GOSFORD HIGH SCHOOL

MATHEMATICS – EXTENSION 1

2005
YEAR 11 PRELIMINARY YEARLY EXAM

Time allowed 2 hours *(Plus 5 minutes reading time)*

NAME :..

TEACHER :..

DIRECTIONS TO CANDIDATES

- ***Attach this cover sheet*** to your answers
- ***All*** questions may be attempted
- All necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.
- ***START*** each question on a new page, clearly marked with your name and the question number
- Approved calculators and templates may be used
- Write using blue or black pen

QUESTION	MARK
1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

Marks

Question 1

a) A is the point $(-5,7)$ and B is $(-1,-2)$. Find the point P which divides AB externally in the ratio 3:1. **3**

b) Solve $x - 5 \geq \frac{6}{x}$ **3**

c) (i) Write down 3 formulae for $\cos 2\theta$ **1**

(ii) Prove that $\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2\theta$ **4**

Question 2 <u>Begin a new sheet of paper</u>

a) For what values of p will $x^2 - (p+3)x + 4p = 0$ have:

(i) Equal roots? **2**

(ii) No real roots? **2**

b) (i) Find the equation of the tangent to the parabola $y = 3x - x^2$ at the point where $x = 2$. **3**

(ii) Draw a sketch of the parabola and this tangent. **2**

c) Use the expansion of $\tan(A + B)$ to find the value of $\tan 75°$ in simplest exact form. **2**

Question 3 <u>Begin a new sheet of paper</u>

a) Solve $3^{2x} + 2(3^x) - 15 = 0$ **4**

b) Find the values of A, B, and C if

$$7x^2 - 5x + 3 \cong A(x + 1)^2 + B(x + 1) + C$$ **3**

c) A parabola has its focus at $(-2,3)$ and its directrix has equation $x = 4$.

(i) Draw a sketch, and use it to find **1**

(ii) the coordinates of its vertex. **1**

(iii) the equation of the parabola. **3**

Question 4 Begin a new sheet of paper **Marks**

a) Differentiate the following (Simplify your answer):

(i) $\dfrac{x}{\sqrt{1-2x}}$ **3**

(ii) $\dfrac{2x^2 - 5x + 1}{\sqrt{x}}$ **3**

b) Differentiate from first principles $f(x) = x - 5x^2$ **4**

c) Find the point on the curve $y = x^2 - 3x - 7$ where the tangent is parallel to the line $y = 5x - 4$. **2**

Question 5 Begin a new sheet of paper

a) For the function $y = x^4 - 4x^3$

(i) Find any stationary points and determine their nature **4**

(ii) Find any points of inflexion **2**

(iii) Find the intercepts with the axes **1**

(iv) Sketch the curve **3**

(v) For what values of x is the curve concave down? **1**

b) Prove that the locus of points equidistant from A(−5,4) and B(3, 2) is the perpendicular bisector of AB. **4**

Question 6 Begin a new sheet of paper

a) Solve $\sin 3x \cos x - \cos 3x \sin x = -\dfrac{\sqrt{3}}{2}$ for $0° \le x \le 360°$ **3**

b) (i) Express $2\sqrt{3}\cos x - 2\sin x$ in the form $R\cos(x + \alpha)$ where R is positive and α is acute. **3**

(ii) Hence find the range of the function $f(x) = 2\sqrt{3}\cos x - 2\sin x$ **1**

c) (i) If $t = \tan\dfrac{x}{2}$, show that $3\sin x - 4\cos x - 4 = \dfrac{6t - 8}{1 + t^2}$ **2**

(ii) Hence solve $3\sin x - 4\cos x - 4 = 0$ for $0° \le x \le 360°$ **3**

Question 7 Begin a new sheet of paper **Marks**

a) The quadratic equation $x^2 - 3x + 7 = 0$ has roots α and β.

(i) Find the value of $\alpha + \beta$ **1**

(ii) Find the value of $\alpha\beta$ **1**

(iii) Form the equation which has roots α^2 and β^2 **4**

b) A discontinuous function is partly defined as $f(x) = 3x - 1$ for $x > 0$.
It is given that $f(x)$ is an odd function.

