GOSFORD HIGH SCHOOL

2007

PRELIMINARY HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 4

Yearly Examination

MATHEMATICS – EXTENSION 1

General Instructions:

- Reading time – 5minutes
- Working time – 2 Hours
- Write using black or blue pen.
- Board-approved calculators may be used
- All necessary working should be shown in every question.

Total marks: - 72

- Attempt Questions 1 - 6
- All Questions are of equal value

QUESTION 1 (12 Marks)

(a) Solve $\frac{5}{x-3} < 7$ and graph the solution on a number line **3**

(b) Find the acute angle between the lines $3x + 2y - 7 = 0$ and $y = 4x - 5$ to the nearest degree **3**

(c) Find the coordinates of the point that divides the interval A(2, 6) and B(9, 3) externally in the ratio 5:2 **3**

(d) (i) Show that $\frac{1}{a^2 + ab} + \frac{1}{b^2 + ab} = \frac{1}{ab}$ **2**

(ii) Hence express $\frac{1}{7}$ in the form $\frac{1}{a} + \frac{1}{b}$ where *a* and *b* are positive integers **1**

END OF QUESTION 1

QUESTION 2 (Start a New Page) (12 Marks)

(a) Calculate the remainder when $P(x) = x^3 + 2$ is divided by $x - 1$ **1**

(b) The polynomial $P(x) = x^3 + ax + 12$ has a factor $(x + 3)$. Find the value of *a*. **2**

(c) Find the quotient Q(*x*) and the remainder, R(*x*), when the polynomial $P(x) = x^4 - x^2 + 1$ is divided by $x^2 + 1$ **3**

(d) The point P(3, 7) lies on the graph of the odd polynomial function $y = P(x)$.
Find, with reasons,

(i) the remainder when P(*x*) is divided by $(x - 3)$ **1**

(ii) the remainder when P(*x*) is divided by $(x + 3)$ **1**

(e) Solve $x^3 - 4x^2 - x + 4 = 0$ **2**

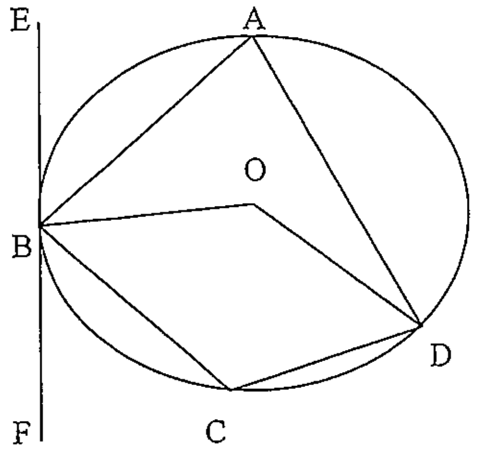
(f) If α, β and γ are roots of the cubic equation $5x^3 + 7x^2 - 3x - 4 = 0$, find the value of **2**

$$\alpha^2 + \beta^2 + \gamma^2$$

END OF QUESTION 2

QUESTION 3 (Start a New Page) (12 Marks)

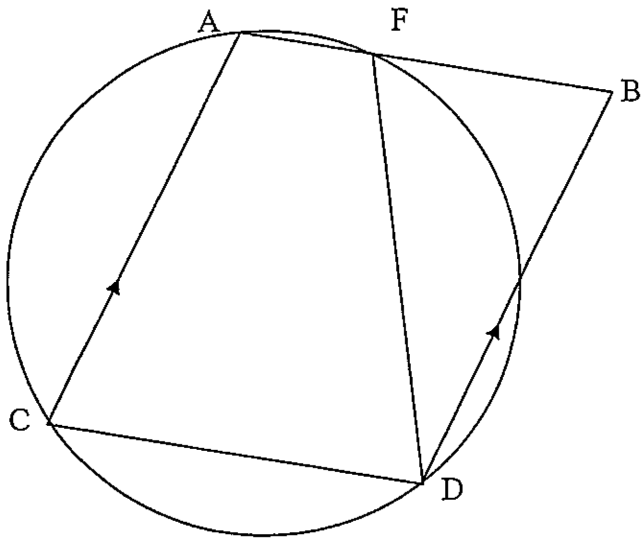
(a)



ABCD is a quadrilateral inscribed in a circle with centre O. Reflex ∠BOD = 240°.
Find, giving reasons

