MATHEMATICS EXTENSION 1

PRELIMINARY EXAMINATION

2004

Time Allowed – 1.5 hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt all Questions.
- Start each Question on a new page.
- Board approved calculators may be used.
- All necessary working should be shown.

OUTCOMES ASSESSED
PE1, PE2, PE3, PE4, PE5, PE6.

QUESTION 1 (12 Marks) **Marks**

a). Solve for x:
$$\frac{3x-1}{x-2} \geq 1$$ 3

b). Find the acute angle between the lines $3x + y = 4$ and $2x - 3y = 1$.
(Write your answer to the nearest minute). 2

c). Find the exact value of $\sin 75°$ 2

d). The equation $x^2 + px + 24 = 0$ has one root which is one and a half times the other. Show that there are two possible values for p and find the roots of each of these values. 3

e). Write down the equation of the locus of a parabola with vertex at (2,3) and focus (2,1). 2

QUESTION 2 (12 Marks) **Use a new page**

a). Find $\lim_{x \to 2} \frac{x^3-8}{x-2}$ 2

b). If $f(x) = \frac{1}{\sqrt{x - x^2}}$ find the domain. 1

c). Find the general solution to the equation $2\cos(2x - 60°) = \sqrt{3}$ 2

d). The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$

(i). Show that the equation of the tangent at P is $y = px - ap^2$ 2

(ii). If $Q(2aq, aq^2)$ also lies on the parabola, and PQ is a focal chord, show that the tangents at P and Q intersect on the directrix at right angles. 3

e). How many different arrangements of the letters of the word *SERIES* are possible if
(i). the two letters S are to be together.
(ii). no two vowels are together. 2

QUESTION 3 (12 Marks) Use a new page **Marks**

a).

(i). In how many ways can 6 men and 5 women sit around a circular table without restrictions? **2**

(ii). How many ways are there if 2 particular women wish to sit next to a particular man? **2**

b). For the curve $y = \dfrac{x^2 - x + 1}{x - 1}$

(i). Find any stationary points and determine their nature.

(ii). Write down the equations of any asymptotes.

(iii). Write down any intercepts made with the co-ordinate axes.

(iv). Sketch the curve. **8**

QUESTION 4 (12 Marks) Use a separate page **Marks**

a).(i). Write the expansion for $\cos(\alpha + \theta)$ **1**

(ii). Hence, or otherwise, prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **2**

(iii). Solve $8\cos^3\theta - 6\cos\theta - \sqrt{3} = 0$ for $0° \le \theta \le 360°$ **2**

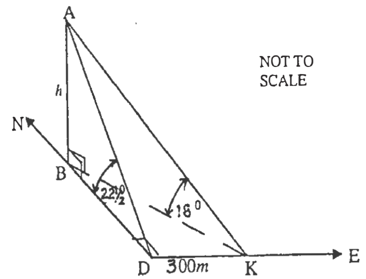
b). Prove $\dfrac{\tan x - \tan y}{\tan x + \tan y} = \dfrac{\sin(x-y)}{\sin(x+y)}$ **3**

c). If P (6, 2) divides the interval AB externally in the ratio 2:7, find the co-ordinates of B, if A has the co-ordinates (4, 0). **2**

d). Find the co-ordinates of Q, given that the chord of contact of the tangents from Q to the parabola $x^2 = 14y$ is $5x - 14y - 21 = 0$ **2**

QUESTION 5 (12 Marks) Use a new page **Marks**

a). Doris is standing at D and observes the angle of elevation of the tip of a flagpole A, on top of a building to be 22½° .Her friend Katie, who is standing at K, 300 metres due east of Doris, finds the angle of elevation of the tip of the flagpole to be 18°. The building is due north of Doris and B is the base of the building. The points B, D and K are on level ground.