(i) Draw a sketch of the graph of $y = f(x)$ **2**

(ii) Complete the definition of $f(x)$ for $x \le 0$. **2**

c) Show that the locus of points equidistant from the two straight lines $y = 2$ and $3x - 4y = 1$ is a pair of perpendicular lines, and find their equations. **4**

(Hint: the distance of a point from a line is the perpendicular distance)

Question 8 Begin a new sheet of paper

a) (i) Expand $-(x-5)(x-1)$ **1**

For the function $y = \dfrac{x-3}{(x+1)(x-2)}$ it is given that $\dfrac{dy}{dx} = \dfrac{-x^2+6x-5}{(x+1)^2(x-2)^2}$

(ii) Find any stationary points and establish their nature. **4**

(iii) Find any discontinuities **2**

(iv) Find $\lim\limits_{x \to \infty} \dfrac{x-3}{x^2-x-2}$. What is its graphical significance? **2**

b) Find, as a relation between k, l and m, the condition for the quadratic equation in x **4**

$$(k^2 + l^2)x^2 + 2l(k+m)x + (l^2 + m^2) = 0$$

to have real roots. Give your answer in simplest form.

2005 Solutions to Extension 1 Preliminary

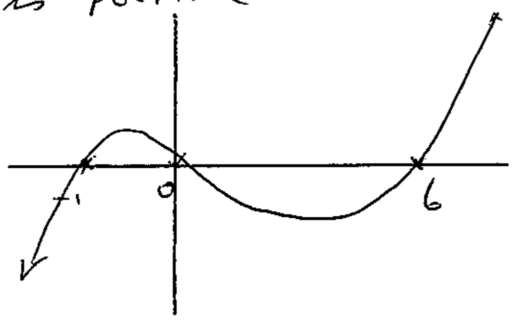
Question 1

$k : l = 3 : -1$

a) $x = \dfrac{3\times 1 + -1\times 5}{3 + -1} = \dfrac{-3+5}{2} = 1$ $\quad (1, -6\tfrac{1}{2})$

$y = \dfrac{3\times -2 + -1\times 7}{3 + -1} = \dfrac{-6-7}{2} = -6\tfrac{1}{2}$

b) multiply b/s by x^2 as it is positive



$x^3 - 5x^2 \geq 6x$

$x^3 - 5x^2 - 6x \geq 0$

$x(x^2 - 5x - 6) \geq 0$

$x(x-6)(x+1) \geq 0$

$-1 \leq x \leq 0$ or $x \geq 6$

But $x \neq 0$ $\therefore$ $\underline{-1 \leq x < 0 \text{ or } x \geq 6}$

c) i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

ii) $LHS = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$

$= \dfrac{1 - (1 - 2\sin^2\theta)}{1 + 2\cos^2\theta - 1}$

$= \dfrac{2\sin^2\theta}{2\cos^2\theta}$

$= \underline{\tan^2\theta} = RHS$

Question 2

a) i) $(p+3)^2 - 16p = 0$ $\quad$ ($\Delta = 0$ for equal roots)

$p^2 + 6p + 9 - 16p = 0$

$p^2 - 10p + 9 = 0$

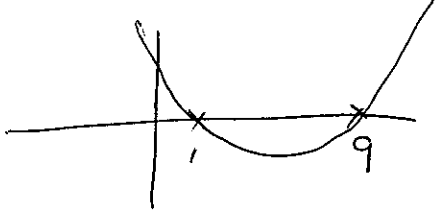
$(p-9)(p-1) = 0$

$\underline{p = 9 \text{ or } 1}$

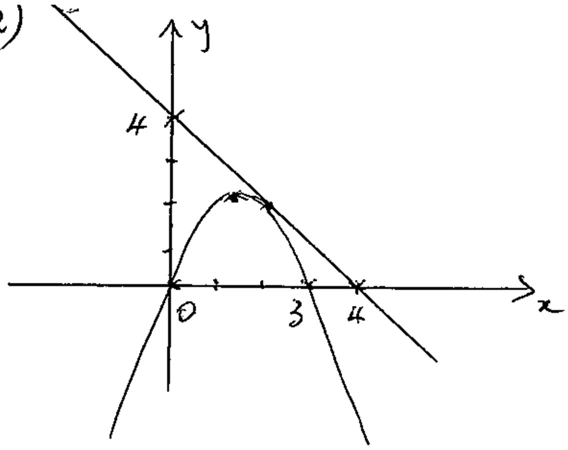
ii) ($\Delta < 0$ for no real roots)