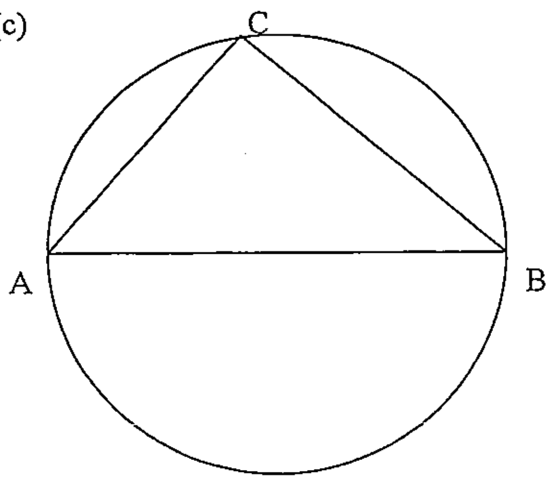
(i) the size of ∠BAD **2**
(ii) the size of ∠BCD **2**
(iii) the size of ∠FBD **2**

(b)



In the diagram (not to scale), AC = BD and AC // BD. Prove DF = DB **4**

(c) **2**



In the diagram (not to scale), AB = 41 cm,
AC = 9 cm and BC = 40 cm
Prove AB is the diameter of the circle drawn

END OF QUESTION 3

QUESTION 4 (Start a New Page) (12 Marks)

(a) By using an expression for $\tan(\alpha + \beta)$, find the exact value of $\tan 105°$ **2**

(b) Prove that $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$ **3**

(c) Solve the equation $\sqrt{3}\sin\theta - \cos\theta = 1$ for $0 \le \theta \le 2\pi$ **4**

(d) A ship, S, is sailing on a straight course for an oil rig, R, on a bearing of 050°. At the same time, a lighthouse, L, is sighted in a direction of 012°. The charts indicate that the bearing from the rig to the lighthouse is 340°, and a distance of 18 km from it.

(i) Draw a diagram showing the above information **1**
(ii) Find the distance from the ship to the oil rig correct to the nearest metre **2**

END OF QUESTION 4

QUESTION 5 (Start a New Page) (12 Marks)

(a) Show $\dfrac{\sin 2\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\cos\theta} = \sec\theta$ **2**

(b) Sketch $f(x) = \left||x-2|-1\right|$ **2**

(c) Using First Principles, find the derivative of **2**

$f(x) = 4x^2 + 5$

(d) For the curve $y = x^2(x^2 - 1)$,

(i) Find the derivative $\dfrac{dy}{dx}$ **1**
(ii) Hence or otherwise, find the equation of the normal at the point (1, 0) **2**

(e) The straight line joining the centre O, of a circle to an external point T, cuts the circumference of the circle at Z. TZ = 32 cm and the tangent to the circle from T is 40 cm long.

(i) Draw a diagram to represent this information **1**
(ii) Calculate the radius of the circle giving reasons for your answer **2**

END OF QUESTION 5

QUESTION 6 (Start a New Page) (12 Marks)

(a) Solve for x: $\left|x^2 - 5\right| \leq 5x + 9$ **2**

(b) The equation $x^3 + px^2 + qx + pq = 0$, where $p \neq 0$ and $q \neq 0$ has 3 real roots α, β and γ.

(i) By considering the relationships between the roots and the coefficients of the equation, show that $(\alpha + \beta + \gamma)\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) = 1$ **2**

(ii) Show that $-p$ is a root of the equation. Hence show that $q<0$. **2**

(c) The quadratic equation $x^2 - x + k = 0$, where k is a real number has 2 distinct, positive real roots.