(i) Show that the height (h) of the flagpole above the ground is given by $h = 300\,(\cot^2 18° - \cot^2 22½°)^{-½}$ **3**

(ii) Find the value of h, correct to 3 significant figures **1**

b) (i) Express $\sqrt{3}\cos\theta + \sin\theta$ in the form $A\cos(\theta - \alpha)$ where A>0 **2**

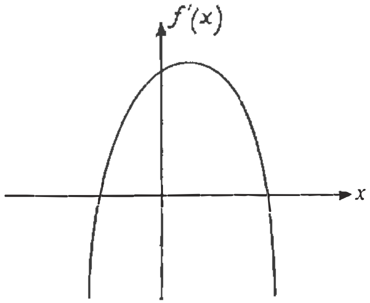
(ii) Hence, solve the equation $\sqrt{3}\cos\theta + \sin\theta = -\sqrt{3}$ for $0° \le \theta \le 360°$ **2**

c) (i) Given that $\dfrac{d^2y}{dx^2} = x - 1$, and the tangent to the curve $y = f(x)$ at the point (0, -1) is inclined at 45° to the positive x - axis. Find the equation of the curve. **2**

(ii) Find $\dfrac{d^2y}{dx^2}$ for the function $y = \sqrt{1-2x}$ **2**

QUESTION 6 **(12 Marks)** **Use a new page** **Marks**

a). Given the gradient function in the diagram below:- **2**



Draw a set of axes and sketch a function to represent f(x).

b) **(i)** If $f(x)=\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$, show that $f'(x)=\dfrac{-1}{\sqrt{x}\left(\sqrt{x}-1\right)^2}$ **2**

(ii) Show that the equation of the tangent at (4,3) on the curve

$f(x)=\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$ is given by $x+2y-10=0$ **2**

c) The running cost (cost of fuel) for a certain ship is \$4 per hour when the ship is not moving, and this cost increases by an amount that is proportional to the cube of its speed, Vkm/h. If the running cost per hour is \$7.25 when the speed is 15 km/h, obtain a formula for the running cost per hour at speed V, and calculate the value of V for which the total running cost for a journey of 500km is a minimum. **4**

d) Find the Cartesian equation of the curve whose parametric equations are $x = 1 + 2^{-t}$ and $y = 2^t$. **2**

Question 1

a) Solve $\dfrac{3x-1}{x-2} \geqslant 1$ $\quad x \neq 2$ (1)

For $x-2>0$

$x>2$

$3x-1 \geqslant x-2$

$2x \geqslant -1$

$x \geqslant -\frac{1}{2}$

Simultaneously: $x>2$

For $x-2<0$

$x<2$

$3x-1 \leq x-2$

$2x \leq -1$

$x \leq -\frac{1}{2}$

Simultaneously: $x \leq -\frac{1}{2}$

Solution is $x \leq -\frac{1}{2}$ or $x>2$. /3

b) Find the acute angle between $3x+y=4$ & $2x-3y=1$.

l_1 : $y=4-3x$ $\quad \therefore m_1=-3$

l_2 : $y=\frac{2x}{3}-\frac{1}{3}$ $\quad \therefore m_2=\frac{2}{3}$

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{-3-\frac{2}{3}}{1+(-3)(\frac{2}{3})}\right| = \frac{11}{3}$$

$\therefore \theta = 74°45'$

Thus, the acute angle is $74°45'$. /2

c) Find $\sin 75°$.

$$\sin 75° = \sin(30°+45°) = \sin 30°\cos 45° + \sin 45° \cos 30° = \frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}$$

$$\therefore \sin 75° = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

d) Let $x^2+px+24=0$ have 2 roots α, β, where $\frac{3}{2}\alpha = \beta$ (1)

then sum of roots: $\frac{3}{2}\alpha + \alpha = \frac{-p}{1}$

$\therefore \frac{5\alpha}{2} = -p$

product of roots: $\frac{3\alpha^2}{2} = 24$

$3\left(\frac{-2p}{5}\right)^2 = 48$

$p^2 = 100$

$\therefore p = \pm 10$ /3

The 2 resulting quadratics are

$x^2-10x+24=0$ where the roots are $x=4, 6$

$x^2+10x+24=0$ where the roots are $x=-4,-6$.