$(p-9)(p-1) < 0$

$\underline{1 < p < 9}$



2b) i) $x=2$, $y=6-4=2$ $(2,2)$



$$\frac{dy}{dx} = 3-2x$$

$\therefore$ when $x=2$, $m=-1$

$\therefore$ $y-2 = -1(x-2)$

$$\underline{x+y-4=0}$$

ii) $y = x(3-x)$

vertex $\left(\frac{3}{2}, \frac{9}{4}\right)$ or $\left(1\frac{1}{2}, 2\frac{1}{4}\right)$

c) $$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$

$$\tan(45^\circ+30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1-\tan 45^\circ \tan 30^\circ} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}\cdot 1} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{4+2\sqrt{3}}{2} = \underline{\underline{2+\sqrt{3}}}$$

Question 3

a) Put $u = 3^x$

$\therefore$ $u^2 + 2u - 15 = 0$

$(u+5)(u-3) = 0$

$u = -5$ or 3

$\therefore$ $3^x = -5$ or $3^x = 3$

No solution

$\underline{\underline{x=1}}$ is only solution.

b) $7x^2 - 5x + 3 \equiv A(x+1)^2 + B(x+1) + C$

Equate coefficients of x^2: $7 = A$ ——— (1)

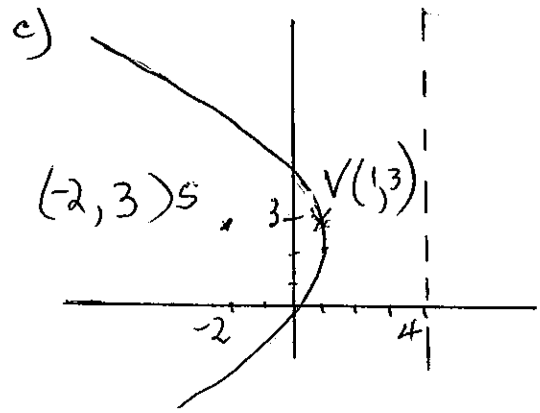
Sub $x=-1$: $7+5+3 = C$

$C = 15$ ——— (2)

Sub $x=0$: $3 = 7 + B + 15$

$B = -19$ ——— (3)

c)



ii) vertex $(1, 3)$ Form $Y^2 = -4aX$

iii) $(y-3)^2 = -4\times 3(x-1)$

$$\underline{(y-3)^2 = -12(x-1)}$$

Question 4

a) i) $y = \dfrac{x}{(1-2x)^{1/2}}$

$$\frac{dy}{dx} = \frac{(1-2x)^{\frac{1}{2}}.1 - x.\frac{1}{2}(1-2x)^{-\frac{1}{2}}.-2}{1-2x}$$

$$= \frac{\sqrt{1-2x} + \frac{x}{\sqrt{1-2x}}}{1-2x}$$

$$\frac{dy}{dx} = \frac{1-2x+x}{(1-2x)\sqrt{1-2x}} = \frac{1-x}{(1-2x)\sqrt{1-2x}}$$

ii) $y = 2x^{\frac{3}{2}} - 5x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = 2.\frac{3}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$= 3\sqrt{x} - \frac{5}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

$$= \frac{6x^2 - 5x - 1}{2x\sqrt{x}}$$ OR

$$\frac{dy}{dx} = \frac{(6x+1)(x-1)}{2x\sqrt{x}}$$

b) $f(x+h) = (x+h) - 5(x+h)^2$

$= x + h - 5x^2 - 10xh - 5h^2$

$f(x+h) - f(x) = h - 10xh - 5h^2$

$$\frac{f(x+h)-f(x)}{h} = 1 - 10x - 5h$$

$$f'(x) = \lim_{h \to 0} 1 - 10x - 5h$$

$$f'(x) = 1 - 10x$$

c) $y = x^2 - 3x - 7$

$\frac{dy}{dx} = 2x - 3$

Put $2x - 3 = 5$

$2x = 8$

$x = 4$

$y = 16 - 12 - 7$

$\therefore$ Point is $(4, -3)$

Question 5 $\quad y = x^4 - 4x^3$

a) i) $\frac{dy}{dx} = 4x^3 - 12x^2$

$= 4x^2(x-3)$

For stationary pts $\frac{dy}{dx} = 0$

$\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\}$ or $\left.\begin{array}{l} 3 \\ -27 \end{array}\right\}$

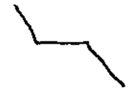
$y'' = 12x^2 - 24x$

when $x = 0$, $y'' = 0$

no further info on $(0, 0)$

$\therefore$ Examine y'

x	0^-	0	0^+
y'	$-$	0	$-$



$\therefore (0,0)$ is a horizontal inflexion.