(i) Show that $0 < k < \frac{1}{4}$ **2**

(ii) Let the two roots be α and β. Show that $\alpha^2 + \beta^2 = 1 - 2k$ and deduce that $\frac{1}{2} < \alpha^2 + \beta^2 < 1$ **4**

END OF EXAMINATION

PRELIMINARY EXTENSION 1 YEARLY SOLUTIONS 2007

QUESTION 1 (12 Marks)

(a) Solve $\frac{5}{x-3} < 7$ and graph the solution on a number line **3**

Note: $x \neq 3$

For $x<3$

$5 > 7x - 21$

$26 > 7x$

$\frac{26}{7} > x$

$x < \frac{26}{7}$

For $x>3$

$5 < 7x - 21$

$26 < 7x$

$\frac{26}{7} < x$

$x > \frac{26}{7}$

Testing points

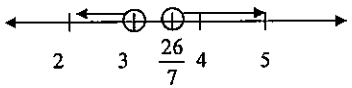
$x \to 3^-$ $\frac{5}{x-3} < 7$

$x \to 3^+$ $\frac{5}{x-3} > 7$

$x \to \frac{26}{7}^-$ $\frac{5}{x-3} > 7$

$x \to \frac{26}{7}^+$ $\frac{5}{x-3} < 7$

$\therefore x < 3$ and $x > \frac{26}{7}$



(b) Find the acute angle between the lines $3x + 2y - 7 = 0$ and $y = 4x - 5$ to the nearest degree **3**

$3x + 2y - 7 = 0$

Gradient $= \frac{-3}{2}$

$y = 4x - 5$

Gradient $= 4$

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$\tan\theta = \frac{\frac{-3}{2} - 4}{1 + \frac{-3}{2} \times 4}$

$\tan\theta = \frac{11}{10}$

$\theta = 47.73°$

$\theta = 48°$

(c) Find the coordinates of the point that divides the interval A(2, 6) and B(9, 3) externally in the ratio 5:2 **3**

The point (X,Y) which divides the interval in the ratio $r_1:r_2$ is given by

$(X,Y) = \left(\frac{r_1 x_2 + r_2 x_1}{r_1 + r_2}, \frac{r_1 y_2 + r_2 y_1}{r_1 + r_2}\right)$

$(X,Y) = \left(\frac{5\times 9 + -2\times 2}{5 + -2}, \frac{5\times 3 + -2\times 6}{5 + -2}\right)$ Here $r_1 = 5$ and $r_2 = -2$ (externally)

$(X,Y) = (13\frac{2}{3}, 1)$

(d) (i) Show that $\frac{1}{a^2 + ab} + \frac{1}{b^2 + ab} = \frac{1}{ab}$ **2**

$LHS = \frac{1}{a^2 + ab} + \frac{1}{b^2 + ab}$

$LHS = \frac{1}{a(a+b)} + \frac{1}{b(a+b)}$

$LHS = \frac{(a+b)}{ab(a+b)}$

$LHS = \frac{1}{ab}$

$\therefore LHS = RHS$

(ii) Hence express $\frac{1}{7}$ in the form $\frac{1}{a} + \frac{1}{b}$ where a and b are positive integers **1**

Choose factors of 7 → 1 and 7 as a and b (from above) and write in reverse

$\frac{1}{1\times 7} = \frac{1}{1^2 + 1\times 7} + \frac{1}{7^2 + 1\times 7}$

$= \frac{1}{8} + \frac{1}{56}$

END OF QUESTION 1

QUESTION 2 (Start a New Page) (12 Marks)

(a) Calculate the remainder when $P(x) = x^3 + 2$ is divided by $x - 1$ **1**

$R(1) = 1^3 + 2$

$R(1) = 3$

(b) The polynomial $P(x)=x^3+ax+12$ has a factor $(x+3)$. Find the value of a. **2**

$R(-3)=(-3)^3+a(-3)+12$
$0=-27-3a+12$
$0=-3a-15$
$a=-5$

As $(x+3)$ is a factor, the remainder is 0 (i.e. $R(-3)=0$)

(c) Find the quotient Q(x) and the remainder, R(x), when the polynomial $P(x)=x^4-x^2+1$ is divided by x^2+1 **3**

$$\begin{array}{r|l} & x^2-2 \\ x^2+0x+1 & x^4+0x^3-x^2+0x+1 \\ & \underline{x^4+0x^3+x^2} \\ & -2x^2+0x+1 \\ & \underline{-2x^2+0x-2} \\ & 3 \end{array}$$

$\therefore Q(x)=x^2-2$
$R(x)=3$

(d) The point P(3, 7) lies on the graph of the odd polynomial function $y=P(x)$. Find, with reasons,

