32' \checkmark$$

Bearing of R from P
$= 90^\circ - 25^\circ 32'$
$= 64^\circ 28'$ ✓
$= 064^\circ$ (correct to nearest degree)

(c) $l = r\theta$ (θ in radians)

$5 = 3\theta$ ✓

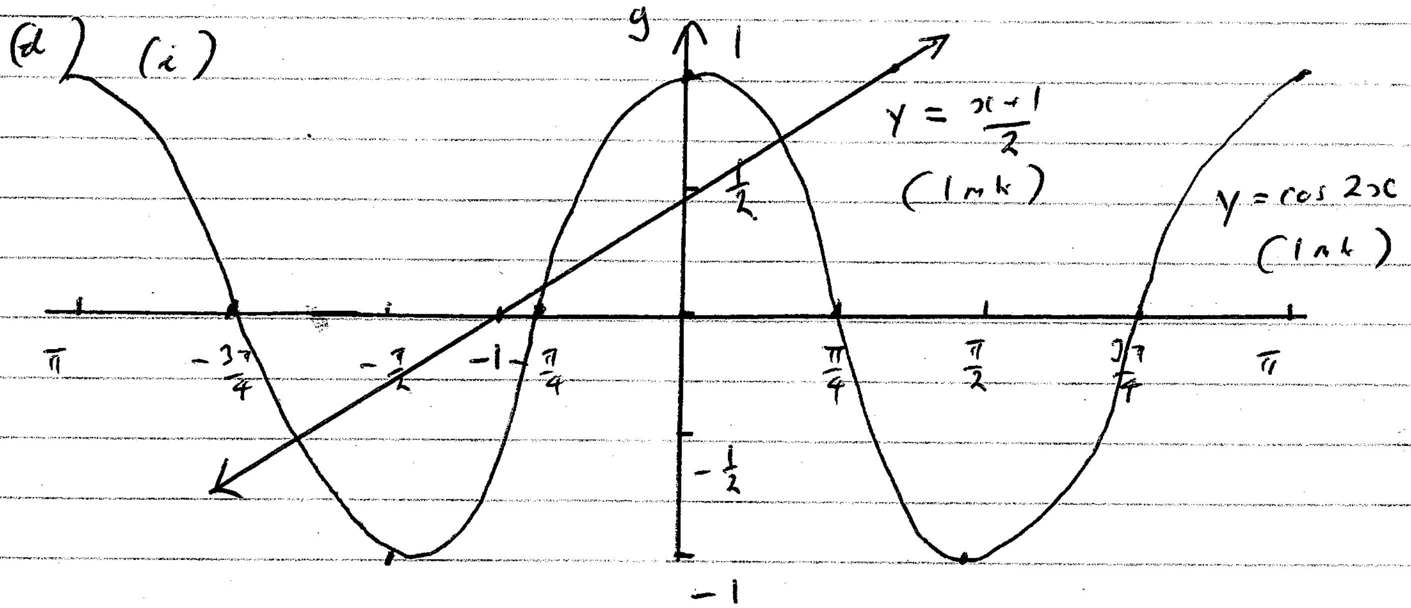
$\theta = \frac{5}{3}$

$= \frac{5}{3} \times \frac{180}{\pi}$

$= 95.49296586...$

$= 95^\circ$ (nearest degree) ✓

(d) (i)



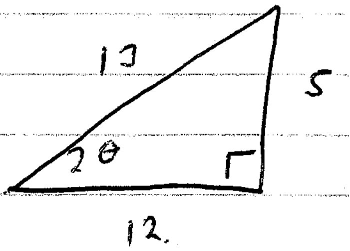
(ii) There are 3 solutions to the equation $2\cos x = x+1$ for $-\pi \leq x \leq \pi$ (Since the two graphs intersect 3 times in the given domain) ✓

QUESTION FOUR

(a) $\cos 5x \cos 3x + \sin 5x \sin 3x = \cos(5x - 3x)$ ✓

$= \cos 2x$ ✓

(b)