Nature of $(3, -27)$

when $x = 3$, $y'' = 108 - 72$

positive

$\therefore (3, -27)$ is a minimum turning point.

ii) Inflexions when $y'' = 0$

$12x(x-2) = 0$

$x = 0$ or 2

already examined.

x	2^-	2	2^+
y''	$-$	0	$+$

Curve changes from concave down to concave up.

$\therefore (2, -16)$ is a point of inflexion

iii) For x intercepts $y = 0$

$x^4 - 4x^3 = 0$

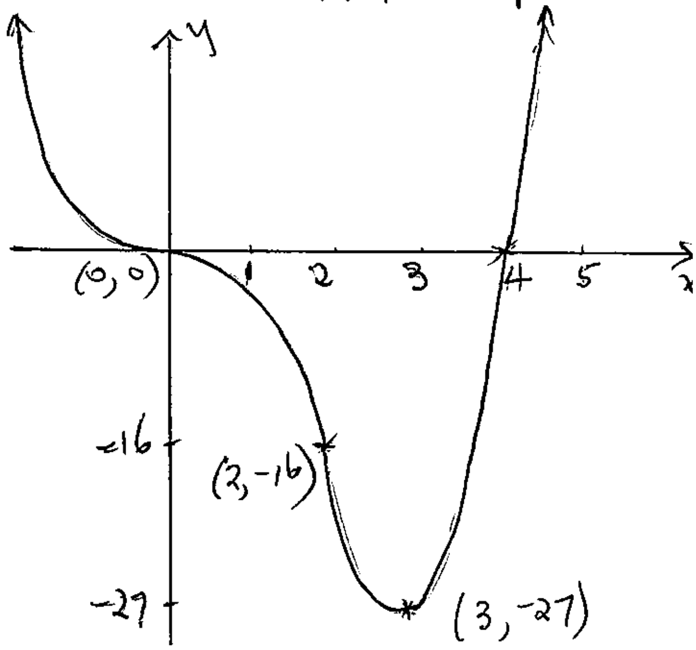
$x^3(x-4) = 0$

$x = 0$ or 4

For y intercepts $x = 0$

$y = 0$.

$\therefore (0,0)$ $(4,0)$ are only intercepts



v) Curve concave down

$0 < x < 2$

b) locus is

$(x+5)^2 + (y-4)^2 = (x-3)^2 + (y-2)^2$

$x^2 + 10x + 25 + y^2 - 8y + 16 = x^2 - 6x + 9 + y^2 - 4y + $

$16x - 4y + 28 = 0$

$4x - y + 7 = 0$ (gradient 4

Gradient of AB $\quad m_1 = \frac{2-4}{3+5} = \frac{-2}{8}$

$m_1 = -\frac{1}{4}$

$\therefore$ Perp line has gradient of 4.

Mid point of AB is

$\left(\frac{-5+3}{2}, \frac{4+2}{2}\right) = (-1, 3)$

$4x - y + 7 = -4 - 3 + 7 = 0$

$\therefore (-1, 3)$ lies on locus.

$\therefore 4x - y + 7 = 0$ is perp bisec

Question 6

a) $\sin(3x - x) = -\frac{\sqrt{3}}{2}$ $\qquad 0 \le x \le 360^\circ$

$\sin 2x = -\frac{\sqrt{3}}{2}$ $\qquad 0 \le 2x \le 720^\circ$

Reference $\angle$ for $2x$ is 60° as $\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\therefore 2x = 180+60,\ 360-60,\ 360+240,\ 360+300$

$= 240,\ 300,\ 600,\ 660$

$\underline{\underline{x^\circ = 120^\circ,\ 150^\circ,\ 300^\circ,\ 330^\circ}}$

b) i) $2\sqrt{3}\cos x - 2\sin x \equiv R\cos(x+\alpha)$

$\equiv R\cos x\cos\alpha - R\sin x\sin\alpha$

$\therefore R\cos\alpha = 2\sqrt{3}$ —— ①

$R\sin\alpha = 2$ —— ②

② ÷ ① $\tan\alpha = \frac{1}{\sqrt{3}} \Longrightarrow \alpha = 30^\circ$ (as α is acute)

①^2 + ②^2 $R^2\cos^2\alpha + R^2\sin^2\alpha = 4\times 3 + 4 = 16$

$R = 4$ (as R is positive)

