GIRRAWEEN HIGH SCHOOL

YEAR 11 YEARLY EXAMINATION

2010

MATHEMATICS EXTENSION

Time allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question.
  Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1, Question 2 , etc. Write on only one side of the paper.
  Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.

Question1 (22 marks)

a) Solve: $\dfrac{2x+1}{x-2} \geq 1$ **3**

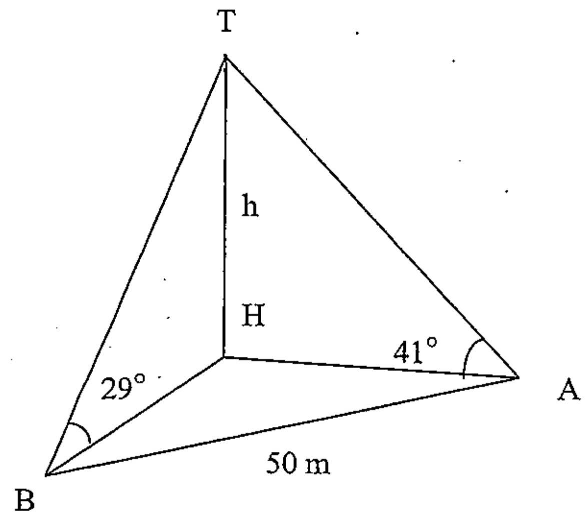
b) Find the acute angle between the lines $x - 2y + 1 = 0$ and $y = 3x - 2$, correct to the nearest minute. **3**

c) Find the coordinates of point P which divides the interval joining $A(-4, 2)$ and $B(2, -3)$ externally in the ratio 5:2. **3**

d) Differentiate: $y = (2x-1)^3(x-1)^2$ **3**

e) Solve for x: $\dfrac{1}{\sqrt{3}+\sqrt{x}} + \dfrac{1}{\sqrt{3}-\sqrt{x}} = 2\sqrt{3}$ **3**

f)



The angle of elevation of a hill at a place A due east of it is 41°. At a place B due south of H the angle of elevation is 29°. If the distance from A to B is 50 metres and H is the foot of the hill,

(i) Show that $50^2 = h^2(\tan^2 49° + \tan^2 61°)$ **4**

(ii) Find the height of the hill (correct to 4 significant figures). **3**

Question 2 (27 marks)

a) Find the value of k if the remainder is 3 when $5x^2 - 10x + k$ is divided by $x - 1$. **2**

b) If $x = 1$ is a double root of $P(x) = 6x^4 - 7x^3 + cx^2 + 13x - 4$.

(i) Show that $c = -8$ **1**

(ii) Hence, or otherwise, fully factorise **3**

$P(x) = 6x^4 - 7x^3 - 8x^2 + 13x - 4$

(iii) Sketch the graph of $P(x) = 6x^4 - 7x^3 - 8x^2 + 13x - 4$ clearly showing the intercepts. **3**

(iv) Hence, or otherwise, solve $6x^4 - 7x^3 - 8x^2 + 13x - 4 \geq 0$ **2**

c) Find the value of k for which $kx^2 + 2kx + 3 = 0$ has roots that differ by 1. **3**

d) If α, β and γ are the roots of the equation $2x^3 - 5x^2 - 2x + 6 = 0$, find the values of:

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta + \beta\gamma + \alpha\gamma$ **1**

(iii) $\alpha\beta\gamma$ **1**

(iv) $\alpha^2 + \beta^2 + \gamma^2$ **3**

(v) $\dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\beta^2\gamma^2} + \dfrac{1}{\alpha^2\gamma^2}$ **3**

e) $P(x)$ is a polynomial of degree 3. When $P(x)$ is divided by $x - 2$, the remainder is 1. When $P(x)$ is divided by $x + 3$, the remainder is 2. When $P(x)$ is divided by $(x-2)(x+3)$, the remainder is $(ax + b)$. Find the values of a and b. **4**

Question 3 (24 marks)

a) Given that $\cos x = \frac{8}{17}$ and $\tan y = \frac{5}{12}$ where x and y are acute, find the exact value of $\sin(x - y)$. **4**

b) Simplify: $\frac{\sin 2A}{1 + \cos 2A}$ **3**

c) Given that $t = \tan\frac{\theta}{2}$, express $\frac{1}{1 + \cos\theta - \sin\theta}$ in terms of t. **3**

d) Solve for $0° \le \theta \le 360°$

(i) $\tan 2\theta = \frac{1}{\sqrt{3}}$ **3**

(ii) $\tan^2\theta + 5\sec^2\theta = 11$ **3**

e) Express $\cos x + 7\sin x$ in the form $A\cos(x + \alpha)$.
Hence, solve $\cos x + 7\sin x = 5$, $0° \le x \le 360°$. **4**

f) Solve $3\sin\theta + \cos\theta = 2$ by using $t = \tan\frac{\theta}{2}$, $0° \le \theta \le 360°$. **4**