$\sin 2\theta = \frac{5}{13}$

$\tan 2\theta = \frac{5}{12}$ ✓

$\tan(\theta + \theta) = \frac{5}{12}$

$$\frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \frac{5}{12}$$

$$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{5}{12}$$ ✓

$24\tan\theta = 5 - 5\tan^2\theta.$

$5\tan^2\theta + 24\tan\theta - 5 = 0$

$(5\tan\theta - 1)(\tan\theta + 5) = 0$ ✓

$\therefore \tan\theta = \frac{1}{5}$ or $\tan\theta = -5$

but since θ is acute (1st quadrant)

then $\tan\theta = \frac{1}{5}$ ✓

(c) (i) RHS $= \frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \frac{1 - (1 - 2\sin^2\theta)}{1 + (2\cos^2\theta - 1)}$ ✓

$$= \frac{2\sin^2\theta}{2\cos^2\theta}$$ ✓

$= \tan^2\theta = \text{LHS}$

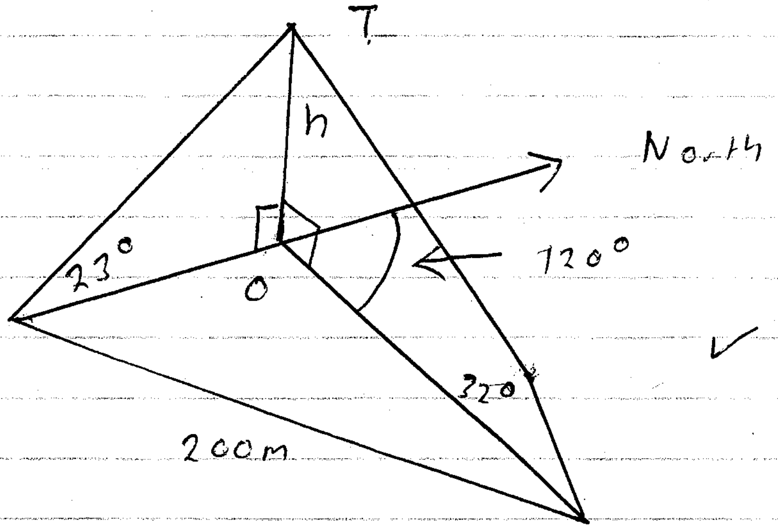
(ii) Let $\theta = \frac{\pi}{8}$ $\quad \tan^2\frac{\pi}{8} = \frac{1 - \cos\frac{\pi}{4}}{1 + \cos\frac{\pi}{4}} = \frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}$ ✓

$$= \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1}$$

$$= (\sqrt{2}-1)^2$$

$$\therefore \tan \frac{\pi}{8} = \sqrt{2}-1$$

(d) (i)



(ii) Let TO = h metres In $\triangle AOT$ $\tan 23° = \frac{h}{AO}$

$$\therefore AO = \frac{h}{\tan 23°}$$

In $\triangle BOT$ $\tan 32° = \frac{h}{BO}$

$$BO = \frac{h}{\tan 32°}$$

$\angle AOB = 180° - 120°$ (from diagram)

$$AB^2 = AO^2 + BO^2 - 2 \times AO \times BO \times \cos 60°$$

$$200^2 = \left(\frac{h}{\tan 23°}\right)^2 + \left(\frac{h}{\tan 32°}\right)^2 - 2 \times \frac{h}{\tan 23°} \times \frac{h}{\tan 32°} \times \cos 60$$

$$= \frac{h^2}{\tan^2 23°} + \frac{h^2}{\tan^2 32°} - \frac{2h^2}{\tan 23° \tan 32°} \times \frac{1}{2}$$

$$h^2\left(\frac{1}{\tan^2 23°} + \frac{1}{\tan^2 32°} - \frac{1}{\tan 23° \tan 32°}\right) = 40\,000$$