$\therefore 2\sqrt{3}\cos x - 2\sin x \equiv \underline{\underline{4\cos(x^\circ + 30^\circ)}}$

ii) Range of $f(x)$: $\underline{\underline{-4 \le y \le 4}}$

c) i) $\sin x = \frac{2t}{1+t^2}$ $\qquad \cos x = \frac{1-t^2}{1+t^2}$

$\therefore 3\sin x - 4\cos x - 4 = \frac{6t}{1+t^2} - \frac{4-4t^2}{1+t^2} - 4$

$= \frac{6t - 4 + 4t^2 - 4 - 4t^2}{1+t^2}$

$= \frac{6t - 8}{1+t^2}$

ii) First check does $x = 180^\circ$

$3\sin 180 - 4\cos 180 - 4$

$= 3\times 0 - 4\times -1 - 4$

$= 0$

$\therefore 180^\circ$ is a solution

Then $\frac{6t-8}{1+t^2} = 0 \Longrightarrow 6t - 8 = 0$

$\tan\frac{x}{2} = t = \frac{4}{3}$

$\frac{x}{2} = 53^\circ 8'$ (only $0 \le \frac{x}{2} \le 180$ needed)

$\underline{\underline{\therefore x = 106^\circ 16' \text{ or } 180^\circ}}$

Question 7

a) i) $\alpha + \beta = \frac{-b}{a} = \underline{\underline{3}}$ ii) $\alpha\beta = \frac{c}{a} = \underline{\underline{7}}$

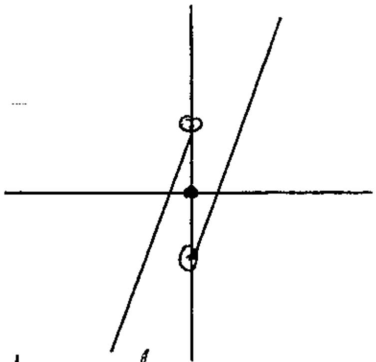
iii) We need $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= 9 - 14 = -5$

& $\alpha^2\beta^2 = (\alpha\beta)^2 = 49$

$\therefore$ Equation is

$$\underline{\underline{x^2 + 5x + 49 = 0}}$$

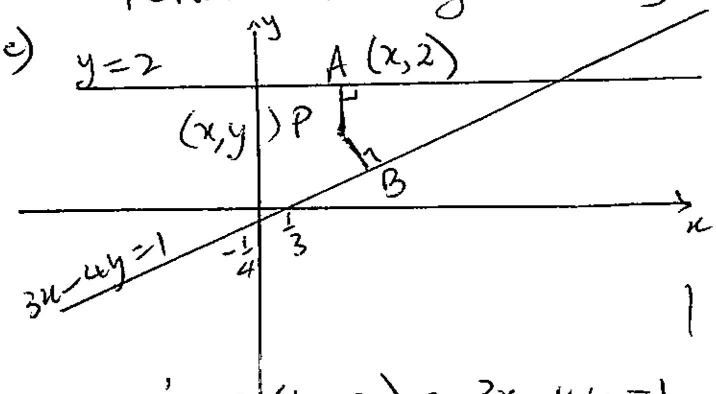
b) i)



ii) $f(x) \begin{cases} = 3x - 1 & \text{for } x > 0 \\ = 0 & \text{for } x = 0 \\ = 3x + 1 & \text{for } x < 0 \end{cases}$

rotational symmetry about O.

c)



$PA = PB$

$PA = |x - 2|$

$PB = \left| \frac{3x - 4y - 1}{\sqrt{3^2 + 4^2}} \right|$

$|y - 2| = \left| \frac{3x - 4y - 1}{5} \right|$

$\therefore 5(y-2) = 3x - 4y - 1$ or $-5(y-2) = 3x - 4y - 1$

$5y - 10 = 3x - 4y - 1$ $\quad$ $-5y + 10 = 3x - 4y - 1$

$3x - 9y - 9 = 0$ $\quad$ $\underline{\underline{11 = 3x + y}}$

or $\underline{\underline{x - 3y - 1 = 0}}$ & $m_2 = -3$

$m_1 = \frac{1}{3}$

$m_1 m_2 = \frac{1}{3} \times -3 = -1$

$\therefore$ lines are $\underline{\underline{\text{perpendicular.}}}$

Question 8

a) i) $-(x^2 - x - 5x + 5) = -x^2 + 6x - 5$

ii) $\frac{dy}{dx} = \frac{-(x-5)(x-1)}{(x+1)^2(x-2)^2}$

For stationary points $\frac{dy}{dx} = 0 \quad \therefore \left.\begin{matrix} x = 5 \\ y = \frac{1}{9} \end{matrix}\right\}$ or $\left.\begin{matrix} 1 \\ 1 \end{matrix}\right\}$