(i) the remainder when P(x) is divided by $(x-3)$ **1**

From the point (3,7) above, when $x=3$, $y=7$, therefore, $R(3)=7$

(ii) the remainder when P(x) is divided by $(x+3)$ **1**

As the polynomial is odd, $P(x)=-P(-x)$, therefore, $R(-3)=-7$

(e) Solve $x^3-4x^2-x+4=0$ **2**

Firstly testing factors of 4 →

$P(1)=1^3-4(1)^2-1+4$
$P(1)=0$
→ $(x-1)$ is a factor and $x=1$ is a solution

$P(-1)=(-1)^3-4(-1)^2-(-1)+4$
$P(-1)=0$
→ $(x+1)$ is a factor $x=-1$ is a solution

$P(4)=4^3-4(4)^2-4+4$
$P(4)=0$
→ $(x-4)$ is a factor $x=4$ is a solution

$\therefore (x+1)(x-1)(x-4)=x^3-4x^2-x+4$

OR

$\therefore x=-1,1,4$

(f) If α, β and γ are roots of the cubic equation $5x^3+7x^2-3x-4=0$, find the value of **2**

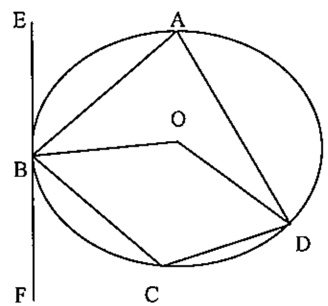
$\alpha^2+\beta^2+\gamma^2$

$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-\alpha\beta-\alpha\gamma-\beta\alpha-\beta\gamma-\gamma\alpha-\gamma\beta$
$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$
$\alpha^2+\beta^2+\gamma^2=\left(\frac{-b}{a}\right)^2-2\left(\frac{c}{a}\right)$
$\alpha^2+\beta^2+\gamma^2=\left(\frac{-7}{5}\right)^2-2\left(\frac{-3}{5}\right)$
$\alpha^2+\beta^2+\gamma^2=\left(\frac{49}{25}\right)+\frac{6}{5}$
$\alpha^2+\beta^2+\gamma^2=\frac{79}{25}$

END OF QUESTION 2

QUESTION 3 (Start a New Page) (12 Marks)

(a)



ABCD is a quadrilateral inscribed in a circle with centre O. Reflex ∠BOD = 240°.
Find, giving reasons

(i) the size of ∠BAD **2**

∠BOD = 120° (∠'s at a point given reflex ∠BOD=240°)
∠BAD = 60° (∠ at centre = twice ∠ at circumference on same arc)

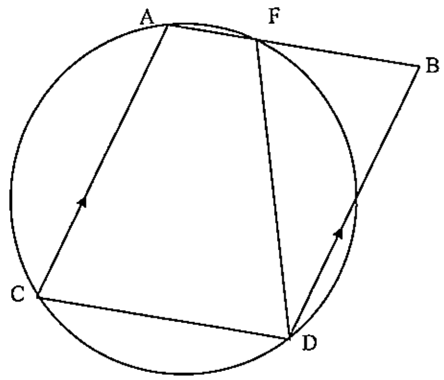
(ii) the size of ∠BCD **2**

∠BCD = 120° (given ∠BOD = 240°, ∠ at centre = twice ∠ at circumference on same arc)

(iii) the size of ∠FBD **2**

∠FBD = ∠BAD (∠ between tangent and chord = ∠ in alternate segment)
∴ ∠FBD = 60° (∠BAD = 60° from above)

(b)



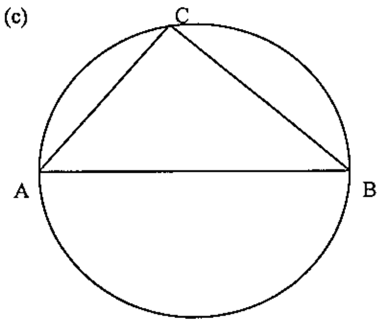
In the diagram (not to scale),

AC = BD and AC // BD. Prove DF = DB

4

ABCD is a //ogram (given 1 pair opp sides = and //)
$\therefore \angle ACD = \angle ABD$ (opp $\angle$'s //ogram ABCD=)
ACDF is a cyclic quad (quad with all vertices on the circumference of same circle)
$\angle BFD = \angle ACD$ (ext $\angle$ cyc. quad. = $\angle$ in alt seg.)
$\therefore \angle BFD = \angle DBF$ (both = $\angle ACD$)
$\therefore$ ΔDBF is isosc. (base $\angle$'s =)
$\therefore$ DF = DB (= sides opp = $\angle$'s of isosc ΔDBF)