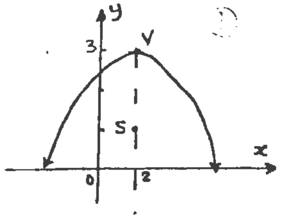
e) Focus $(2,1)$, Vertex $(2,3)$

$\therefore a=2$

Equation has the form:

$(x-h)^2 = -4a(y-k)$

$\therefore (x-2)^2 = -8(y-3)$ # /2



Question 2

a) $\lim\limits_{x \to 2} \frac{x^3-8}{x-2} = \lim\limits_{x \to 2} \frac{(\cancel{x-2})(x^2+2x+4)}{\cancel{x-2}}$ for $x \neq 2$ 1

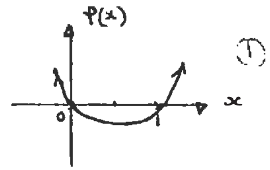
$= \lim\limits_{x \to 2} x^2 + 2x + 4$

$= 2^2 + 2 \times 2 + 4$

$= 12$

/2

b) Given $f(x) = \frac{1}{\sqrt{x - x^2}}$ 1



then $x - x^2 > 0$

ie $x^2 - x < 0$

$x(x-1) < 0$

$\therefore 0 < x < 1$

/1

Domain $= \{x : 0 < x < 1\}$

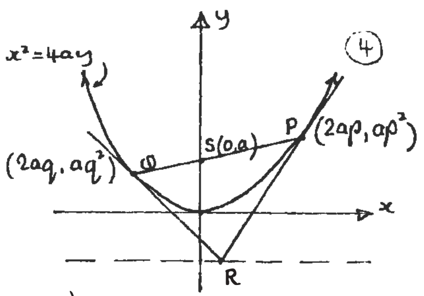
c) $2\cos(2x - 60^\circ) = \sqrt{3}$ $\quad \cos 30^\circ = \frac{\sqrt{3}}{2}$

$2x - 60^\circ = n.360^\circ \pm 30^\circ$

$2x = n.360^\circ + 90^\circ,\ n.360^\circ + 30^\circ$

$\therefore x = n.180^\circ \pm 15^\circ + 30^\circ$

) (i) Show the eqn. of the tangent at P is $y = px - ap^2$.



$y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$ at $x = 2ap$

$m = p$

$y - ap^2 = p(x - 2ap)$

$y = px - ap^2$ (as required)

/2

b) (iii) Intercepts : if $y = 0$ then $\frac{x^2 - x + 1}{x - 1} = 0$

has no solutions, thus there are no x intercepts.

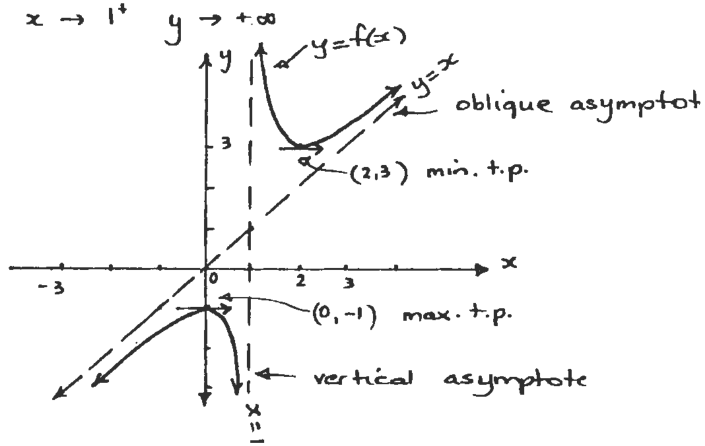
if $x = 0 \therefore y = -1$.

(iv) Sketch

Check the curve near the asymptote $x = 1$

as $x \to 1^-$ $\quad y \to -\infty$

as $x \to 1^+$ $\quad y \to +\infty$



Question 3a

(i) The first person sits anywhere, and then treating the circle now as a line, the remaining 10 people sit in any spot. That is $10!$ possibilities.