Note denominator of $\frac{dy}{dx}$ is always positive

Nature of $(5, \frac{1}{9})$

x	5^-	5	5^+
y'	$+$	0	$-$

$(5, \frac{1}{9})$ is a maximum

Nature of $(1, 1)$

x	1^-	1	1^+
y'	$-$	0	$+$

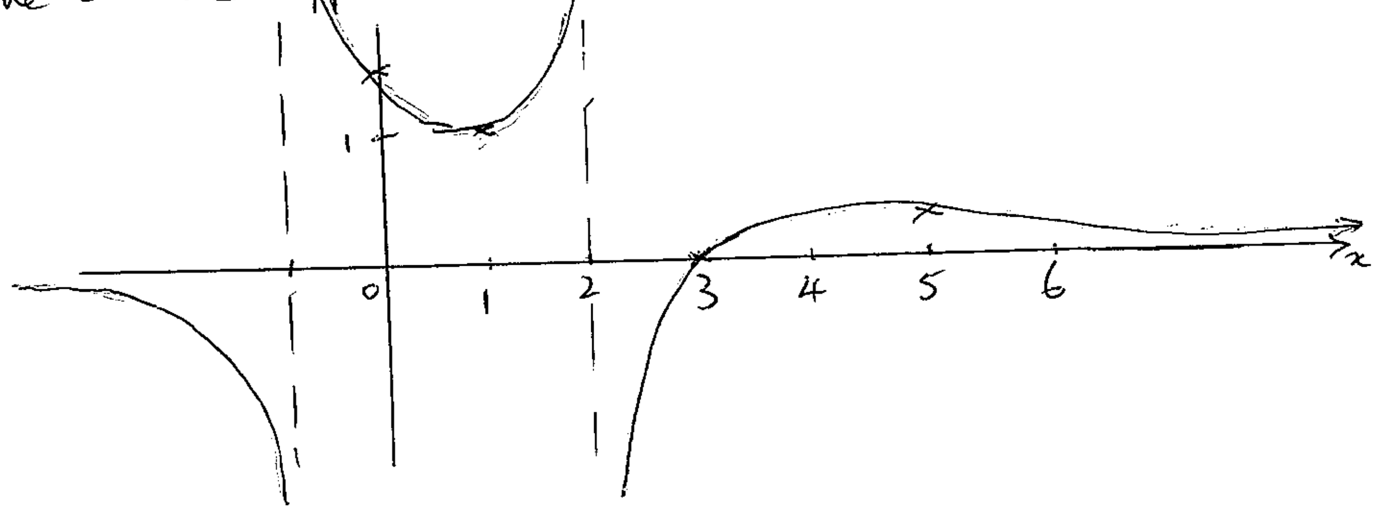
$(1,1)$ is a minimum

iii) for Discontinuities $(x+1)(x-2) = 0$

$x = -1$, $x = 2$ are asymptotes

iv) $\lim\limits_{x \to \infty} \dfrac{x-3}{x^2-x-2} = \lim\limits_{x \to \infty} \dfrac{\frac{1}{x} - \frac{3}{x^2}}{1 - \frac{1}{x} - \frac{2}{x}} = 0$

The curve approaches the x axis as $x \to \infty$.



b) For real roots, $\Delta \geq 0$

$$4l^2(k+m)^2 - 4(k^2+l^2)(l^2+m^2) \geq 0$$

$$l^2(k^2+2km+m^2) - (k^2l^2+k^2m^2+l^4+l^2m^2) \geq 0$$

$$l^2k^2 + 2kml^2 + m^2l^2 - k^2l^2 - k^2m^2 - l^4 - l^2m^2 \geq 0$$

$$-l^4 + 2kml^2 - k^2m^2 \geq 0$$

$$0 \geq l^4 - 2kml^2 + k^2m^2$$

$$0 \geq (l^2 - km)^2 \quad \text{or} \quad (l^2-km)^2 \leq 0$$

which is only possible if $l^2 - km = 0$

ie $l^2 = km$