(c)



2

In the diagram (not to scale), AB = 41 cm,
AC = 9 cm and BC = 40 cm
Prove AB is the diameter of the circle drawn

$$AB^2 = 41^2$$
$$AB^2 = 1681$$

$$AC^2 + BC^2 = 40^2 + 9^2$$
$$AC^2 + BC^2 = 1681$$
$$AC^2 + BC^2 = AB^2$$

$\therefore$ ΔABC is a right angled triangle (square of hyp = sum of squares of other 2 sides)
$\therefore \angle ACB = 90°$ (opp hypotenuse)
$\therefore$ AB is a diameter ($\angle$ at the circumference in a semi circle is 90°)

END OF QUESTION 3

QUESTION 4 (Start a New Page) (12 Marks)

(a) By using an expression for $\tan(\alpha + \beta)$, find the exact value of tan 105° **2**

$$\tan 105° = \tan(60° + 45°)$$
$$\tan 105° = \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$$
$$\tan 105° = \frac{\sqrt{3}+1}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$
$$\tan 105° = \frac{3+1+2\sqrt{3}}{1-3}$$
$$\tan 105° = \frac{4+2\sqrt{3}}{-2}$$
$$\tan 105° = -2-\sqrt{3}$$

(b) Prove that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$ **3**

Using the *t* Method

Let $\tan\theta = t$ then $\tan 2\theta = \frac{2t}{1-t^2}$,

$\sin 2\theta = \frac{2t}{1+t^2}$ and $\cos 2\theta = \frac{1-t^2}{1+t^2}$

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$$
$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{\frac{2t}{1+t^2}}{\frac{1+t^2}{1+t^2}+\frac{1-t^2}{1+t^2}}$$
$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{\frac{2t}{1+t^2}}{\frac{2}{1+t^2}}$$
$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2t}{1+t^2} \times \frac{1+t^2}{2}$$
$$\frac{\sin 2\theta}{1+\cos 2\theta} = t$$
$$\therefore \frac{\sin 2\theta}{1+\cos 2\theta} = \tan 2\theta$$

Using Sum/Diff of Angles Method

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+2\cos^2\theta - 1}$$
$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$$
$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{\sin\theta}{\cos\theta}$$
$$\therefore \frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$$

(c) Solve the equation $\sqrt{3}\sin\ \theta - \cos\ \theta = 1$ for $0 \le \theta \le 2\pi$
4

Using $a\sin\theta - b\cos\theta = r\sin(\theta-\alpha)$ where $r=\sqrt{a^2+b^2}$ and $\tan\alpha = \frac{b}{a}$, α acute

$a=\sqrt{3}$, $b=-1$ → $r=\sqrt{3+1}$, $r=2$ and $\tan\alpha = \frac{-1}{\sqrt{3}}$, $\alpha = \frac{\pi}{6}$

$\therefore \sqrt{3}\sin\theta - \cos\theta = 2\sin(\theta - \frac{\pi}{6})$

$\therefore 2\sin(\theta-\frac{\pi}{6}) = 1$

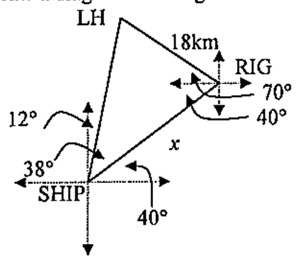
$\sin(\theta - \frac{\pi}{6}) = \frac{1}{2}$ O

$\theta - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = \frac{\pi}{3}, 2\pi$

(d) A ship, S, is sailing on a straight course for an oil rig, R, on a bearing of 050°. At the same time, a lighthouse, L, is sighted in a direction of 012°. The charts indicate that the bearing from the rig to the lighthouse is 340°, and a distance of 18 km from it.