(ii) Consider the woman-man-woman group as a single 'unit' in the circle (two possible arrangements), leaving 5 men and 3 women i.e. 8 people.
Using a similar argument to (i), in total there are $8! \times 2$ possibilities (the 2 comes from arrangement within the group)

Question 3b

(i) *Stationary points:*

$$y = \frac{x^2 - x}{x-1} + \frac{1}{x-1} \quad = x + \frac{1}{x-1}$$

$$\frac{dy}{dx} = 1 - \frac{1}{(x-1)^2} = \frac{(x-1)^2 - 1}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{(x-1)^3}$$

Therefore, stationary points occur at $x = 0$ and $x = 2$.

The second derivative at $x = 0$ is negative, so $x = 0$ is a **local maxima,**

The second derivative at $x = 2$ so $x = 2$ is a **local minima**.

(ii) *Asymptotes:*

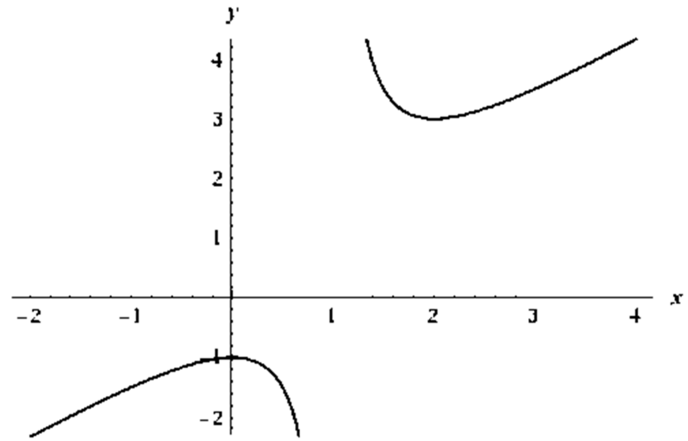
The equation of the asymptote is $x = 1$

(iii) *Intercepts:*

There is no x-intercept as the discriminant of $x^2 - x - 1$ is negative which implies that $x^2 - x - 1 = 0$ has no solution.

The y-intercept is $y = -1$

(iv) *Graph of function:*



Question 4

a) (i) $\cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta$.

(ii) Prove $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Proof: $L.HS = \cos 3\theta$

$= \cos(2\theta+\theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta.\sin\theta$

$= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta.\sin\theta$

$= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$

$= \cos^3\theta - 3\sin^2\theta\cos\theta$

$= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$= 4\cos^3\theta - 3\cos\theta$.

$\therefore$ LHS = RHS

(iii) Solve $8\cos^3\theta - 6\cos\theta - \sqrt{3} = 0 \quad 0^\circ \le \theta \le 360^\circ$.

divide b.s. by 2 giving

$4\cos^3\theta - 3\cos\theta = \frac{\sqrt{3}}{2}$

Now, from above

$4\cos^3\theta - 3\cos\theta = \cos 3\theta$

$\therefore \quad \cos 3\theta = \frac{\sqrt{3}}{2}$

$3\theta = 30^\circ, 330^\circ, 390^\circ, 690^\circ, 750^\circ, 1050^\circ$

$\therefore \theta = 10^\circ, 110^\circ, 130^\circ, 230^\circ, 250^\circ, 350^\circ$.

b) Prove $\dfrac{\tan x - \tan y}{\tan x + \tan y} = \dfrac{\sin(x-y)}{\sin(x+y)}$

L.HS. $= \dfrac{\frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}}{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}$

$= \dfrac{\frac{\sin x\cos y - \sin y\cos x}{\cos x\cos y}}{\frac{\sin x\cos y + \sin y\cos x}{\cos x\cos y}}$

$= \dfrac{\sin x\cos y - \sin y\cos x}{\sin x\cos y + \sin y\cos x}$

$= \sin(x-y)$ ...

c) $P(6,2)$ divides AB externally in the ratio 2:7

If $A(4,0)$ find $B(x_2, y_2)$

$6 = \dfrac{(2)(x_2) - (7)(4)}{2-7} \qquad 2 = \dfrac{(2)(y_2) - (7)(0)}{2-7}$

$\therefore x_2 = -1 \qquad y_2 = -5$

Thus, the coordinates of B are $(-1,-5)$ /2

d) Find the coordinates of Q, given that the chord of contact of the tangents from Q to $x^2 = 14y$ is $5x - 14y - 21 = 0$.