Question 4 (23 marks)

a) How many numbers greater than 500 can be formed from the digits 2, 3, 6 and 9 if each digit can only be used once? **3**

b) In how many ways can the letters of the word ISOSCELES be arranged using all the letters? **2**

c) In how many ways can the batting order of an 11 man cricket team be arranged if:

(i) Monjee has to bat first? **2**

(ii) Monjee has to bat first and Arnel has to bat last? **2**

d) A committee of 3 is to be formed from 4 boys and 5 girls. In how many ways can the committee be formed if:

(i) the committee comprises all boys? **2**

(ii) the committee contains 1 boy and 2 girls? **2**

(iii) a particular girl will be on the committee? **2**

e) Seven people are to be seated at a round table.

(i) How many seating arrangements are possible? **2**

(ii) How many arrangements are possible if Clarissa and Rojda refuse to sit together? **3**

f) In how many ways can eight boys sit in an eight-oared boat (4 on each side) if 3 of the crew can only row on the stroke side and one particular boy must be on the bow side? **3**

Question 5 (24 marks)

a) Find the Cartesian equation of the parabola $x = 8t, y = 4t^2$. **3**

b) Find the parametric equations for the parabola $x^2 = 12y$ **3**

c) Consider the parabola $x^2 = 4ay$ where $a > 0$, and suppose the tangent at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ intersect at the point T. Let $S(0, a)$ be the co-ordinates of the focus of the parabola.

(i) Show that the equation of the chord PQ is **4**

$$y = \frac{1}{2}(p+q)x - apq$$

(ii) Find the coordinates of M, the midpoint of PQ. **2**

(iii) If $\angle POQ = 90°$, where O is the origin, prove that $pq = -4$. **3**

(iv) Show that the equation of the tangent at P is $y = px - ap^2$ **3**

(v) Show that the coordinates of T are $(a(p+q), apq)$ **3**

(vi) Show that $SP = a(p^2 + 1)$. **3**

Question 1 (22 marks)

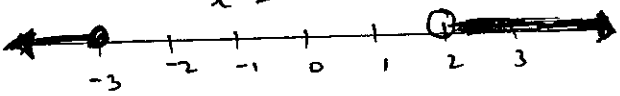
a) $\dfrac{2x+1}{x-2} \geq 1$

$x \neq 2$

Solve $\dfrac{2x+1}{x-2} = 1$

$2x+1 = x-2$

$x = -3$



Test: $x=-4$ $\quad \dfrac{2(-4)+1}{-6} = \dfrac{7}{6} \geq 1$ ✓

$x=0$ $\quad \dfrac{1}{-2} \geq 1$ ✗

$x=3$ $\quad \dfrac{7}{1} \geq 1$ ✓

(3)

$\therefore$ Solution: $x \leq -3,\ x > 2$

b) $x - 2y + 1 = 0$

$2y = x+1$

$y = \dfrac{x}{2} + \dfrac{1}{2}$ $\qquad m_1 = \dfrac{1}{2}$

$y = 3x - 2$ $\qquad m_2 = 3$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{\frac{1}{2} - 3}{1 + \frac{1}{2} \times 3}\right|$

$= \left|\dfrac{-5/2}{5/2}\right| = 1$

(3)

$\tan\theta = 1$

$\theta = 45^\circ$

c) $A(-4, 2)$ $B(2, -3)$ $\quad -5 : 2$

$x = \dfrac{2 \times -4 + -5 \times 2}{-3} = 6$

$y = \dfrac{2 \times 2 + -5 \times -3}{-3} = -6\frac{1}{3}$

$P\left(6, -6\frac{1}{3}\right)$

(3)

d) $y = (2x-1)^3 (x-1)^2$

$u = (2x-1)^3$
$u' = 3(2x-1)^2 \times 2 = 6(2x-1)^2$
$v = (x-1)^2$
$v' = 2(x-1)$

$\dfrac{dy}{dx} = vu' + uv'$

$= 6(2x-1)^2(x-1)^2 + 2(2x-3)^3(x-1)$

$= 2(2x-1)^2(x-1)(3x-3+2x-1)$

$= 2(2x-1)^2(x-1)(5x-4)$

(3)