(i) Draw a diagram showing the above information **1**



(ii) Find the distance from the ship to the oil rig correct to the nearest metre **2**

$\frac{x}{\sin 32^\circ} = \frac{18}{\sin 38^\circ}$

$x = \frac{18\sin 32^\circ}{\sin 38^\circ}$

$x = 15.493 km$

END OF QUESTION 4

QUESTION 5 (Start a New Page) (12 Marks)

(a) Show $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$ **2**

$\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \frac{\cos\theta\sin 2\theta}{\sin\theta\cos\theta} - \frac{\sin\theta\cos 2\theta}{\sin\theta\cos\theta}$

$\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \frac{\sin(2\theta-\theta)}{\sin\theta\cos\theta}$

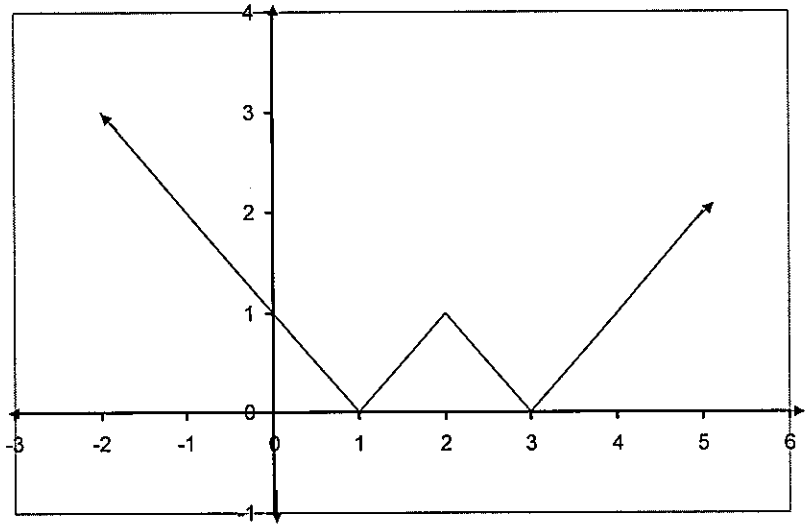
$\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \frac{\sin\theta}{\sin\theta\cos\theta}$

$\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \frac{1}{\cos\theta}$

$\therefore \frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$

(b) Sketch $f(x) = \left||x-2|-1\right|$ **2**

By inspection of critical points $x=1$, $x=2$, $x=3$ and sketching



(c) Using First Principles, find the derivative of **2**

$f(x) = 4x^2 + 5$

Let there be a small increase in x, by h

$f(x+h) = 4(x+h)^2 + 5$
$f(x+h) = 4(x^2 + 2hx + h^2) + 5$
$f(x+h) = 4x^2 + 5 + 8hx + 4h^2$

Now the gradient of the tangent is given by

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$f'(x) = \lim_{h \to 0} \frac{4x^2 + 5 + 8hx + 4h^2 - (4x^2 + 5)}{h}$
$f'(x) = \lim_{h \to 0} \frac{8hx + 4h^2}{h}$
$f'(x) = \lim_{h \to 0} 8x + 4h$
$f'(x) = 8x$

(d) For the curve $y = x^2(x^2 - 1)$,

(i) Find the derivative $\frac{dy}{dx}$ **1**

$\frac{dy}{dx} = 2x(x^2 - 1) + x^2(2x)$
$\frac{dy}{dx} = 2x(x^2 - 1 + x^2)$
$\frac{dy}{dx} = 2x(2x^2 - 1)$

(ii) Hence or otherwise, find the equation of the normal at the point (1, 0) **2**

At x=1

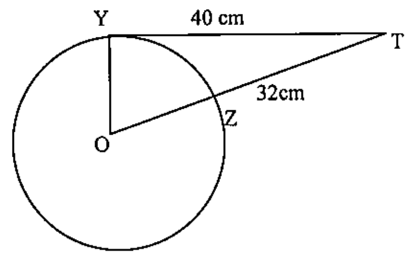
$\frac{dy}{dx} = 2(2-1)$
$\frac{dy}{dx} = 2$

At (1, 0) and $m = \frac{-1}{2}$

$y - y_1 = m(x - x_1)$
$y - 0 = \frac{-1}{2}(x - 1)$
$y = \frac{-x}{2} + \frac{1}{2}$
or
$x + 2y - 1 = 0$

(e) The straight line joining the centre O, of a circle to an external point T, cuts the circumference of the circle at Z. TZ = 32 cm and the tangent to the circle from T is 40 cm long.