The chord of contact has the form:

$xx_0 = 2a(y+y_0) \qquad$ where $(x_0, y_0) = Q$

$\therefore \quad xx_0 = 7y + 7y_0 \qquad a = \frac{7}{2}$

ie $\quad xx_0 - 7y - 7y_0 = 0$

given $\quad 5x - 14y - 21 = 0$

The ratio of the coefficients are equal

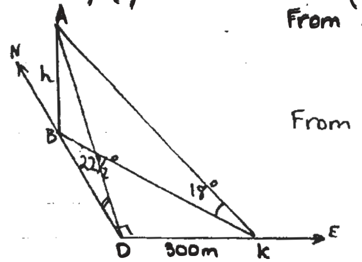
$\therefore \dfrac{x_0}{5} = \dfrac{-7}{-14} \qquad \dfrac{-7y_0}{-21} = \dfrac{-7}{-14}$

$\therefore x_0 = 5/2 \qquad y_0 = \frac{3}{2}$

Thus, Q has coordinates $(5/2, 3/2)$

Question 5 (3)

a) (i) show $h = 300(\cot^2 18° - \cot^2 22\tfrac{1}{2}°)^{-1/2}$.



From ΔBAD: $\cot 22\tfrac{1}{2}° = \frac{BD}{h}$

$\therefore BD = h \cot 22\tfrac{1}{2}°$

From ΔBAK: $\cot 18° = \frac{BK}{h}$

$\therefore BK = h \cot 18°$

Using Pythagoras' Theorem:-

$h^2\cot^2 18° - h^2 \cot^2 22\tfrac{1}{2}° = 300^2$

$\therefore h = 300(\cot^2 18° - \cot^2 22\tfrac{1}{2}°)^{-1/2}$ /3

(ii) $h = 157.16268\ldots$

≈ 157 metres /1

b) (i) Express $\sqrt{3}\cos\theta + \sin\theta$ in the form $A\cos(\theta - \alpha)$.

$A\cos(\theta-\alpha) = A(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= \sqrt{3}\cos\theta + \sin\theta$

Equating like parts,

$A\cos\alpha = \sqrt{3}$ $\quad A\sin\alpha = 1$ $\quad \tan\alpha = \frac{1}{\sqrt{3}}$

$A^2(\cos^2\alpha + \sin^2\alpha) = \sqrt{3}^2 + 1^2$ $\quad \therefore \alpha = 30°$

$A^2 = 4$

$\therefore A = 2$ (take +ve root only!)

$\therefore \cos\alpha = \sqrt{3}/2$ & $\sin\alpha = 1/2$

$\therefore \sqrt{3}\cos\theta + \sin\theta = 2.\cos(\theta - 30°)$ #

(ii) Solve $\sqrt{3}\cos\theta + \sin\theta = -\sqrt{3}$ $\quad 0° \le \theta \le 360°$

From (i) $\quad 2\cos(\theta - 30°) = -\sqrt{3}$

$\therefore \cos(\theta-30°) = -\sqrt{3}/2$

$\theta - 30° = 150°, 210°$

$\therefore \theta = 180°, 240°$ #

$180-\theta$ S	A
$180+\theta$ T	C

$\cos 30° = \frac{\sqrt{3}}{2}$

c) (i) $\frac{d^2y}{dx^2} = x - 1$ (1)

$\frac{dy}{dx} = \frac{x^2}{2} - x + c$ $\quad$ when $x = 0$ $\frac{dy}{dx} = -1$