$$h^2 = \frac{4000}{\left(\frac{1}{\tan^2 23°} + \frac{1}{\tan^2 32°} - \frac{1}{\tan 23° \tan 32°}\right)}$$

$$h = \sqrt{\frac{40000}{4,3409\ldots}}$$

$$= 95,9924\ldots \ m$$

$$= 96m \ (\text{to nearest } m)$$ ✓

∴ Height of the tower is 96m

QUESTION FIVE

(a) (i) $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

(ii) $\sin x + \cos x = -1$

$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = -1$$

$$2t + 1 - t^2 = -1 - t^2$$

$$2t = -2$$

$$t = -1$$

$$\therefore \tan \frac{x}{2} = -1$$

$$\frac{x}{2} = 135^\circ, 315^\circ$$

$$x = 270^\circ, 630^\circ$$

$$x = \frac{3\pi}{2} \quad \text{for } 0 \le x \le 2\pi$$

Final check sub $x = \pi$ in original eqn.

$$\sin \pi + \cos \pi = -1 \quad \text{TRUE}$$

$$\therefore x = \pi, \frac{3\pi}{2} \quad \text{for } 0 \le x \le 2\pi$$

(b) (i) $3\sin x + 4\cos x = A\sin(x+\alpha)$

$A = \sqrt{4^2+3^2} = 5 \qquad \alpha = \tan^{-1}\left(\frac{4}{3}\right)$ ✓

$3\sin x + 4\cos x = 5\sin\left(x + \tan^{-1}\left(\frac{4}{3}\right)\right)$ ✓

(ii) $5\sin\left[x + \tan^{-1}\left(\frac{4}{3}\right)\right] = 5$

$\sin\left[x + \tan^{-1}\left(\frac{4}{3}\right)\right] = 1$

$x + \tan^{-1}\left(\frac{4}{3}\right) = \frac{\pi}{2}$ ✓

$x = \frac{\pi}{2} - \tan^{-1}\left(\frac{4}{3}\right)$

$= \frac{\pi}{2} - 0.9272\ldots$

$= 0.64$ (to 2 dec pl) ✓

(c) $\sqrt{2}\cos 2\theta + 1 = 0$

$\cos 2\theta = -\frac{1}{\sqrt{2}}$ ✓

$2\theta = 2n\pi \pm \frac{3\pi}{4}$ ✓

$\theta = n\pi \pm \frac{3\pi}{8}$ ✓

QUESTION SIX

a) i) $y = \frac{1}{x} + \frac{1}{x^2}$

$= x^{-1} + x^{-2}$

$y' = -x^{-2} - 2x^{-3}$

$= -\frac{1}{x^2} - \frac{2}{x^3}$

ii) $y = \sqrt{2}$

$= x^{\frac{1}{2}}$

$y' = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

iii) $y = (3x-1)(3x^2+1) = (3x-1)6x + (3x^2+1)3$

$= 18x^2 - 6x + 9x^2 + 3$

$= 27x^2 - 6x + 3$

iv) $y = (3x^2 - 2x - 1)^4 = 4(3x^2 - 2x - 1)^3 \times (6x - 2)$

$= 8(3x-1)(3x^2 - 2x - 1)^3$

v) $y = \frac{4x^2}{x^2+5} = \frac{(x^2+5)8x - 4x^2 \times 2x}{(x^2+5)^2}$

$= \frac{8x^2 + 40x - 8x^2}{(x^2+5)^2}$

$= \frac{40x}{(x^2+5)^2}$

b) (i) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

(ii) $f'(x) = \lim_{h \to 0} \dfrac{\left[3(x+h)^2 - 4(x+h)\right] - \left[3x^2 - 4x\right]}{h}$

$= \lim_{h \to 0} \dfrac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{4x} - 4h - \cancel{3x^2} + \cancel{4x}}{h}$

$= \lim_{h \to 0} \dfrac{6xh - 4h + 3h^2}{h}$

$= \lim_{h \to 0} 6x - 4 + 3h$

$= 6x - 4$