(i) Draw a diagram to represent this information **1**



(ii) Calculate the radius of the circle giving reasons for your answer **2**

Let the radius of the circle be r and the pt of contact with the radius and the tangent be Y

$TY \perp OY$ (radius of circle $\perp$ tangent at pt of contact)

Therefore
$r^2 + 40^2 = (r + 32)^2$
$r^2 + 1600 = r^2 + 64r + 1024$
$64r = 576$
$r = 9cm$

END OF QUESTION 5

QUESTION 6 (Start a New Page) (12 Marks)

(a) Solve for x: $|x^2 - 5| \le 5x + 9$ **2**

When $|x^2 - 5| > 0$
$x^2 - 5 \le 5x + 9$
$x^2 - 5x - 14 \le 0$
$(x - 7)(x + 2) \le 0$

Critical Points
-2 and 7

Testing Points

As $x \to -2^-$ $(x-7)(x+2) \ge 0$
As $x \to -2^+$ $(x-7)(x+2) \le 0$
As $x \to 7^-$ $(x-7)(x+2) \le 0$
As $x \to 7^+$ $(x-7)(x+2) \ge 0$
i.e. $-2 \le x \le 7$

When $|x^2 - 5| < 0$
$x^2 - 5 \ge -5x - 9$
$x^2 + 5x + 4 \ge 0$
$(x + 4)(x + 1) \ge 0$

Critical Points
-4 and -1

As $x \to -4^-$ $(x+4)(x+1) \ge 0$
As $x \to -4^+$ $(x+4)(x+1) \le 0$
As $x \to -1^-$ $(x+4)(x+1) \le 0$
As $x \to -1^+$ $(x+4)(x+1) \ge 0$
i.e. $-4 \ge x \ge -1$

$\therefore -1 \le x \le 7$

(b) The equation $x^3 + px^2 + qx + pq = 0$, where $p \neq 0$ and $q \neq 0$ has 3 real roots α, β and γ.

(i) By considering the relationships between the roots and the coefficients of the equation, show that $(\alpha+\beta+\gamma)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)=1$ **2**

$(\alpha+\beta+\gamma)=\frac{-b}{a}$

$\alpha+\beta+\gamma=-p$

$\therefore(\alpha+\beta+\gamma)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)=-p\times\frac{-1}{p}$

$\therefore(\alpha+\beta+\gamma)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)=1$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\frac{c}{a}}{\frac{-d}{a}}$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{q}{-pq}$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{-1}{p}$

(ii) Show that $-p$ is a root of the equation. Hence show that $q<0$. **2**

If $-p$ is a root, then $P(-p)=0$

$P(-p)=(-p)^3+p(-p)^2+q(-p)+pq$

$P(-p)=-p^3+p^3-pq+pq$

$P(-p)=0$

Therefore $-p$ is a root

$x^3+px^2+qx+pq=x^2(x+p)+q(x+p)$

$x^3+px^2+qx+pq=(x+p)(x^2+q)$

Therefore $P(x)=0$ when $(x+p)=0$ OR $(x^2+q)=0$

$x^2=-q$

BUT

So either $x=-p$ OR $x^2\geq 0$

$\therefore -q>0$

$\therefore q<0$

(c) The quadratic equation $x^2-x+k=0$, where k is a real number has 2 distinct, positive real roots.

(i) Show that $0<k<\frac{1}{4}$ **2**

$\Delta=b^2-4ac$

$\Delta=1-4k$

For 2 distinct, positive real roots, $\Delta>0$ and the product of the roots is positive

$1-4k>0$

$1>4k$

So $\frac{1}{4}>k$

$k<\frac{1}{4}$

and

$\alpha\beta>0$

$\therefore\frac{c}{a}>0$

$\therefore k>0$

Therefore $0<k<\frac{1}{4}$

(ii) Let the two roots be α and β. Show that $\alpha^2+\beta^2=1-2k$ and deduce that $\frac{1}{2}<\alpha^2+\beta^2<1$ **4**

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$\alpha^2+\beta^2=1^2-2k$

$\alpha+\beta=\frac{-b}{a}$

$\alpha+\beta=1$

From (i) above

$0<k<\frac{1}{4}$

$0<2k<\frac{1}{2}$

$0>-2k>\frac{-1}{2}$

$1>1-2k>\frac{1}{2}$

$\frac{1}{2}<1-2k<1$

$\frac{1}{2}<\alpha^2+\beta^2<1$

END OF EXAMINATION