$-1 = c$

$\therefore \frac{dy}{dx} = \frac{x^2}{2} - x - 1$

$y = \frac{x^3}{6} - \frac{x^2}{2} - x + K$ $\quad$ when $x = 0, y = -1$

$-1 = K$

$\therefore y = \frac{x^3}{6} - \frac{x^2}{2} - x - 1$ # /2

(ii) $y = \sqrt{1-2x}$

$= (1-2x)^{1/2}$

$\frac{dy}{dx} = \frac{1}{\cancel{2}}(1-2x)^{-1/2} . -\cancel{2}$

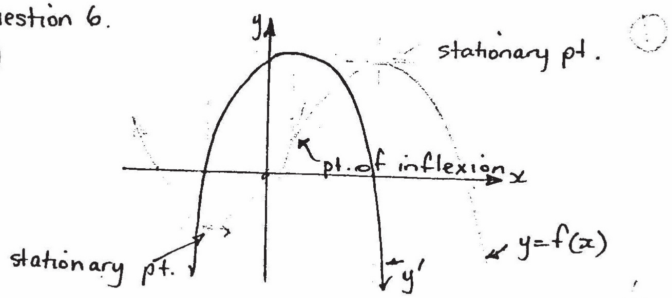
$= \frac{-1}{\sqrt{1-2x}}$

$\frac{d^2y}{dx^2} = -1. -\frac{1}{\cancel{2}}(1-2x)^{-3/2} . -\cancel{2}$

$= \frac{-1}{\sqrt{1-2x}^3}$ /2.

Question 6.

a)



b) (i) $f(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}-1}$

$f'(x) = \dfrac{\dfrac{\sqrt{x}-1}{2\sqrt{x}} - \dfrac{(\sqrt{x}+1)}{2\sqrt{x}}}{(\sqrt{x}-1)^2}$

$= \dfrac{-2}{2\sqrt{x}(\sqrt{x}-1)^2}$

$\therefore f'(x) = \dfrac{-1}{\sqrt{x}(\sqrt{x}-1)^2}$ (as required)

(ii) The equation of the tangent has the form: $y - y_1 = m(x - x_1)$ where $(x_1, y_1) = (4, 3)$

ie $y - 3 = -\frac{1}{2}(x - 4)$ $m = \dfrac{-1}{\sqrt{4}(\sqrt{4}-1)^2} = -\dfrac{1}{2}$

$2y - 6 = -x + 4$

$\therefore x + 2y - 10 = 0$ (as required)

c) (i) Weekly rent for n increases

$= \$(300 + 20 \times n)$

No of units occupied

$= 100 - 4 \times n$

$\therefore$ Rent (R) $= (300 + 20n)(100 - 4n)$

$= 30000 + 800n - 80n^2$ (as required)

(ii) To maximize revenue/rent:

$\dfrac{dR}{dn} = 800 - 160n$

$\dfrac{d^2R}{dn^2} = -160$ ($<0 \Rightarrow$ maximum)

For turning pts $\dfrac{dR}{dn} = 0$

ie $800 = 160n$

$\therefore n = 5$

$R = \dfrac{30000 + 800 \times 5 - 80 \times 5^2}{100}$

$\therefore R = \$320.00$

So the weekly rent should be \$320 in order to maximize revenue.

d) Find the cartesian equation of the parametric equations

$x = 1 + 2^{-t}$ — ①

$y = 2^t$ — ②

From ① $(2^{-t})^{-1} = (x-1)^{-1}$

$2^t = \dfrac{1}{x-1}$

where $y = 2^t$.

$\therefore y = \dfrac{1}{x-1}$ #

c) Cost/hour $= \$(3 + kV^3)$

$6.75 = 3 + 3375k$

$\therefore k = \dfrac{1}{900}$.

Let x be total running costs:-

$x = \left(3 + \dfrac{V^3}{900}\right) \times \dfrac{450}{V}$ ↙ no. of hours

$\dfrac{dx}{dV} = -\dfrac{1350}{V^2} + V = 0$

$V^3 = 1350$

$\therefore V = 11.052\ldots$

when $V < 11$ $\dfrac{dx}{dV} < 0$

when $V > 11$ $\dfrac{dx}{dV} > 0$

Thus, we have a min. cost