e) $\dfrac{1}{\sqrt{3}+\sqrt{x}} + \dfrac{1}{\sqrt{3}-\sqrt{x}} = 2\sqrt{3}$

$\dfrac{1}{\sqrt{3}+\sqrt{x}} \times \dfrac{\sqrt{3}-\sqrt{x}}{\sqrt{3}-\sqrt{x}} + \dfrac{1}{\sqrt{3}-\sqrt{x}} \times \dfrac{\sqrt{3}+\sqrt{x}}{\sqrt{3}+\sqrt{x}} = 2\sqrt{3}$

$\dfrac{\sqrt{3}-\sqrt{x}}{3-x} + \dfrac{\sqrt{3}+\sqrt{x}}{3-x} = 2\sqrt{3}$

$\dfrac{2\sqrt{3}}{3-x} = 2\sqrt{3}$

$3 - x = 1$

$x = 2$

(3)

f) i) $\angle ATH = 49^\circ$, $\angle BTH = 61^\circ$

In ΔATH,

$\tan 49^\circ = \dfrac{AH}{h}$

$AH = h\tan 49^\circ$

In ΔBTH

$\tan 61^\circ = \dfrac{BH}{h}$

$BH = h\tan 61^\circ$

In ΔBHA, using Pythagoras' thm

$BA^2 = AH^2 + BH^2$

$50^2 = h^2\tan^2 49^\circ + h^2\tan^2 61^\circ$

$50^2 = h^2(\tan^2 49^\circ + \tan^2 61^\circ)$

(4)

ii) $h = \dfrac{50}{\sqrt{\tan^2 49^\circ + \tan^2 61^\circ}}$

$= 23.37\text{ m}$

(3)

1 mark for 4 sig figs

Question 2 (27 marks)

a) $P(1) = 5(1)^2 - 10(1) + k = 3$

$5 - 10 + k = 3$

$k = 8$ (2)

b) $P(x) = 6x^4 - 7x^3 + cx^2 + 13x - 4$

i) $P(1) = 6 - 7 + c + 13 - 4 = 0$

$c = -8$ (1)

ii) Long division of $6x^4 - 7x^3 - 8x^2 + 13x - 4$ by $x^2 - 2x + 1$, quotient $6x^2 + 5x - 4$:

$$\begin{array}{r} 6x^4 - 7x^3 - 8x^2 + 13x - 4 \\ -\ (6x^4 - 12x^3 + 6x^2) \\ \hline 5x^3 - 14x^2 + 13x \\ -\ (5x^3 - 10x^2 + 5x) \\ \hline -4x^2 + 8x - 4 \\ -4x^2 + 8x - 4 \\ \hline \cdot\cdot \end{array}$$

$\therefore P(x) = (x-1)^2(6x^2 + 5x - 4)$

$= (x-1)^2(3x+4)(2x-1)$ (3)

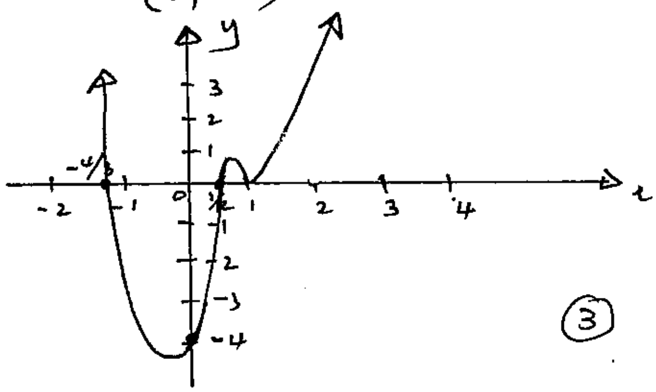
iii) x int $\Rightarrow y = 0$

$(1,0)$, $(-4/3, 0)$ $(\frac{1}{2}, 0)$

$(1,0)$ ↑ double

y int $\Rightarrow x = 0$

$(0, -4)$



(3)

iv) $P(x) \geq 0$ when

$x \leq -4/3$, $x \geq \frac{1}{2}$ (2)

c) $kx^3 + 2kx + 3 = 0$

roots : α, $\alpha + 1$

Sum : $2\alpha + 1 = -2$

$2\alpha = -3$

$\alpha = -\frac{3}{2}$

Prod : $\alpha^2 + \alpha = \frac{3}{k}$

$\left(\frac{-3}{2}\right)^2 + \left(\frac{-3}{2}\right) = \frac{3}{k}$

$\frac{9}{4} - \frac{3}{2} = \frac{3}{k}$

$\frac{3}{4} = \frac{3}{k}$ (3)

$k = 4$

d) $2x^3 - 5x^2 - 2x + 6 = 0$

i) $\alpha + \beta + \gamma = \frac{5}{2}$ (1)

ii) $\alpha\beta + \beta\gamma + \alpha\gamma = -1$ (1)

iii) $\alpha\beta\gamma = -3$ (1)

iv) $\alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= \left(\frac{5}{2}\right)^2 - 2(-1)$

$= \frac{25}{4} + 2$ (3)

$= 8\frac{1}{4}$

v) $\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\alpha^2\gamma^2}$

$= \frac{\gamma^2 + \alpha^2 + \beta^2}{(\alpha\beta\gamma)^2} = \frac{33/4}{(-3)^2}$ (3)

$= \frac{11}{12}$

e) $P(x) = (x-2)(x+3)Q(x) + ax + b$

$P(2) = 2a + b = 1$ —— (1)

$P(-3) = -3a + b = 2$ —— (2)

(1) − (2) $5a = -1$

$a = -\frac{1}{5}$

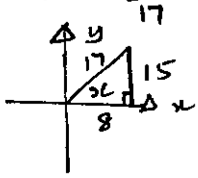
$2\left(-\frac{1}{5}\right) + b = 1$

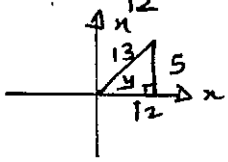
$b = \frac{7}{5}$ (4)

$a = -\frac{1}{5}$, $b = \frac{7}{5}$

Question 3 (24 marks)

a) $\cos x = \frac{8}{17}$; $\tan y = \frac{5}{12}$





$\sin(x-y) = \sin x \cos y - \cos x \sin y$

$= \frac{15}{17} \cdot \frac{12}{13} - \frac{8}{17} \cdot \frac{5}{13}$

$= \frac{140}{221}$ (4)

b) $\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{1+(1-2\sin^2 A)}$

$= \frac{2\sin A\cos A}{2-2\sin^2 A} = \frac{2\sin A\cos A}{2(1-\sin^2 A)}$

$= \frac{\sin A\cos A}{\cos^2 A} = \frac{\sin A}{\cos A}$

$= \tan A$ (3)

c) $\frac{1}{1+\cos\theta - \sin\theta}$

$= \frac{1}{1+\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}}$

$= \frac{1}{\frac{1+t^2+1-t^2-2t}{1+t^2}}$

$= \frac{1}{\frac{2-2t}{1+t^2}} = 1 \times \frac{1+t^2}{2-2t}$ (3)

$= \frac{1+t^2}{2(1-t)}$

d) (i) $\tan 2\theta = \frac{1}{\sqrt{3}}$ Q1, 3 $0° \le 2\theta \le 720°$

$2\theta = 30°, 210°, 390°, 570°$

$\theta = 15°, 105°, 195°, 285°$ (3)

d) (ii) $\tan^2\theta + 5\sec^2\theta = 11$

$\tan^2\theta + 5(1+\tan^2\theta) = 11$

$\tan^2\theta + 5 + 5\tan^2\theta = 11$

$6\tan^2\theta = 6$

$\tan^2\theta = 1$ (3)

$\tan\theta = \pm 1$

$\theta = 45°, 135°, 225°, 315°$

e) $\cos x + 7\sin x = A\cos(x-\alpha)$

$A = \sqrt{50}$

$\frac{1}{\sqrt{50}}\cos x + \frac{7}{\sqrt{50}}\sin x = \cos(x-\alpha)$

$= \cos x\cos\alpha + \sin x \sin$

$\cos\alpha = \frac{1}{\sqrt{50}}$; $\sin\alpha = \frac{7}{\sqrt{50}}$

$\alpha < 90°$

$\tan\alpha = 7$

$\alpha = 81°52'$

$\therefore \cos x + 7\sin x = \sqrt{50}\cos(x+81°5$

$\sqrt{50}\cos(x+81°52') = 5$

$\cos(x+81°52') = \frac{1}{\sqrt{2}}$

$x + 81°52' = 45°, 315°, 405°$

$x = 233°08', 323°08'$ (4)

f) $3\sin\theta + \cos\theta = 2$

$3\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2} = 2$

$6t + 1 - t^2 = 2 + 2t^2$

$3t^2 - 6t + 1 = 0$

$t = \frac{6 \pm \sqrt{24}}{6}$

$\tan\frac{\theta}{2} = \frac{3 \pm \sqrt{6}}{3}$ $0° \le \frac{\theta}{2} \le 1$

$\frac{\theta}{2} = 10°24', 61°10'$

$\theta = 20°48', 122°20'$

Test $\theta = 180°$

$3\sin 180° + \cos 180° = -1 \ne 2$

$\therefore$ not a solution (4)

Solution : $\theta = 20°48', 122°20'$

Question 4 (23 marks)

a) 3 digit nos $\Rightarrow$ $2 \times 3 \times 2 = 12$
4 digit nos $\Rightarrow$ $4 \times 3 \times 2 = 24$
$\therefore$ Total $= 36$ (3)

b) ISOSCELES 9, 3S, 2E

$$\frac{9!}{3!\,2!} = 30240$$ (2)

c) i) $10! = 3\,628\,800$ (2)

ii) $9! = 362\,880$ (2)

d) 4B 5G

i) 3B : ${}^4C_3 = 4$ (2)

ii) 1B 2G : ${}^4C_1 \times {}^5C_2$
$= 40$ (2)

iii) ${}^8C_2 = 28$ (2)

e) i) $6! = 720$ (2)

ii) not together
$= 6! -$ together
$= 6! - 2 \times 5!$
$= 720 - 240$ (3)
$= 480$

f) Stroke side : $4 \times 3!$ ← 3B
Bow side : 4 ← 1B
Remainder $= 4!$

$\therefore\ 4 \times 3! \times 4 \times 4!$
$= 2304$ (3)

Question 5 (24 marks)

a) $x = 8t$; $y = 4t^2$

$t = \frac{x}{8}$

$y = 4\left(\frac{x}{8}\right)^2$

$= \frac{4x^2}{64} = \frac{x^2}{16}$

$x^2 = 16y$ ③

b) $x^2 = 12y$

$4a = 12$

$a = 3$

$\therefore x = 2at$; $y = at^2$

$x = 6t$; $y = 3t^2$ ③

c) $x^2 = 4ay$

i) $P(2ap, ap^2)$ $Q(2aq, aq^2)$

$m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{a(p+q)(p-q)}{2a(p-q)}$

$m = \frac{p+q}{2}$

$E_{PQ}: y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$2y - 2ap^2 = px + qx - 2ap^2 - 2apq$

$2y = (p+q)x - 2apq$

$y = \frac{1}{2}(p+q)x - apq$ ④

ii) $M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(\frac{2a(p+q)}{2}, \frac{a(p^2+q^2)}{2}\right)$

$= \left(a(p+q), \frac{a}{2}(p^2+q^2)\right)$ ②

iii) $\angle POQ = 90°$

$\therefore m_{OP} \cdot m_{OQ} = -1$

$m_{OP} = \frac{ap^2}{2ap} = \frac{p}{2}$

$m_{OQ} = \frac{aq^2}{2aq} = \frac{q}{2}$

$\frac{p}{2} \times \frac{q}{2} = -1$

$\therefore pq = -4$ ③

iv) $m_{tangent}$ at $P = p$ pt $(2ap, ap^2)$

$E: y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$ ③

v) $E_{tangent\ at\ P}: y = px - ap^2$ —— ①

$E_{tangent\ at\ Q}: y = qx - aq^2$ —— ②

① − ② $\quad 0 = (p-q)x - (p^2 - q^2)a$

$(p-q)x = (p+q)(p-q)a$

$x = a(p+q)$

substitute $x = a(p+q)$ into ①

$y = p \cdot a(p+q) - ap^2$

$y = ap^2 + apq - ap^2$

$y = apq$ ③

$\therefore T = (a(p+q), apq)$

vi) $S(0,a)$ $P(2ap, ap^2)$

$SP = \sqrt{(2ap)^2 + (ap^2 - a)^2}$

$= \sqrt{4a^2p^2 + a(p^2-1)^2}$

$= \sqrt{4a^2p^2 + a^2(p^4 - 2p^2 + 1)}$

$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$ ③

$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$

$= \sqrt{a^2(p^2+1)^2} = a(p^2+1)$